ATTRACTORS OF THE REACTION-DIFFUSION SYSTEMS WITH RAPIDLY OSCILLATING COEFFICIENTS AND THEIR HOMOGENIZATION.

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Dedicated to Prof. W.Jaeger
on the occasion of his 60th birthday

Introduction

We consider the reaction-diffusion system

$$\begin{cases} \partial_t u = A_\varepsilon u - f(u) + g, \quad x \in \Omega, \\ u\big|_{t=0} = u_0, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{0.1}$$

in a bounded domain $\Omega \subset\subset \mathbb{R}^3$ with a sufficiently smooth boundary. Here $u = u(t,x) = (u^1, \cdots, u^k)$ is an unknown vector-valued function, the functions $f(u) = (f^1(u), \cdots, f^k(u))$ and $g = g(x) = (g^1, \cdots, g^k)$ are given and the second order elliptic differential operator A_ε has the following form:

$$A_\varepsilon u := \operatorname{diag}\left(A^1_\varepsilon u^1, \cdots, A^k_\varepsilon u^k\right) \tag{0.2}$$

with

$$A^l_\varepsilon u^l := \sum_{i,j=1}^{3} \partial_{x_i}\left(a^l_{ij}\left(\varepsilon^{-1}x\right)\partial_{x_j} u^l(x)\right), \quad \varepsilon \ll 1 \tag{0.3}$$

where the functions $a^l_{ij}(y)$, $y \in \mathbb{R}^3$ are assumed to be symmetric ($a^l_{ij}(y) \equiv a^l_{ji}(y)$) and almost-periodic with respect to $y \in \mathbb{R}^3$ (i.e. $a^l_{ij} \in AP(\mathbb{R}^3)$ see e.g. [10]) for every fixed indexes i, j, l and the uniform ellipticity condition

$$\sum_{i,j} a^l_{ij}(y)\xi_i\xi_j \ge \nu|\xi|^2, \quad y, \xi \in \mathbb{R}^3 \tag{0.4}$$

is also assumed (with the appropriate $\nu > 0$) to be valid for every operator A^l_ε.

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We impose the regularity conditions to the non-linear term f:

$$\begin{cases} 1. \ \ f \in C^1(\mathbb{R}^k, \mathbb{R}^k); \\ 2. \ \ |f(u)| \le C(1+|u|^p); \end{cases} \tag{0.5}$$

for a certain $p \ge 1$ and the following anisotropic dissipativity condition: there are the exponents $p_i \ge 2(p-1)$, $i = 1, \cdots, k$ such that

$$\sum_{i=1}^{k} f_i(v) v^i |v^i|^{p_i} \ge -C, \quad \text{for all } v \in \mathbb{R}^k \tag{0.6}$$

which generalizes the standard isotropic one ($p_i = 0$) and from the ours point of view is more adopted to study the problems of the type (0.1) with diagonal leading part A_ε.

For example if $k = 2$, $u = (v, w)$ the anisotropic dissipativity assumption is satisfied for the following non-linearities:

$$f(u) = \begin{pmatrix} v^3 - \alpha v - \beta w \\ w - \gamma v \end{pmatrix}, \quad \text{or} \quad f(u) = \begin{pmatrix} v^3 - w \\ w - v^3 \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}$$

Note that the first example (which corresponds to the Fitz-Nagumo equation see [8]) satisfies the standard isotropic dissipativity assumption as well; the second non-linearity satisfies the relation $f^1(u) + f^2(u) = 0$ (which is natural for chemical kinetics) and evidently does not satisfy the isotropic dissipativity condition (see the end of Section 2 for the further examples).

It is assumed also that the external force $g \in L^2(\Omega)$, the initial date $u_0 \in L^\infty(\Omega)$ and the solution $u(t)$ of the problem (0.1) is defined to be a function

$$u \in L^\infty([0,T] \times \Omega) \cap L^2([0,T], W_0^{1,2}(\Omega)) \cap C([0,T], L^2(\Omega)) \tag{0.7}$$

which satisfies the equation (0.1) in the sense of distributions. (Here and below we denote by $W^{l,p}$ the Sobolev space of functions whose derivatives up to the order l inclusively belong to L^p and $\|\cdot\|_{l,p} := \|\cdot\|_{W^{l,p}}$).

The problems of the type (0.1) has been intensively studied by many authors. The long-time behaviour of solutions of (0.1) for *fixed* $\varepsilon > 0$ are considered under the various assumptions on the nonlinear term f and the operator $A_\varepsilon = A$ in [1], [6], [13].

The homogenization problems for individual solutions of linear and nonlinear elliptic or parabolic equations of the form (0.1) has been investigated in [2], [5], [7], [20] (see also the references therein).

The long-time behaviour of solutions of RDE and even hyperbolic equations in the non-homogenized periodic media with asymptotic degeneracy has been studied in [3], [12].

In the present paper we study the case where we have a system of reaction-diffusion equations (0.1) in the non-homogenized almost-periodic media. (For simplicity we restrict ourselves to consider only the most relevant from the applications point of view 3-dimensional case but the applied methods works after the minor changings for an arbitrary dimension n.)

It is proved that under the assumptions (0.2)–(0.6) for every fixed $\varepsilon > 0$ the problem (0.1) possesses a global attractor $\mathcal{A}^\varepsilon$ in the phase space $u_0 \in \Phi = L^\infty(\Omega)$. Moreover, these attractors are occured to be uniformly bounded with respect to $\varepsilon \to 0$ in the space $C^{2\beta}(\Omega)$ of Hölder continuous functions with an appropriate Holder exponent $1 > 2\beta > 0$:

$$\|\mathcal{A}^\varepsilon\|_{C^{2\beta}(\Omega)} \le C \quad \text{for } \varepsilon \le \varepsilon_0 \tag{0.8}$$

In order to study the behaviour of attractors $\mathcal{A}^\varepsilon$ when $\varepsilon \to 0$ we introduce the homogenized problem for the equation (0.1)

$$\begin{cases} \partial_t u = A_0 u - f(u) + g; \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0 \end{cases} \tag{0.9}$$

where $A_0 = \operatorname{diag}\{A_0^1, \cdots, A_0^k\}$ and the elliptic operators A_0^l are constructed by the standard formulae of the almost-periodic homogenization theory (see e.g. [20]).

It is proved that this limit problem also possesses the attractor $\mathcal{A}^0$ in the phase space $u_0 \in \Phi$ and the family of attractors $\mathcal{A}^\varepsilon$ tends to the limit attractor $\mathcal{A}^0$ in the following sense:

$$\operatorname{dist}_{C^{2\beta'}(\Omega)}\left(\mathcal{A}^\varepsilon, \mathcal{A}^0\right) \to 0 \text{ when } \varepsilon \to 0 \tag{0.10}$$

here the Hölder exponent $0 < 2\beta' < 2\beta < 1$ and dist_V means the non-symmetric Hausdorff distance between subsets of the space V:

$$\operatorname{dist}_V(X, Y) := \sup_{x\in X} \inf_{y \in Y} \|x - y\|_V \tag{0.11}$$

Thus, the attractors $\mathcal{A}^\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$ are occured to be upper-semicontinious at $\varepsilon = 0$ in the space $C^{2\beta'}(\Omega)$.

In order to illustrate the obtained results we give a more detailed consideration in the case where the coefficients a_{ij}^l are assumed to be periodic. In this case, imposing some additional requirements on the structure of the limit attractor $\mathcal{A}^0$ and using the method of asymptotic expansions we give the explicite estimate of the error of approximation of the non-homogenized attractors of (0.1) by the attractor $\mathcal{A}^0$ of the homogenized equation (0.9). Namely, assume in addition that the limit attractor is exponential, i.e. there is a positive number $\nu > 0$ and the function Q such that for every bounded subset $B \subset \Phi$ the following is true:

$$\operatorname{dist}_{L^2(\Omega)}\left(S_t^0 B, \mathcal{A}^0\right) \le Q(\|B\|_\Phi) e^{-\nu t} \tag{0.12}$$

where S_t^0 is a semigroup in Φ generated by the limit equation (0.9).

Theorem. *Let the coefficients a_{ij}^l be periodic and smooth enough and let the limit attractor $\mathcal{A}^0$ be exponential. Then*

$$\operatorname{dist}_{C(\Omega)}\left(\mathcal{A}^\varepsilon, \mathcal{A}^0\right) \le C\varepsilon^\kappa \tag{0.13}$$

where the constant C and exponent $0 < \kappa < 1$ can be calculated explicitly.

It is worth to emphasize also that the exponential rate of converges (0.12) has been already established for some classes of equations (0.9). For instance, let the non-linear term $f(u)$ be potential, i.e.

$$f(u) = \nabla_u F(u), \quad F \in C^2(\mathbb{R}^k, \mathbb{R}) \tag{0.14}$$

which is always true in a scalar case $k = 1$. Then the equation (0.9) evidently possesses the global Lyapunov function and consequently for generic g's (for which the equation (0.9) has a finite number of equilibria points and all of them are hyperbolic) the attractor $\mathcal{A}^0$ is regular and exponential (see [1], [15]). Therefore, due to the theorem, the estimate (0.13) is valid in this case.

Note that the estimates of the form (0.13) (for a symmetric or non-symmetric distance) for the regular attractors of the abstract semigroups which possess the Lyapunov functions and depend regularly on a parameter ε has been obtained in [1] (see Remark 4.3 for the estimate (0.13) of the symmetric distance in our case).

These results has been recently applied in [16] for obtaining the estimates of the form (0.13) for the reaction diffusion equations with $A_\varepsilon \equiv \Delta_x$ and spatial rapid quasiperiodic oscilations in the subbordinated terms (i.e. $f(u) := a(\varepsilon^{-1}x)\tilde{f}(u)$ or a more general non-linearities $f(u, \varepsilon^{-1}x)$). The analogues of these estimates for a singular perturbed non-autonomous parabolic systems with rapid *temporal* oscilations in subbordinated terms (for instance with $g = g(\varepsilon^{-1}t, x)$) has been obtained in [19].

§1 Uniform a priori estimates.

In this Section we derive a number of estimates for the solutions of the equation (0.1) which are of fundamental significance for our purposes.

We start with the uniform (with respect to $\varepsilon \to 0$) L^{p_i}- estimates for the solutions of (0.1).

Theorem 1.1. *Let the above assumptions be satisfied and let u be a solution of the problem* (0.1)*. Then the following estimate is valid:*

$$(1.1)\qquad \sum_{i=1}^{k} \|u^i(t)\|_{0,p_i+2}^{p_i+2} \le C\left(\sum_{i=1}^{k} \|u^i(0)\|_{0,p_i+2}^{p_i+2}\right)e^{-\alpha t} + C\left(1 + \sum_{i=1}^{k} \|g\|_{0,2}^{p_i+2}\right)$$

where $C, \alpha > 0$ are independent of ε.

Proof. Let us multiply the l-th equation of (0.1) by $u^l(t)|u^l(t)|^{p_l}$, integrate over $x \in \Omega$ and integrate by parts the leading term A_ε^l in the following way

$$(1.2)\qquad \left(A_\varepsilon^l u^l, u^l|u^l|^{p_l}\right) = \\ = -\frac{4(p_l+1)}{p_l^2}\left(\sum_{i,j} a_{ij}^l \partial_{x_j}\left(|u^l|^{(p_l+2)/2}\right), \partial_{x_i}\left(|u^l|^{(p_l+2)/2}\right)\right) \le \\ \le -C\nu\|\nabla_x\left(|u^l|^{(p_l+2)/2}\right)\|_{0,2}^2$$

(here we have used also the uniform ellipticity (0.4)).

Taking the sum of all obtained inequalities and taking into the account the dissipativity assumption (0.6) we derive that

$$(1.3)\qquad \partial_t\left(\sum_{i=1}^{k} \|u^i(t)\|_{0,p_i+2}^{p_i+2}\right) + C\nu\sum_{i=1}^{k} \|\nabla_x\left(|u^i(t)|^{(p_i+2)/2}\right)\|_{0,2}^2 \le \\ \le C_1 + \sum_{i=1}^{k}\left(g^i, u^i(t)|u^i(t)|^{p_i}\right) \equiv C_1 + G_u(t)$$

Denote also

$$(1.4)\qquad F_u(t) := \sum_{i=1}^{k} \|u^i(t)\|_{0,p_i+2}^{p_i+2}, \quad \Phi_u(t) := \sum_{i=1}^{k} \|\nabla_x \left(|u^i(t)|^{(p_l+2)/2}\right)\|_{0,2}^2$$

Note now that the Fridrich's inequality implies that $F_u(t) \le C_1\Phi_u(t)$ and consequently applying the Gronwall inequality to (1.3) we derive after the standard computations that

$$(1.5)\qquad \sup_{t\in[T,T+1]} F_u(t) + \int_T^{T+1} \Phi_u(t)\,dt \le \\ \le CF_u(0)e^{-\alpha T} + C_1 + \int_0^{T+1} e^{-\alpha(T+1-t)}|G_u(t)|\,dt$$

with the appropriate $\alpha > 0$.

Thus, the main problem now is to estimate the integral into the right-hand side of (1.5). To this end we transform it to the following form which is more convenient for our purposes:

$$(1.6)\qquad \int_0^{T+1} |G_u(t)|\,dt \le C_2 \sup_{t\in[0,T]} \left(e^{-\alpha(t-T)/2} \int_t^{t+1} |G_u(s)|\,ds\right)$$

It is essential for us that C_2 in this estimate is independent of T.

Let $W_i([t,t+1]) := L^{p_i+1}([t,t+1], L^{2(p_i+1)}(\Omega))$. Then due to Hölder inequality

$$(1.7)\qquad \int_t^{t+1} |(g^i, u^i(s)|u^i(s)|^{p_i})|\,ds \le \|u^i\|_{W_i([t,t+1])}^{p_i+1}\|g\|_{0,2} \le \\ \le \mu\|u^i\|_{W_i([t,t+1])}^{p_i+2} + C_\mu\|g\|_{0,2}^{p_i+2}$$

where the constant μ can be choosen arbitrary small. In order to estimate the first term in the left-hand side of (1.7) we need the following interpolation result.

Lemma 1.1. *Let $\Omega \subset\subset \mathbb{R}^3$ be smooth domain. Then*

$$(1.8)\qquad \|v\|_{L^q([t,t+1],L^{2q}(\Omega))} \le C\|v\|_{L^\infty([t,t+1],L^2(\Omega))}^{1-\theta}\|v\|_{L^2([t,t+1],W^{1,2}(\Omega))}^{\theta}$$

where $q = \frac{7}{3}$ and $\theta = \frac{6}{7}$.

Proof. Indeed, according to the standard interplolation theorem (see e.g [14]) the inequality (1.8) is valid with exponents q and θ satisfying the relations

$$(1.9)\qquad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}; \quad \frac{1}{2q} = \frac{1-\theta}{2} + \theta(\frac{1}{2} - \frac{1}{3})$$

Solving the system (1.9) we obtain the exponents from the lemma. Lemma 1.1 is proved.

The embedding (1.8) implies particularly that

$$(1.10)\quad \|u^i\|^{p_i+2}_{W_i([t,t+1])} \le C\|u^i\|^{p_i+2}_{L^{q(p_i+2)/2}([t,t+1],L^{q(p_i+2)}(\Omega))} \le$$

$$\le C_1\left(\sup_{s\in[t,t+1]}\|u^i(s)\|^{p_i+2}_{0,p_i+2} + \int_t^{t+1}\|\nabla_x\left(|u^i(s)|^{(p_i+2)/2}\right)\|^2_{0,2}\,ds\right) \le$$

$$\le C_2\left(\sup_{s\in[t,t+1]}F_u(s) + \int_t^{t+1}\Phi_u(s)\,ds\right)$$

(here we have used the evident fact that $q(p_i+2)/2 = \frac{7}{6}(p_i+2) > p_i+1$).

Inserting this estimate to the right-hand side of (1.7) and taking a sum over the indexes $i = 1, \cdots, k$ we derive the estimate

$$(1.11)\quad \int_t^{t+1}|G_u(s)|\,ds \le \mu\left(\sup_{s\in[t,t+1]}F_u(s) + \int_t^{t+1}\Phi_u(s)\,ds\right) + C_\mu\sum_{i=1}^k\|g\|^{p_i+2}_{0,2}$$

where $\mu > 0$ can be chosen arbitrary small.

Denoting the left-hand side of (1.5) by $Z_u(T)$ and inserting the estimate (1.11) into the right-hand side of (1.5) we derive that

$$(1.12)\quad Z_u(T) \le CF_u(0)e^{-\alpha T} + \mu\sup_{t\in[0,T]}\left(e^{-\alpha(T-t)/2}Z_u(t)\right) + C_\mu\left(\sum_{i=1}^k\|g\|^{p_i+2}_{0,2} + 1\right)$$

It can be easily proved (see e.g. [17]) that if $\mu < 1/2$ then (1.12) implies the estimate

$$(1.13)\quad Z_u(T) \le C_1F_u(0)e^{-\alpha T/2} + C_2\left(\sum_{i=1}^k\|g\|^{p_i+2}_{0,2} + 1\right)$$

Theorem 1.1 is proved.

Corollary 1.1. *Let the assumptions of Theorem 1.1 hold. Then*

$$(1.14)\quad \|f(u(t))\|_{0,2} \le Q(\|u_0\|_{0,\infty})e^{-\alpha t} + Q(\|g\|_{0,2})$$

for the appropriate $\alpha > 0$ and monotonic function Q which are independent of ε.

Indeed, (1.14) is an immediate corollary of (1.1), the second assumption of (0.5) and our choice of the exponents p_i $(2p \le p_i + 2)$.

Remark 1.1. Note that the estimate (1.14) (and consequently all corollaries of it which will be formulated below) remains valid if we replace the growth restriction 2) from (0.5) by the following one:

$$(1.15)\quad |f(v)|^2 \le C\left(1 + \sum_{i=1}^k|v^i|^{p_i+2}\right)$$

Having the uniform estimate (1.14) for the nonlinear term $f(u)$ in the equation (0.1) we can apply the standard methods (see e.g. [1], [9]) in order to obtain more convenient estimates. Since these methods are well known we formulate below the estimates and indicate only the main ideas of their proofs (in order to show that they are really uniform with respect to ε).

Firstly, taking the inner product in $\mathbb{R}^k$ of the equation (0.1) with the function $u(t)$, integrating by $x \in \Omega$ (with the integration by parts in A_ε and using the uniform ellipticity conditions), estimating the non-linear term by (1.14) and applying the Gronwall inequality we obtain the following estimate.

Proposition 1.1. *Let the above assumptions hold. Then*

$$\|u(T)\|_{0,2}^2 + \int_T^{T+1} \|u(t)\|_{1,2}^2\,dt \le Q(\|u_0\|_{0,\infty})e^{-\alpha T} + Q(\|g\|_{0,2}) \tag{1.16}$$

where the constant $\alpha > 0$ and the monotonic function Q (which are not the same as in (1.14)*) are independent of ε.*

Corollary 1.2. *Let the above assumptions hold. Then*

$$\begin{aligned} \|u(T)\|_{1,2}^2 + \int_T^{T+1} \|A_\varepsilon u(t)\|_{0,2}^2\,dt + \int_T^{T+1} \|\partial_t u(t)\|_{0,2}^2\,dt \le \\ \le \left(Q(\|u_0\|_{0,\infty}) + C\|u_0\|_{1,2}^2\right) e^{-\alpha T} + Q(\|g\|_{0,2}) \end{aligned} \tag{1.17}$$

where the constants $\alpha, C > 0$ and the function Q are independent of ε.

Proof. Indeed, taking the scalar product in $\mathbb{R}^k$ of the equation (0.1) with the function $A_\varepsilon u(t)$ and integrating by parts the term $(\partial_t u, A_\varepsilon u)$ and using the fact that A_ε^l are symmetric we obtain the inequality

$$\begin{aligned} \partial_t(\sum_{i,j,l} a_{ij}^l \partial_i u^l(t), \partial_j u^l(t)) + (A_\varepsilon u^l(t), A_\varepsilon u^l(t)) + \sum_{i,j,l} (a_{ij}^l \partial_i u^l(t), \partial_j u^l(t)) \le \\ \le \sum_{i,j,l} (a_{ij}^l \partial_i u^l(t), \partial_j u^l(t)) + C\left(\|f(u(t))\|_{0,2}^2 + \|g\|_{0,2}^2\right) \end{aligned} \tag{1.18}$$

Recall now that the coefficients $a_{ij}^l(y)$ are assumed to be almost-periodic and consequently uniformly bounded in $\mathbb{R}^3$. Therefore the first term in the right-hand side of (1.18) can be estimated by $C_a\|u(t)\|_{1,2}^2$ with the constant C_a independent of ε. Inserting now the estimate (1.14) into the right-hand side of (1.18) applying the Gronwall inequality and using (1.16) for estimating the integral of the $W^{1,2}$-norm we derive the estimate (1.17), but without the norm of $\partial_t u$ in the left-hand side:

$$\begin{aligned} \|u(T)\|_{1,2}^2 + \int_T^{T+1} \|A_\varepsilon u(t)\|_{0,2}^2\,dt \le \\ \le \left(Q(\|u_0\|_{0,\infty}) + C\|u_0\|_{1,2}^2\right) e^{-\alpha t} + Q(\|g\|_{0,2}) \end{aligned} \tag{1.19}$$

The estimate for the time derivative $\partial_t u$ can be easily obtained now from the equation (0.1) and from the estimates (1.14) and (1.19). Corollary 1.2 is proved.

Analogously, taking the inner product of the equation (0.1) with the function $tA_\varepsilon^l u^l(t)$ we derive in a standard way the following version of the smoothing property.

Corollary 1.3. *Let the above assumptions hold. Then*

$$\begin{aligned} \|u(T)\|_{1,2}^2 + \int_T^{T+1} \|\partial_t u(t)\|_{0,2}^2\,dt + \int_T^{T+1} \|A_\varepsilon u(t)\|_{0,2}^2\,dt \le \\ \le \frac{T+1}{T}\left(Q(\|u_0\|_{0,\infty})e^{-\alpha T} + Q(\|g\|_{0,2})\right) \end{aligned} \tag{1.20}$$

where $\alpha > 0$ and Q are independent of $\varepsilon > 0$.

Corollary 1.4. *Let the previous assumptions be valid. Then the following estimate holds:*

$$\|u(t)\|_{0,\infty} \le Q(\|u_0\|_{0,\infty})e^{-\alpha t} + Q(\|g\|_{0,2}) \tag{1.21}$$

where the monotonic function Q and the constant $\alpha > 0$ are independent of ε. Moreover, there is a positive number $\beta > 0$ (independent of ε) such that for every $T \ge 1$ the solutions $u(t)$ of (0.1) belongs to the space $C^{\beta,2\beta}([T,T+1]\times\Omega)$ of Hölder continuous functions with Hölder constants β and 2β with respect to the variables t and x correspondingly (see [9]) and the following estimate is valid:

$$\|u\|_{C^{\beta,2\beta}([T,T+1]\times\Omega)} \le Q_1(\|u_0\|_{0,\infty})e^{-\alpha T} + Q_1(\|g\|_{0,2}), \quad T \ge 1 \tag{1.22}$$

where the constant α and the function Q_1 are independent of ε.

Proof. Indeed, let us consider the l-th equation of (0.1)

$$\partial_t u^l - A^l_\varepsilon u^l = f^l(u(t)) + g^l := h^l(t) \tag{1.23}$$

Applying the maximum principle for solutions from the class (0.7) of this equation (see [9, Th. 3.7.1]) we obtain that for $t \in [0,T]$

$$\|u^l(t)\|_{0,\infty} \le C\left(\|u^l_0\|_{0,\infty} + \|h^l\|_{L^\infty([0,T],L^2(\Omega))}\right) \le \\ \le C\|g\|_{0,2} + C\|f(u)\|_{L^\infty([0,T],L^2(\Omega))} \tag{1.24}$$

where the constant C depends only on the L^∞-norm of a^l_{ij} and on the uniform ellipticity constant (0.4) (and independent of ε).

Inserting the estimate (1.14) into the rigth-hand side of (1.24) we deduce that

$$\|u(t)\|_{0,\infty} \le Q(\|u_0\|_{0,\infty}) + Q(\|g\|_{0,2}) \tag{1.25}$$

Note however that the estimate (1.25) is not dissipative with respect to the initial conditions u_0. In order to obtain the dissipative one we fix $T \ge 1$ and consider the function $v^l(t) = (t-T+1)u^l(t)$ which evidently satisfies the equation

$$\partial_t v^l - A^\varepsilon v^l = u^l(t) + (t-T+1)f^l(u) + (t-T+1)g^l := h^l_T(t), \quad v^l(T-1) = 0 \tag{1.26}$$

It is proved in [9, Th. 3.10.1] that there is a positive Hölder constant $\beta > 0$ such that

$$\|v^l\|_{C^{\beta,2\beta}([T,T+1]\times\Omega)} \le C\|h^l_T\|_{L^\infty([T-1,T+1],L^2(\Omega))} \le \\ \le C\|g\|_{0,2} + C\|u\|_{L^\infty([T-1,T+1],L^2(\Omega))} + C\|f(u)\|_{L^\infty([T-1,T+1],L^2(\Omega))} \tag{1.27}$$

Moreover the constants $\beta > 0$ and C are independent of ε.

Inserting the estimate (1.14) and (1.16) into the right-hand side of (1.27) we derive the following estimate

$$\|u\|_{C^{\beta,2\beta}([T,T+1]\times\Omega)} \le C_1\|v\|_{C^{\beta,2\beta}([T,T+1]\times\Omega)} \le \\ \le Q(\|u_0\|_{0,\infty})e^{-\alpha T} + Q(\|g\|_{0,2}) \tag{1.28}$$

which holds for $T \ge 1$. The estimates (1.21) and (1.22) are immediate corollaries of (1.25) and (1.28). Corollary 1.4 is proved.

We obtain now some estimates for the time derivative $\partial_t u(t)$ for the solutions of (0.1). To this end we differentiate the equation (0.1) with respect to t and denote $\theta(t) = \partial_t u(t)$. Then we obtain the equation

$$\partial_t\theta(t) = A_\varepsilon\theta(t) - f'(u(t))\theta(t) \tag{1.29}$$

Recall that $f \in C^1$, consequently (1.21) implies that

$$\|f'(u(t))\|_{0,\infty} \le Q_1(\|u_0\|_{0,\infty})e^{-\alpha t} + Q_1(\|g\|_{0,2}) \tag{1.30}$$

therefore the equation (1.29) also can be treated as the linear one.

Corollary 1.5. *Let $t \geq 1$. Then*

$$\|\partial_t u(t)\|_{0,2} \leq Q(\|u_0\|_{0,\infty})e^{-\alpha t} + Q(\|g\|_{0,2}) \tag{1.31}$$

where the constant $\alpha > 0$ and the function Q is independent of ε.

Proof. According to the smoothing property (1.20) we have

$$\int_T^{T+1} \|\theta(t)\|_{0,2}^2\,dt \leq Q(\|u_0\|_{0,\infty})e^{-\alpha T} + Q(\|g\|_{0,2}) \tag{1.32}$$

for $T \geq 1/2$. Taking the inner product in $\mathbb{R}^k$ of the equation (1.29) with $(t-1/2)\theta(t)$ and integrating over $x \in \Omega$ we obtain using the integration by parts, and the uniform ellipticity assumption (0.4), that

$$\begin{aligned}(1.33)\quad \partial_t\left((t-1/2)\|\theta(t)\|_{0,2}^2\right) + \nu\left((t-1/2)\|\theta(t)\|_{0,2}^2\right) \leq \\ \leq C\|\theta(t)\|_{0,2}^2 + C_1(t-1/2)\|f'(u(t)\|_{0,\infty}\|\theta(t)\|_{0,2}^2\end{aligned}$$

Applying now the Gronwall inequality (starting with the time moment $t = 1/2$) to the estimate (1.33) and estimating the right-hand side of it by (1.32) and (1.30) we derive after the standard computatins the following estimate for $t \geq 1/2$

$$\|\theta(t)\|_{0,2}^2 \leq \frac{t+1}{t-1/2}\left(Q(\|u_0\|_{0,\infty})e^{-\alpha t} + Q(\|g\|_{0,2})\right) \tag{1.34}$$

Restricting in (1.34) $t \geq 1$ we obtain the estimate (1.31). Corollary 1.5 is proved.

Having the estimate (1.31) for the time derivative $\theta = \partial_t u$ of the solutions of (1.29) and arguing as in the proof of the estimate (1.22) one can proof the following result.

Corollary 1.6. *Let the above assumptions hold and let $u(t)$ be a solution of the equation* (0.1). *Then there is the exponent $\beta > 0$ such that $\partial_t u \in C^{\beta,2\beta}([T,T+1]\times\Omega)$ for $T \geq 2$ and the following estimate is valid:*

$$\|\partial_t u\|_{C^{\beta,2\beta}([T,T+1]\times\Omega)} \leq Q(\|u_0\|_{0,\infty})e^{-\alpha T} + Q(\|g\|_{0,2}),\quad T \geq 2 \tag{1.35}$$

where the exponents $\beta, \alpha > 0$ and the function Q are independent of ε.

We summiraze the obtained results in the following theorem.

Theorem 1.2. *Let the assumptions* (0.2)–(0.6) *hold. Then for every fixed $\varepsilon > 0$ the problem* (0.1) *possesses a unique (in the class* (0.7)*) solution $u(t)$ and the following uniform with respect to ε estimate is valid:*

$$\|u(T)\|_{0,\infty}^2 + \int_T^{T+1} \|u(t)\|_{1,2}^2\,dt \leq Q(\|u_0\|_{0,\infty})e^{-\alpha T} + Q(\|g\|_{0,2}) \tag{1.36}$$

where $\alpha > 0$ and Q are independent of ε.

Moreover, the following smoothing property is valid: if $T \geq 2$ then the function $u \in C^{1+\beta,2\beta}([T,T+1]\times\Omega)$ and

$$\begin{aligned}(1.37)\quad \|u(T)\|_{1,2}^2 + \|u\|_{C^{1+\beta,2\beta}([T,T+1]\times\Omega)}^2 + \int_T^{T+1} \|A_\varepsilon u(t)\|_{0,2}^2\,dt \leq \\ \leq Q(\|u_0\|_{0,\infty})e^{-\alpha T} + Q(\|g\|_{0,2})\end{aligned}$$

where the exponents $\alpha, \beta > 0$ and the function Q are independent of ε.

Proof. Indeed, the estimates (1.36) and (1.37) has been obtained in Corollaries 1.1–1.6. The uniqueness of a solution in the class (0.7) is evident. The existence of a solution can be deduced from the a priori estimates (1.36) and (1.37) by standard arguments (e.g. for the smooth initial data u_0 and smooth coefficients a^l_{ij} the existence of a solution can be obtained by Leray-Schauder principle as in [9], the existence of a solution u in general situation can be obtained approximating the initial data u^0 and the coefficients a^l_{ij} by smooth ones u^n_0 and $a^{l,(n)}_{ij}$ and then passing to the limit $n \to \infty$). Theorem 1.2 is proved.

Remark 1.2. As it has been already mentioned in Remark 1.1 the result of the Theorem remains valid if the condition 2) of (0.5) will be replaced by (1.15).

§2 The attractors.

In the previous Section we have proved that for every fixed $\varepsilon > 0$ and every $u_0 \in \Phi := L^\infty(\Omega)$ the problem (0.1) possesses a unique solution $u(t) \in \Phi$ and consequently the semi-group

$$S^\varepsilon_t : \Phi \to \Phi, \quad S^\varepsilon_t u_0 := u(t) \tag{2.1}$$

where $u(t)$ is a solution of (0.1), is correctly defined.

In this Section we prove that for every $\varepsilon > 0$ the semigroup (2.1) generated by the equation (0.1) possesses an attractor $\mathcal{A}^\varepsilon$ in the phase space Φ and obtain a number of uniform with respect to $\varepsilon > 0$ estimates for these attractors which will be used in the next Sections in order to study the homogenization limit $\varepsilon \to 0$. Moreover, in order to clarify the anisotropic dissipativity assumption (0.6) we will give a number of examples of the right-hand sides f which admit the above theory.

For the convenience of the reader we recall shortly the definition of the attractor and it's basic properties (see e.g. [1], [6], [13] for the detailed exposition).

Definition 2.1. Let $S^\varepsilon_t : \Phi \to \Phi$ be a semigroup, acting in a B-space Φ. Then a set $\mathcal{A}^\varepsilon$ is defined to be the attractor for S^ε_t if the following is true:

1. The set $\mathcal{A}^\varepsilon$ is compact in Φ.
2. The set $\mathcal{A}^\varepsilon$ is strictly invariant, i.e. $S^\varepsilon_t \mathcal{A}^\varepsilon = \mathcal{A}^\varepsilon$.
3. The set $\mathcal{A}^\varepsilon$ is an attracting set for the semigroup S^ε_t, i.e. for every open neighbourhood $\mathcal{O}(\mathcal{A}^\varepsilon)$ of the set $\mathcal{A}^\varepsilon$ in Φ and for every bounded subset $B \subset \Phi$ there is $T = T(\mathcal{O}, B)$ such that

$$S^\varepsilon_t B \subset \mathcal{O}(\mathcal{A}^\varepsilon), \quad \text{if } t \geq T \tag{2.2}$$

Remark 2.1. The punkt 3) of the previous Definition is equivalent to the following: for every bounded subset $B \subset \Phi$

$$\lim_{t\to\infty} \operatorname{dist}_\Phi (S^\varepsilon_t B, \mathcal{A}^\varepsilon) = 0 \tag{2.3}$$

where dist_Φ means the nonsymmetric Hausdorff distance, i.e.

$$\operatorname{dist}_\Phi (V, W) := \sup_{v\in V} \inf_{w\in W} \|v - w\|_\Phi \tag{2.4}$$

It is also know (see e.g. [1]) that if the attractor $\mathcal{A}^\varepsilon$ exists then it is generated by the complete bounded trajectories of the semigroup S_t^ε, i.e.

$$\text{(2.5)}\quad \mathcal{A}^\varepsilon = \{u_0 \in \Phi : \exists u \in L^\infty(R,\Phi),\ \ u(0)=u_0; \\ S_t^\varepsilon u(s) = u(t+s),\ \ \forall s \in \mathbb{R},\ t \in \mathbb{R}_+\}$$

The following proposition gives the sufficient conditions which imply the existence of the attractor.

Proposition 2.1 [1]. *Let $S_t^\varepsilon : \Phi \to \Phi$ be a semigroup. Then it possesses an attractor in $\mathcal{A}^\varepsilon$ in Φ if*

1. The semigroup S_t^ε possesses a compact attracting set K in Φ (in the sence of punkt 3) of Definition 2.1).

2. The operators $S_t^\varepsilon : \Phi \to \Phi$ have closed grafs for every fixed $t \geq 0$ (as usual, it means that the set $\{(u_0, S_t^\varepsilon u_0) : u_0 \in \Phi\}$ is closed in $\Phi \times \Phi$).

Moreover, in this case $\mathcal{A}^\varepsilon \subset K$.

We are in a position now to state the main result of this Section.

Theorem 2.1. *Let the assumptions of Theorem 1.2 hold. Then for every $\varepsilon > 0$ the semigroup (2.1) generated by the equation (0.1) possesses an attractor $\mathcal{A}^\varepsilon$ in the phase space $\Phi := L^\infty(\Omega)$ which admits the following description:*

$$\text{(2.6)}\qquad \mathcal{A}^\varepsilon = K^\varepsilon\big|_{t=0}$$

where K^ε is a collection of all bounded with respect to $t \in \mathbb{R}$ solutions $\widehat{u}(t)$ of the equation

$$\text{(2.7)}\qquad \partial_t \widehat{u}(t) = A_\varepsilon \widehat{u}(t) - f(\widehat{u}(t)) + g,\quad t \in \mathbb{R},\quad \widehat{u}\big|_{\partial\Omega} = 0$$

Proof. Let us verify the assumptions of Proposition 2.1. Indeed, it follows from the estimate (1.37) that the ball K_R of radius R in the space $\Phi_\beta := C^{2\beta}(\Omega)$ will be the attracting set for the semigroup (2.1) if R is large enough (namely if $R > Q(\|g\|_{0,2})$). Since the embedding $\Phi_\beta \subset \Phi$ is compact then the set K_R is a (pre)compact attracting set for the semigroup (2.1). Taking it's closure in Φ we will construct the compact attracting set for S_t^ε. Thus, the first assumption of Proposition 2.1 is verified.

Let us verify the second one. Recall, that we should prove that if $u_0^n \to u_0$ in Φ and $u_n(T) := S_T^\varepsilon u_0 \to v$ in Φ then $v = u(T) := S_T^\varepsilon u_0$.

Note, for the first that the operator S_T^ε is uniformly Liptshitz continuous with respect to the L^2-norm on every bounded subset B in Φ. Indeed, let $u_0^1, u_0^2 \in B$ be an arbitrary initial data, $u_i(t)$ be the corresponding solutions of the problem (0.1) and $w(t) = u_1(t) - u_2(t)$. Then this function satisfies the equation

$$\text{(2.8)}\qquad \partial_t w = A_\varepsilon w - l(t)w,\quad w\big|_{t=0} = u_0^1 - u_2^0,\quad w\big|_{\partial\Omega} = 0$$

where $l(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds$. The estimate (1.21) implies that $\|u_i(t)\|_{0,\infty} \leq C_B$ where the constant C_B depends only on the norm $\|B\|_\Phi$ and consequently

$$\text{(2.9)}\qquad \|l(t)\|_{0,\infty} \leq C_1(B)$$

Taking the inner product in $\mathbb{R}^k$ with the function $w(t)$, integrating over $x \in \Omega$, estimating $l(t)$ by (2.9) and applying the Gronwall inequality we derive the estimate

$$\|u_1(T) - u_2(T)\|_{0,2}^2 \le Ce^{KT}\|u_1(0) - u_2(0)\|_{0,2}^2 \tag{2.10}$$

where the constants C and K depend only on $\|B\|_\Phi$.

Having the L^2-Lipschitz continuity (2.10) we immediately conclude that S_T^ε has a closed graph in Φ. Indeed, let $u_0^n \to u_0$ in Φ and $u_n(T) \to v$ in Φ. Then according to (2.10) $u_n(T) \to u(T)$ in $L^2(\Omega)$ and consequently $v = u(T)$.

Thus, all assumptions of Proposition 2.1 are verified for the semi-group (2.1) generated by the equation (0.1) and therefore this semi-group possesses an attractor $\mathcal{A}^\varepsilon$. The description (2.6) is an immediate corollary of the formula (2.5). Theorem 2.1 is proved.

The following Corollary is of fundamental significance for our study the homogenization limit: $\lim_{\varepsilon\to 0}\mathcal{A}^\varepsilon$ in the next Sections.

Corollary 2.1. *The attractors $\mathcal{A}^\varepsilon \in \Phi_\beta \cap W_0^{1,2}(\Omega)$ and uniformly bounded with respect to $\varepsilon > 0$ in this space*

$$\|\mathcal{A}^\varepsilon\|_{1,2} + \|\mathcal{A}^\varepsilon\|_{C^{2\beta}(\Omega)} \le Q(\|g\|_{0,2}) \tag{2.11}$$

where the exponent $\beta > 0$ and the function Q are independent of ε.

Moreover, let $\widehat{u}(t) \in K^\varepsilon$ be a bounded solution of the equation (2.7). Then

$$\|\widehat{u}\|_{C^{1+\beta,2\beta}([T,T+1]\times\Omega)}^2 + \|\widehat{u}(T)\|_{1,2}^2 + \int_T^{T+1}\|A_\varepsilon\widehat{u}(t)\|_{0,2}^2\,dt \le Q(\|g\|_{0,2}) \tag{2.12}$$

where the function Q is independent of ε, $\widehat{u}$, and $T \in \mathbb{R}$.

Indeed, the first estimate is an immediate corollary of the second one together with the representation (2.6) and the estimate (2.12) can be easily derived from the estimate (1.37) and from the fact that $\|\widehat{u}(t)\|_{0,\infty}$ remains bounded when $t \to -\infty$.

Recall that we have proved the attractor's existence Theorem 2.1 under anisotropic dissipativity assumption (0.6) on the non-linear term f which looks a little unusual. In conclusion of this Section we discuss this assumption and give a number of examples where it is satisfied.

Note for the first that in a scalar case $k = 1$ we need only the following natural assumption on the non-liner term f: there is $R_f < \infty$ such that

$$\operatorname{sgn} v f(v) \ge 0 \text{ if } |v| > R_f \tag{2.13}$$

(which as known is essential in order to obtain the global existence of solutions of (0.1)). Note also that in this case the polynomial growth restriction 2) of (0.5) is also can be removed due to the maximum principle.

Consider now the case of systems ($k \ge 2$). The usual (for the attractor's industry) assumptions for the non-linearity f are the following

$$\begin{cases} 1.\ f(u).u \ge -C + C_1|u|^{p+1} \\ 2.\ |f(u)| \le C(1 + |u|^p),\ p < p_c \\ 3.\ |f'(u)| \le C(1 + |u|^{p-1}) \end{cases} \tag{2.14}$$

(see e.g. [1], [13]) which involve the *growth* restrictions $p < p_c$. Unfortunately the limit exponent p_c under the assumptions (2.14) is too restrictive ($p_c = 1 + 4/n = 7/3 < 3$ (see [1])) and consequently even the cubic nonlinearities are out of the consideration).

Another standard possibility is to impose the additional quasi-monotonicity assumption

$$f'(u) \geq -K \tag{2.15}$$

to the non-linear term f. In this case the global existence and uniqueness of weak solutions and even the existence of the attractors $\mathcal{A}^\varepsilon$ in $L^2(\Omega)$ can be obtained without the growth restriction $p < p_c$ (analogously to [1]). Moreover, in the case where $A_\varepsilon = A = a\Delta_x$, $a \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$ with $a + a^* > 0$ the considerable theory (which includes the L^∞-bounds for weak solutions, their smoothness, the differentiability of the corresponding semi-group, the finite dimensionality of the attractors, etc.) can be constructed for the equation (0.1) with essentially weaker growth restriction $p < 1 + 4/(n-4)$, e.g. if $n = 3$ the following assumptions on f

$$1.\ f \in C^1(\mathbb{R}^k, \mathbb{R}^k), \quad 2.\ f(u).u \geq -C, \quad 3.\ f'(u) \geq -K \tag{2.16}$$

are sufficient in the case $A_\varepsilon := a\Delta_x$ (see [18]). But the proof of these results is essentially based on the trick with multiplication of the equation (0.1) by $\Delta_x u$ and does not work in our situation where the operator A_ε has the form (0.2), (0.3). Therefore, even under the additional assumption (2.15) we do not know how to obtain the additional regularity of weak solutions of (0.1) (which is necessary to study the limit behaviour as $\varepsilon \to 0$ of attractors $\mathcal{A}^\varepsilon$) without the growth restriction $p < p_c < 7/3$.

In order to remove this extremely restrictive growth condition we suggest to use the dissipativity assumption in a new form (0.6) which from the one side looks not very restrictive (as the examples given below show) and from the other side admits to obtain the L^∞-bounds of solutions of (0.1) (with the diagonal leading part A_ε, see (0.2)) without the growth restrictions. We illustrate this anisotropic dissipativity assumption on a number of examples.

Assume for the first that the nonlinearity f possesses the following decomposition:

$$f(v) = f_1(v) + f_2(v), \text{ where } f_1(v) := \operatorname{diag}\{f_1^1(v_1), \cdots, f_1^k(v_k)\} \tag{2.17}$$

and the functions $f_1^i(v^i)$ satisfy the assumptions

$$f_1^i(v^i) \cdot v^i \geq -C_i + \alpha_i |v^i|^{q_i+2}, i = 1, \cdots, k \tag{2.18}$$

for the appropriate $q_i \geq 0$, $\alpha_i > 0$ and the functions $f_2(v)$ satisfy the following growth restrictions

$$|f_2^i(v)| \leq C(1 + |v|^{l_i}), \ i = 1, \cdots, k, \quad l_i \geq 0 \tag{2.19}$$

Assume also that

$$l_i < 1 + q_i, \quad i = 1, \cdots, k \tag{2.20}$$

Lemma 2.1. *Let the assumptions* (2.17)–(2.20) *hold. Then for every* $q \geq q_i$, $i = 1, \cdots, k$ *the non-linearity* (2.17) *satisfies the anisotropic dissipativity condition* (0.6) *with the exponents* $p_i = q - q_i$.

Proof. Indeed, due to (2.18)

$$\sum_{i=1}^{k} f_1^i(v^i) \cdot v^i |v^i|^{p_i} \geq -C + 1/2 \sum_{i=1}^{k} \alpha_i |v^i|^{q+2} \geq -C + \beta |v|^{q+2} \tag{2.21}$$

It follows from the restrictions (2.19) and (2.20) and Hölder inequality that

$$\begin{aligned} |f_2^i(v)| \cdot |v^i| v^i|^{q-q_i} &\leq \mu |v^i|^{q+2} + C_\mu |f_2^i(v)|^{(q+2)/(q_i+1)} \leq \\ &\leq \mu |v|^{q+2} + C_\mu + C_\mu |v|^{(q+2)l_i/(q_i+1)} \leq 2\mu |v|^{q+2} + C'_\mu \end{aligned} \tag{2.22}$$

where $\mu > 0$ is an arbitrary positive number. The estimates (2.21) and (2.22) prove the lemma.

Remark 2.2. Note that under the assumptions of Lemma 2.1 (0.6) is valid with $p_i = q - q_i$ where q may be arbitrary large. Consequently, for every nonlinearity (2.17) with the polynomial rate of growth (p. 2 of (0.5) is satisfied) we may satisfy also the assumption $p_i \geq 2(p-1)$ and therefore Theorem 2.1 is valid for such nonlinearities.

Example 2.1. The simplest example of such non-linearities is the following:

$$f_1(v) = \operatorname{diag}\{\alpha_1 v^1 |v^1|^{q_1}, \cdots, \alpha_k v^k |v^k|^{q_k}\} \tag{2.23}$$

with $q_i, \alpha_i > 0$ and f_2 is linear $f_2(v) = Lv$. Then all assumptions of Lemma 2.1 are evidently satisfied and consequently the result of Theorem 2.1 hold for such non-linearities.

Example 2.2. Consider the case $k = 2$, $v = (v_1, v_2)$ and the non-linearity

$$f(v) = \begin{pmatrix} v_1^3 - \alpha v_1 - \beta v_2 \\ v_2 - \gamma v_1 \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R} \tag{2.24}$$

which corresponds to the Fitz-Nagumo system (see [8]). Note that although the assumptions of Lemma 2.1 are violated ($q_2 = 0$, $l_2 = 1$, $l_2 = q_2 + 1$). Nevertheless a simple checking shows that the dissipativity assumption (0.6) is valid with the exponents $p_1 = p_2 = q$ for every $q \geq 0$ and consequently the result of Theorem 2.1 remains true for the nonlinearity (2.24).

Example 2.3. Consider the case $k = 2$ and the nonlinearity

$$f(v) = \begin{pmatrix} v_1^3 - \alpha v_2 \\ v_2 - \beta v_1^3 \end{pmatrix}, \quad 2|\alpha| + |\beta| \leq 3, \quad 2|\beta| + |\alpha| \leq 3 \tag{2.25}$$

If $\alpha = \beta = 1$ then (2.25) satisfies the equation $f^1(v) + f^2(v) = 0$ which is natural from the reaciton-diffusion point of view. Note that this non-linearity evidently does not satisfy the standard dissipativity assumption ($f(v).v \geq -C$). Nevertheless the elementary computation shows that the anisotropic dissipativity assumption is satisfied with $p_1 = 5$ and $p_2 = 1$. Note also that the function (2.24) satisfies also the condition (1.15) with these exponents. Consequently the assertion of Theorem 2.1 holds for this non-linearity.

Example 2.4. We conclude the section by considering the following 'exotic' example

$$f(v) = \begin{pmatrix} v_1^3 - v_2 \\ v_2^3 - v_1^5 \end{pmatrix} \tag{2.26}$$

of two RDEs coupled by the monom v_1^5 of the higest order. Nevertheless the anisotropic dissipativity condition is valid with the exponents $p_2 = 4$ and $p_1 = 10$. Since (1.15) is also hold for this exponents then the assertion of Theorem 2.1 is valid for the non-linearity (2.26).

§3 The homogenization.

This Section is devoted to study the behaviour of the attractors $\mathcal{A}^\varepsilon$ constructed in the previous Section when $\varepsilon \to 0$. The main task of the section is to prove that these attractors tend as $\varepsilon \to 0$ to the attractor $\mathcal{A}^0$ of the homogenized problem (0.9). In order to write this homogenized system we recall briefly some known results from the theory of almost-periodic homogenization (see e.g. [20] for the detailed exposition).

Recall, that every almost-periodic function $w \in AP(\mathbb{R}^3)$ possesses the mean value which can be calculated by the following formula:

$$\langle w \rangle := \lim_{T\to\infty} \frac{1}{2^3 T^3} \int_{[-T,T]^3} w(x)\, dx$$

and the following Fourier expansion

$$w(x) = \sum_{\widehat{w}(\xi)\neq 0} \widehat{w}(\xi) e^{i(x,\xi)} \tag{3.1}$$

where the amplitudes $\widehat{w}(\xi) \in \mathbb{C}$, $\xi \in \mathbb{R}^3$ can be found by the expression

$$\widehat{w}(\xi) := \left\langle w(x) e^{-i(x,\xi)} \right\rangle \tag{3.2}$$

(see e.g. [10], [11]). It is known that the set $\sigma(w) := \{\xi \in \mathbb{R}^3, \quad \widehat{w}(\xi) \neq 0\}$ is not greater than countable therefore the sum (3.1) has a sense. Moreover,

$$\sum_{\xi\in\sigma(w)} |\widehat{w}(\xi)|^2 < \infty \tag{3.3}$$

and the series (3.1) converges to the function w in the sense of the Bezikovich norm $\|v\|^2_{B^2(\mathbb{R}^3)} := \langle v(x)\overline{v}(x)\rangle$ (see [10] for details).

As usual, we denote by $\mathrm{Trig}^\circ(\mathbb{R}^3)$ the space of all *finite* trigonometric polynomials of the form (3.1)

$$\mathrm{Trig}^\circ(\mathbb{R}^3) := \left\{ w(x) = \sum_{k=1}^{N} w_k e^{i(x,\xi_k)} \ : \right. \\ \left. N \in \mathbb{N},\ \xi_k \in \mathbb{R}^3,\ w_k \in \mathbb{C},\ k = 1, \cdots, N \right\} \tag{3.4}$$

Now we are in a position to write the formula for the homogenized operator A_0 for the problem (0.1). To this end we define firstly the functions $\mathcal{B}^l(\xi)$, $\xi \in \mathbb{R}^3$, $l = 1, \cdots, k$ by expressions

$$\mathcal{B}^l(\xi) := \inf_{N \in \mathrm{Trig}^\circ(\mathbb{R}^3)} \left\langle \sum_{i,j} a^l_{ij}(y)(\xi_i + \partial_{y_i} N(y))(\xi_j + \partial_{y_j} N(y)) \right\rangle \tag{3.5}$$

where $a^l_{ij}(y)$ has been defined in (0.3). Note, that (3.5) has a sense since the expression inside of $\langle \cdot \rangle$ is evidently an almost-periodic function.

It is known (see [20]) that the functions $\mathcal{B}^l(\xi)$ are the positive definite quadratic forms with respect to ξ, i.e.

$$\mathcal{B}^l(\xi) = \sum_{i,j} \tilde{a}^l_{ij} \xi_i \xi_j, \quad \tilde{a}^l_{ij} \in \mathbb{R} \tag{3.6}$$

Define now the operators A^l_0 and the operator A_0 in the following way:

$$A^l_0 u^l := \sum_{i,j} \partial_{x_j}(\tilde{a}^l_{ij} \partial_{x_i} u^l), \quad A_0 u := \mathrm{diag}\{A^1_0 u^1, \cdots, A^k_0 u^k\} \tag{3.7}$$

This choice of the leading part of the homogenized equation for (0.1) is justified by the following proposition.

Proposition 3.1 [20]. *Let the functions $v^\varepsilon \in W^{1,2}_0(\Omega)$ be the solutions of the following problem:*

$$A^l_\varepsilon v^\varepsilon = h, \quad h \in W^{-1,2}(\Omega) \tag{3.8}$$

where $l \in \{1, \cdots, k\}$ is fixed and the operator A^l_ε is defined by (0.3). Then $v^\varepsilon \rightharpoonup v^0$ weakly in $W^{1,2}_0(\Omega)$ as $\varepsilon \to 0$ and the function $v^0 \in W^{1,2}_0(\Omega)$ is a solution of the limit problem

$$A^l_0 v^0 = h \tag{3.9}$$

where the operator A^l_0 is defined by (3.5)–(3.7).

Let us consider the homogenized equation

$$\begin{cases} \partial_t u = A_0 u - f(u) + g \\ u\big|_{t=0} = u_0, \quad u\big|_{\partial\Omega} = 0 \end{cases} \tag{3.10}$$

for the non-homogeneous equation (0.1). Note that this equation satisfies all assumptions of Theorem 2.1 (due to the fact that the forms $\mathcal{B}^l(\xi)$ are positive defined) and consequently possesses the attractor $\mathcal{A}^0$ in the phase space $\Phi = L^\infty(\Omega)$. Moreover, the estimates (1.11) and (2.12) remains valid for the limit equation (3.10).

The main result of this Section is the following theorem.

Theorem 3.1. *Let the assumptions of Theorem 2.1 hold and let $\mathcal{A}^0$ be the attractor of the limit ($\varepsilon = 0$) problem* (3.10). *Then the attractors $\mathcal{A}^\varepsilon$ of* (0.1) *converges to $\mathcal{A}^0$ when $\varepsilon \to 0$ in the following sense*

$$\lim_{\varepsilon\to 0} \operatorname{dist}_{C^{2\beta'}(\Omega)}(\mathcal{A}^\varepsilon, \mathcal{A}^0) = 0 \tag{3.11}$$

for the appropriate sufficiently small positive exponent $\beta' > 0$.

Proof. As usual (see [1]) in order to prove the upper-semicontinuity (3.11) one should consider an arbitrary sequence $\varepsilon_n \to 0$ and $u_{0,n} \in \mathcal{A}^{\varepsilon_n}$ and prove that it is possible to extract from it a subsequence $u_{0,n_k} \to u_0 \in \mathcal{A}^0$ in $\Phi_{\beta'}$.

Let us fix an arbitrary sequence $\varepsilon_n \to 0$ and an arbitrary sequence $u_{0,n} \in \mathcal{A}^{\varepsilon_n}$. Let $\widehat{u}_n(t) \in K^{\varepsilon_n}$ be the corresponding bounded solutions of the equation (2.7) with ε replaced by ε_n such that $u_{0,n} = \widehat{u}_n(0)$ (which exist due to the representation (2.6)). Then, according to (2.12)

$$\|\widehat{u}_n\|_{C^{1+\beta,2\beta}([T,T+1]\times\Omega)} \le Q(\|g\|_{0,2}) \tag{3.12}$$

with the appropriate $\beta > 0$ and the function Q independent of $T \in \mathbb{R}$ and $n \in \mathbb{N}$. Let us fix $0 < \beta' < \beta$. Then due to the compactness of the embedding

$$C^{1+\beta',2\beta'}([T,T+1]\times\Omega) \subset\subset C^{1+\beta,2\beta}([T,T+1]\times\Omega)$$

and due to Cantor diagonal procedure we may assume (passing to a subsequence if necessary) that there is a function $\widehat{u} \in C^{1+\beta',2\beta'}([T,T+1]\times\Omega)$ such that

$$\widehat{u}_n \to \widehat{u}, \ \ \partial_t\widehat{u}_n \to \partial_t\widehat{u} \text{ as } n \to \infty \text{ in the space } C^{\beta',2\beta'}([T,T+1]\times\Omega) \tag{3.13}$$

for every fixed $T \in \mathbb{R}$. Particularly, $\widehat{u}_n(0) \to \widehat{u}(0)$ in $\Phi_{\beta'}$. Therefore if $\widehat{u} \in K^0$ then due to (2.6) $u_0 = \widehat{u}(0) \in \mathcal{A}^0$. Thus, it remains to prove that the limit function $\widehat{u}(t)$ is a bounded solution of the limit equation (3.10). The latter can be easily verified using Proposition 3.1 and the convergence (3.13). Indeed, let us verify that the function $u^l(t)$ satisfies the l-th equation of (3.10). Since $\widehat{u}$ and $\partial_t\widehat{u}$ are continuous with respect to (t,x) it is sufficient to verify this identity for every *fixed* $T \in \mathbb{R}$. To this end we rewrite the l-th equation of (1.2) in the form of elliptic boundary problem:

$$A^l_{\varepsilon_n}\widehat{u}^l_n(T) = h^l_n(T) := \partial_t\widehat{u}^l_n(T) + f^l(\widehat{u}(T)) - g^l, \ \ \widehat{u}^l(T)\big|_{\partial\Omega} = 0 \tag{3.14}$$

Note that the convergence (3.13) implies that

$$h^l_n(T) \to h^l_0(T) := \partial_t\widehat{u}^l(T) + f^l(\widehat{u}(T)) - g^l \tag{3.15}$$

in the space $C^{2\beta_1}(\Omega)$.

Let $v_n \in W^{1,2}_0(\Omega)$ be a solution of the following elliptic boundary problem

$$A^l_{\varepsilon_n} v_n = h^l_0(T) \tag{3.16}$$

Then from the one side

$$\|\widehat{u}^l_n(T) - v_n\|_{1,2} \le C\|h^l_n(T) - h^l_0(T)\|_{-1,2} \to 0 \tag{3.17}$$

due to (3.15) and due to the uniform with respect to ε boundedness of $(A_\varepsilon)^{-1}$: $W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$. From the other side due to Proposition 3.1

$$v^n \rightharpoonup v_0, \text{ in } W_0^{1,2}(\Omega) \text{ where } A_0^l v_0 = h_0^l(T) \tag{3.18}$$

The convergences (3.17) and (3.18) imply that $\widehat{u}^l(T) = v_0$ and consequently

$$A_0^l \widehat{u}^l(T) = h_0^l(T) = \partial_t u^l(T) + f^l(\widehat{u}(T)) - g^l \tag{3.19}$$

since T is arbitrary then the function $u^l(t)$ really satisfies the l-th equation of (3.10) and therefore (since $l \in \{1, \cdots, k\}$ is arbitrary) $\widehat{u}$ satisfies the homogenized equation (3.10). Note also that the uniform estimate (3.12) and the convergence (3.13) imply that $\widehat{u}$ is bounded. Thus, $\widehat{u} \in K^0$. Theorem 3.1 is proved.

§4 The case of periodic coefficients: estimates of the error.

This Section is devoted to a more detailed consideration of the particular case where the coefficients $a_{ij}^l(y)$ are periodic and smooth $(C^2(\mathbb{R}^3))$ functions in $\mathbb{R}^3$, i.e it is assumed that there are positive numbers $\vec{T} = (T_1, T_2, T_3) > 0$ such that

$$a_{ij}^l(y + (\vec{T}, m)) \equiv a_{ij}^l(y), \quad \text{for all } i, j, l \text{ and for all } m \in \mathbb{Z}^3,\ y \in \mathbb{R}^3 \tag{4.1}$$

In this case using the method of asymptotic expansions (see [2], [20]) we obtain the error estimates for the approximation of the individual solutions of (0.1) by the solutions of the homogenized problem (3.10) and basing on these estimates we derive after that the estimates for the distance between the global attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ under some additional assumptions on the limit attractor $\mathcal{A}^0$.

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold and let in addition* (4.1) *be also valid. Then for every $\varepsilon > 0$ small enough and every $u_0 \in \Phi \cap W_0^{1,2}(\Omega)$ the following estimate is valid*

$$\|u_\varepsilon(t) - \widehat{u}(t)\|_{0,2} \le Q(\|u_0\|_{L^\infty \cap W_0^{1,2}})\varepsilon^{1/3} e^{Kt} \tag{4.2}$$

where $u_\varepsilon(t) = S_t^\varepsilon u_0$, $\widehat{u}(t) = S_t^0 u_0$ are the solutions of the problems (0.1) *and* (3.10) *respectively, the function Q and the constant $K = K(\|u_0\|_{0,\infty})$ are independent of ε.*

Proof. Note for the first that due to the fact that the elliptic operator A_0 has the constant coefficients one can derive from (1.17) that

$$\|\widehat{u}(T)\|_{1,2}^2 + \int_T^{T+1} \|\widehat{u}(t)\|_{2,2}^2\,dt + \int_T^{T+1} \|\partial_t \widehat{u}(t)\|_{0,2}^2\,dt \le \\ \le Q(\|u_0\|_{0,\infty} + \|u_0\|_{1,2}) + Q(\|g\|_{0,2}) \tag{4.3}$$

for the appropriate function Q independent of $T \ge 0$ (here we have implicitly used the elliptic regularity estimate $\|u\|_{2,2} \le C\|A_0 u\|_{0,2}$).

Let us fix $l \in \{1, \cdots, k\}$ and consider the l-th equation of (0.1):

$$\partial_t u_\varepsilon^l = A_\varepsilon^l u_\varepsilon^l - f^l(u_\varepsilon) + g^l; \quad u_\varepsilon^l\big|_{t=0} = u_0^l \tag{4.4}$$

and introduce the correctors $N_k^l(y)$, $k=1,2,3$ as solutions of the following auxiliary periodic problems:

$$
\left\{\begin{array}{l}\sum_{i,j=1}^3 \partial_{y_i}\left(a_{ij}^l(y)\partial_{y_j} N_k^l(y)\right) = -\sum_{i=1}^3 \partial_{y_i} a_{ik}^l(y), \ \ y\in\mathbb{R}^3 \\ N_k^l(y+(\vec{T},m)) \equiv N_k^l(y), \quad m\in\mathbb{Z}^3\end{array}\right. \tag{4.5}
$$

It is well-known that the periodic problems (4.5) have unique solutions (due to the uniform ellipticity assumption (0.4)) and since $a_{ij}^l(y)$ are smooth (C^2) then

$$
\|N_k^l\|_{C_b^1(\mathbb{R}^3)} \le C \tag{4.6}
$$

Moreover, if the solutions of (4.5) are known then the coefficients $\tilde{a}_{ij}^l$ of the limit elliptic operator A_0^l can be calculated by the following formulae (see e.g. [20]):

$$
\tilde{a}_{ij}^l = \left\langle a_{ij}^l(y)\right\rangle + \sum_{k=1}^3 \left\langle a_{ik}^l(y)\partial_{y_k} N_j(y)\right\rangle \tag{4.7}
$$

Lemma 4.1. *Define the functions*

$$
\tilde{u}_\varepsilon^l(t) := \widehat{u}^l(t) + \varepsilon\sum_{k=1}^3 N_k^l(\varepsilon^{-1}x)\partial_{x_k}\widehat{u}^l(t) \tag{4.8}
$$

Then

$$
\|A_\varepsilon^l \tilde{u}_\varepsilon^l(t) - A_0^l\widehat{u}^l(t)\|_{-1,2} \le C\varepsilon\|\widehat{u}^l(t)\|_{2,2} \tag{4.9}
$$

where the constant C is independent of ε.

Proof. Indeed, it is computed in [20, page 27] that

$$
\sum_{j=1}^3 \left(a_{ij}^l(\varepsilon^{-1}x)\partial_{x_j}\tilde{u}_\varepsilon^l - \tilde{a}_{ij}^l\partial_{x_j}\widehat{u}^l\right) = \varepsilon\sum_{j=1}^3 \partial_{x_j}\left(\alpha_{ij}^{lk}(\varepsilon^{-1}x)\partial_{x_k}\widehat{u}^l\right) + r_\varepsilon^{il} \tag{4.10}
$$

where $\alpha_{ij}^{lk}(y)$ are certain periodic functions such that

$$
\alpha_{ij}^{lk}(y) = -\alpha_{ji}^{lk}(y) \tag{4.11}
$$

and the remainders r_ε^{il} are

$$
r_\varepsilon^{il} := \varepsilon\sum_{k,j=1}^3 N_k^l(\varepsilon^{-1}x)\partial_{x_jx_k}^2\widehat{u}^l - \varepsilon\alpha_{ij}^{kl}(\varepsilon^{-1}x)\partial_{x_jx_k}^2\widehat{u}^l \tag{4.12}
$$

Note that due to (4.11) the divergence from the first term into the right-hand side of (4.10) equals zero and consequently

$$
\|A_\varepsilon^l\tilde{u}_\varepsilon^l(t) - A_0^l\widehat{u}^l(t)\|_{-1,2} \le \sum_{i=1}^3 \|r_\varepsilon^{il}(t)\|_{0,2} \le C\varepsilon\|\widehat{u}^l(t)\|_{2,2}
$$

Lemma 4.1 is proved.

Note that the constructed functions $\tilde{u}^l_\varepsilon(t)$ do not satisfy the boundary conditions ($\tilde{u}^l_\varepsilon(t) \notin W^{1,2}_0(\Omega)$) which is inconvenient for our purposes (we are planning to multiply the equation (4.4) in $L^2(\Omega)$ by $\tilde{u}^l_\varepsilon(t) - \widehat{u}^l(t)$ and integrate by parts). In order to avoid this difficulty we introduce (following to [20]) the family $\tau_\varepsilon(x) \in C^\infty_0(\Omega)$ of cut-off functions satisfying the following conditions

1. $0 \le \tau_\varepsilon \le 1$, and $\tau_\varepsilon(x) \equiv 1$ if $x \in \Omega \backslash \mathcal{O}_\varepsilon(\partial\Omega)$, where $\mathcal{O}_\varepsilon(\partial\Omega)$ means the ε-neighbourhood of the boundary $\partial\Omega$.
2. $\varepsilon|\nabla_x \tau_\varepsilon(x)| \le C$ for every $x \in \Omega$ and $\varepsilon > 0$.

(Such family exists because the boundary $\partial\Omega$ is assumed to be smooth (see [20])) and make the following boundary correction of the functions $\tilde{u}^l_\varepsilon(t)$:

$$w^l_\varepsilon(t) := \tilde{u}^l_\varepsilon(t) - \varepsilon(1-\tau_\varepsilon(x))\sum_{k=1}^3 N_k(\varepsilon^{-1}x)\partial_{x_k}\widehat{u}^l(t) \tag{4.13}$$

Then, evidently, $w^l_\varepsilon(t) \in W^{1,2}_0(\Omega)$ (to be more precise $w^l_\varepsilon \in L^2([0,T], W^{1,2}_0(\Omega)))$ and the $W^{1,2}$-distance between $\tilde{u}^l_\varepsilon$ and w^l_ε is sufficiently small as the following lemma shows.

Lemma 4.2. *Let the functions $\tilde{u}^l_\varepsilon(t)$ and $w^l_\varepsilon(t)$ be defined by (4.8) and (4.13) respectively. Then*

$$\|\tilde{u}^l_\varepsilon - w^l_\varepsilon\|_{1,2} \le C\varepsilon^{1/3}\|\widehat{u}^l(t)\|_{2,2} \tag{4.14}$$

where the constant C is independent of ε.

Proof. Indeed, since $N^l_k(y) \in C^1_b(\mathbb{R}^3)$ then due to our choice of cut-off functions τ_ε

$$\begin{aligned} \|\nabla_x(\tilde{u}^l_\varepsilon - w^l_\varepsilon)\|^2_{1,2} \le \\ \le \int_\Omega |\varepsilon\nabla_x\tau_\varepsilon|^2|\nabla_x\widehat{u}^l(t)|^2 + (1-\tau_\varepsilon)^2|\nabla_y N(\varepsilon^{-1}x)|^2|\nabla_x\widehat{u}(t)|^2 + C\varepsilon^2|\nabla^2_x\widehat{u}(t)|^2\,dx \le \\ \le C\int_{x\in\mathcal{O}_\varepsilon(\partial\Omega)}|\nabla_x\widehat{u}^l(t)|^2\,dx + C\varepsilon^2\|\widehat{u}^l(t)\|^2_{2,2} \end{aligned} \tag{4.15}$$

Applying the Hölder inequality to the first term in the right-hand side of (4.15) and using the embedding $W^{2,2} \subset W^{1,6}$ we derive that

$$\int_{x\in\mathcal{O}_\varepsilon(\partial\Omega)}|\nabla_x\widehat{u}^l(t)|^2\,dx \le |\mathcal{O}_\varepsilon(\partial\Omega)|^{2/3}\|\widehat{u}^l(t)\|^2_{1,6} \le C\varepsilon^{2/3}\|\widehat{u}^l(t)\|^2_{2,2} \tag{4.16}$$

(here we have used the fact that the volume of the ε-neighbourhood $|\mathcal{O}_\varepsilon(\partial\Omega)| \le C\varepsilon$ because Ω is assumed to be smooth). The estimates (4.15) and (4.16) prove the lemma.

Now we are in a position to complete the proof of the theorem. To this end we introduce the function $v_\varepsilon(t) := u_\varepsilon(t) - \widehat{u}(t)$. Then

$$\partial_t v^l_\varepsilon(t) = [A^l_\varepsilon u^l_\varepsilon - A^l_0\widehat{u}^l(t)] - [f^l(u_\varepsilon(t)) - f^l(\widehat{u}(t))], \quad v^l_\varepsilon\big|_{\partial\Omega} = 0 \tag{4.17}$$

Let us take the inner product in $L^2(\Omega)$ of this equation with the function $u_\varepsilon^l(t) - w_\varepsilon^l(t)$:

$$(4.18)\quad \int_0^T \left(\partial_t v_\varepsilon^l, (u_\varepsilon^l - w_\varepsilon^l)\right)\, dt = \int_0^T \left(A_\varepsilon^l u_\varepsilon^l(t) - A_0^l \widehat{u}^l(t), u_\varepsilon^l(t) - w_\varepsilon^l(t)\right)\, dt - \\ - \int_0^T \left(f^l(u_\varepsilon(t)) - f^l(\widehat{u}(t)), u_\varepsilon^l(t) - w_\varepsilon^l(t)\right)\, dt$$

Note that the definitions (4.8) and (4.13) imply the estimate

$$(4.19)\quad \|u_\varepsilon^l(t) - w_\varepsilon^l(t) - v_\varepsilon^l(t)\|_{0,2} \le C\varepsilon \|\widehat{u}^l(t)\|_{1,2}$$

Consequently, the estimate (1.17) and (4.3) imply the inequalty

$$(4.20)\quad \left|\int_0^T \left(\partial_t v_\varepsilon^l(t), u_\varepsilon^l(t) - w_\varepsilon^l(t)\right)\, dt - \int_0^T \left(\partial_t v_\varepsilon^l(t), v_\varepsilon^l(t)\right)\, dt\right| \le \\ \le C\varepsilon \left(\int_0^T \|\partial_t v_\varepsilon^l(t)\|_{0,2}^2\right)^{1/2} \left(\int_0^T \|\widehat{u}^l(t)\|_{1,2}\, dt\right)^{1/2} \le Q(\|u_0\|_{L^\infty \cap W^{1,2}})\varepsilon T$$

for a certain monotonic function Q independent of ε.

The third term in (4.18) can be estimated analogously using the fact that we have the uniform with respect to ε L^∞-estimate for the solutions $u_\varepsilon(t)$ and $\widehat{u}(t)$ (due to Corollary 1.4):

$$(4.21)\quad \left|\int_0^T \left(f^l(u_\varepsilon(t)) - f^l(\widehat{u}(t)), u_\varepsilon^l(t) - w_\varepsilon^l(t)\right)\, dt - \right. \\ \left. - \int_0^T \left(f^l(u_\varepsilon(t)) - f^l(\widehat{u}(t)), v_\varepsilon^l(t)\right)\, dt\right| \le Q(\|u_0\|_{0,\infty})\varepsilon T$$

with the appropriate Q independent of ε.

Thus, it remains only to estimate the most complicated second term of (4.18). We will do so using the results of Lemmata 4.1 and 4.2 and the uniform ellipticity (with the constant $\nu > 0$) of the operators A_ε^l:

$$(4.22)\quad -\left(A_\varepsilon^l u_\varepsilon^l(t) - A_0^l \widehat{u}^l(t), u_\varepsilon^l - w_\varepsilon^l\right) = -\left(A_\varepsilon^l (u_\varepsilon^l(t) - \tilde{u}_\varepsilon^l(t)), u_\varepsilon^l(t) - w_\varepsilon^l(t)\right) - \\ - \left(A_\varepsilon^l \tilde{u}_\varepsilon^l(t) - A_0^l \widehat{u}^l(t), u_\varepsilon^l(t) - w_\varepsilon^l(t)\right) \ge \\ \ge \nu \|u_\varepsilon^l(t) - \tilde{u}_\varepsilon^l(t)\|_{1,2}^2 - C\|\tilde{u}_\varepsilon^l(t) - w_\varepsilon^l(t)\|_{1,2}^2 - \nu/4 \|u_\varepsilon^l(t) - \tilde{u}_\varepsilon^l(t)\|_{1,2}^2 - \\ - C\varepsilon^2 \|\widehat{u}^l(t)\|_{2,2}^2 - \nu/4 \|u_\varepsilon^l(t) - w_\varepsilon^l(t)\|_{1,2}^2 \ge -C_1 \varepsilon^{2/3} \|\widehat{u}^l(t)\|_{2,2}^2$$

Intergating the inequalty (4.22) over $t \in [0, T]$ and taking into the account the estimate (4.3) we derive that

$$(4.23)\quad \int_0^T \left(A_\varepsilon^l u_\varepsilon^l(t) - A_0 \widehat{u}^l(t), u_\varepsilon^l(t) - w_\varepsilon^l(t)\right)\, dt \le Q\left(\|u_0\|_{L^\infty \cap W^{1,2}}\right) \varepsilon^{2/3} T$$

Inserting the estimates (4.20), (4.21) and (4.23) in the relation (4.18) and taking into the account the fact that $v_\varepsilon^l(0) = 0$ we derive the estimate:

$$(4.24) \quad 1/2\|v_\varepsilon^l(T)\|_{0,2}^2 \le Q\left(\|u_0\|_{L_\infty \cap W^{1,2}}\right)\varepsilon^{2/3}T - \\ - \int_0^T \left(f^l(u_\varepsilon(t)) - f^l(\widehat{u}(t)), v_\varepsilon^l(t)\right)\, dt$$

Summing the inequalities (4.24) for $l = 1, \cdots, k$ we finally obtain that

$$(4.25) \qquad \|v_\varepsilon(T)\|_{0,2}^2 \le 2kQ\varepsilon^{2/3}T - 2\int_0^T \left(f(u_\varepsilon(t)) - f(\widehat{u}(t)), v_\varepsilon(t)\right)\, dt$$

Recall that the function $f \in C^1$ and we have the uniform with respect to ε L^∞-estimates of solutions $u_\varepsilon(t)$ and $\widehat{u}(t)$. That is why we can estimate the intergal in (4.25) in a standard way and obtain the estimate

$$(4.26) \qquad \|v_\varepsilon(T)\|_{0,2}^2 \le 2K\int_0^T \|v_\varepsilon(t)\|_{0,2}^2\, dt + 2kQ\varepsilon^{2/3}T$$

where $K = K(\|u_0\|_{0,\infty})$.

The Gronwall inequality applied to (4.26) complets the proof of the theorem.

Remark 4.1. Note that in the case where the limit solution $\widehat{u}(t)$ is smooth enough (e.g. $\widehat{u} \in C^{1,2}(\Omega_T)$, it will be so for example if in addition $g \in C^\beta(\Omega)$ and the initial value $u_0 \in C^{2+\beta}(\Omega)$) one can expect much better estimate then (4.2) (with exponent 1/2 or even 1 instead of 1/3). But in this case the function Q will also depend on $\|u_0\|_{C^{2+\beta}}$ which is not permit to apply this result for estimation the difference between the attractors. So, taking in mind the application of this estimate to the attractors we cannot consider the initial data more regular than $C^{2\beta} \cap W_0^{1,2}$ and this was the main difficulty in the proof of Theorem 4.1.

Now, having the error's estimate for approximation the individual solutions of (0.1) by the solutions of homogenized equation (3.10), we are in a position to derive the analogous error's estimates for the (global) attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$. To this end we need some additional information about the attractor $\mathcal{A}^0$ of the limit equation (3.10). Namely, we require the rate of convergence of images of bounded sets to the attractor $\mathcal{A}^0$ to be exponential, i.e.

$$(4.27) \qquad \operatorname{dist}_{L^2(\Omega)}\left(S_t^0 B, \mathcal{A}^0\right) \le Q\left(\|B\|_\Phi\right) e^{-\nu t}$$

for a certain *positive* exponent $\nu > 0$ and the appropriate function Q.

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold and let the limit attractor $\mathcal{A}^0$ be exponential. Then the nonsymmetric Hausdorff distance between $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ possesses the following estimate*

$$(4.28) \qquad \operatorname{dist}_{L^2(\Omega)}\left(\mathcal{A}^\varepsilon, \mathcal{A}^0\right) \le C\varepsilon^\kappa$$

where the constant C and the exponent $0 < \kappa < 1$ can be calculated explicitly.

Proof. The assertion of the theorem is a simple corollary of the estimates (4.2) and (4.27). Indeed, let $u_\varepsilon^0 \in \mathcal{A}^\varepsilon$ be an arbitrary point of the attractor $\mathcal{A}^\varepsilon$. Then due to (2.6) there is a complete bounded trajectory $u_\varepsilon(t)\in K^\varepsilon$ such that $u_\varepsilon(0)=u^0_\varepsilon$. Moreover, according to the estimate (2.12)

$$\|u_\varepsilon(t)\|_{L^\infty\cap W^{1,2}(\Omega)}\le C \tag{4.29}$$

where C is independent of $t\in\mathbb{R}$, ε and u^0_ε. Let us fix now an arbitrary $T\in\mathbb{R}_+$ and consider the solution $\widehat{u}(t)$ of the homogenized problem (3.10) with the initial value $\widehat{u}(0):=u_\varepsilon(-T)$. Then, according to the estimate (4.2),

$$\|u_\varepsilon(0)-\widehat{u}(T)\|_{0,2}\le C_1\varepsilon^{1/3}e^{KT} \tag{4.30}$$

where the constants C_1 and K depend only on C from (4.29) and on the functions Q and K in (4.2) and independent of u^0_ε, T and ε.

From the other side since the attractor $\mathcal{A}^0$ is exponential then

$$\operatorname{dist}_{L^2(\Omega)}\left(\widehat{u}(T),\mathcal{A}^0\right)\le C_2e^{-\nu T} \tag{4.31}$$

where the constant C_2 is also independent of u^0_ε, T and ε (due to the uniform estimate (4.29)). Combining the estimates (4.30) and (4.31) we derive that

$$\operatorname{dist}_{L^2(\Omega)}\left(u^0_\varepsilon,\mathcal{A}^0\right)\le C_1\varepsilon^{1/3}e^{KT}+C_2e^{-\nu T} \tag{4.32}$$

Recall that $T\ge 0$ is arbitrary, therefore we chose it in order to minimize the right-hand side of (4.33), i.e. from the equation

$$C_1\varepsilon^{1/3}e^{KT}=C_2e^{-\nu T}$$

solving this equation and inserting $T=T(\varepsilon)$ in the right-hand side of (4.32) we obtain that

$$\operatorname{dist}_{L^2(\Omega)}\left(u^0_\varepsilon,\mathcal{A}^0\right)\le C_3\varepsilon^\kappa \tag{4.33}$$

where $\kappa=\frac{\nu}{3(\nu+\kappa)}$. Since $u^0_\varepsilon\in\mathcal{A}^\varepsilon$ is arbitrary then (4.33) implies (4.28). Theorem 4.2 is proved.

Corollary 4.1. *Let the assumptions of Theorem 4.2 hold. Then*

$$\operatorname{dist}_{C(\Omega)}\left(\mathcal{A}^\varepsilon,\mathcal{A}^0\right)\le C\varepsilon^{\kappa_1} \tag{4.34}$$

for the appropriate $0<\kappa_1<\kappa<1$.

Indeed, due to Corollary 2.1 the attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$ are uniformly bounded in $C^{2\beta'}(\Omega)$, $\beta>0$ therefore the estimate (4.27) together with the appropriate interpolation inequality implies (4.34).

Remark 4.2. Assume in addition that the attractors $\mathcal{A}^\varepsilon$, $\varepsilon<\varepsilon_0$ are uniformly with respect to ε exponential, i.e.

$$\operatorname{dist}_{L^2(\Omega)}\left(S^\varepsilon_tB,\mathcal{A}^\varepsilon\right)\le Q(\|B\|_\Phi)e^{-\nu t} \tag{4.35}$$

where the positive constant $\nu > 0$ and the function Q *are independent of* ε. Then arguing as in the proof of Theorem 4.2 one can easily obtain the lower semicontinuity of the attractors $\mathcal{A}^\varepsilon$ and the estimate

$$\operatorname{dist}_{L^2(\Omega)}\left(\mathcal{A}^0, \mathcal{A}^\varepsilon\right) \le C\varepsilon^\kappa \tag{4.36}$$

and consequently in this case we have the estimate (4.2) not only for symmetric Hausdorff distance but for the non-symmetric one as well.

Note in conclusion that there is a large class of systems of the form (0.1) for which the estimate (4.27) is known. Indeed, assume in addition that the nonlinear function f has a gradient structure (which is always true in a scalar case $k = 1$)

$$f(u) = \nabla_u F(u) \tag{4.37}$$

Then as known (see e.g. [1]) the equation (3.10) possesses a global Lyapunov function

$$L(u) = \int_\Omega \sum_{i,j,l} \tilde{a}^l_{ij} \partial_{x_i} u^l \partial_{x_j} u^l + 2F(u) - 2g.u\, dx \tag{4.38}$$

and consequently in the generic case where we have only finite number of equilibria points $\mathcal{R} := \{z_1, \cdots, z_N\}$ for the equation (3.10) and all of them are hyperbolic the attractor of the equation (3.10) is regular i.e. consists of a finite collection of the finite dimensional unstable manifolds $\mathcal{M}^+(z_i)$ of the equilibria points $z_i \in \mathcal{R}$:

$$\mathcal{A}^0 = \cup_{i=1}^N \mathcal{M}^+(z_i) \tag{4.39}$$

(see e.g. [1] or [15]). Moreover, it is also known (see [1]) that the regular attractor is exponential i.e. (4.27) holds for the regular attractors. Thus, we have proved the following theorem.

Theorem 4.3. *Let the assumptions of Theorem 4.2 holds and let in addition* (4.37) *is satisfied and all equilibria points of the homogenized equation are hyperbolic. Then the estimate* (4.28) *holds.*

Let us consider now several examples of reaction-diffusion systems arising in mathematical physics for which the assumptions of Theorem 4.2 are fulfilled.

Example 4.1. One of the simplest examples is a Chaffee-Infante equation in the nonhomogenized almost-periodic media:

$$\partial_t u = A_\varepsilon u - u^3 + \alpha u + g(x), \quad u\big|_{\partial\Omega} = 0 \tag{4.40}$$

Here u is a scalar function ($k = 1$), $g \in L^2(\Omega)$, $\alpha \in \mathbb{R}$ is a given constant and the operator A_ε is defined via (0.3). All assumptions o Theorem 3.1 are evidently satisfied for this equation and consequently (4.40) possesses a global attractor $\mathcal{A}^\varepsilon$ for every $\varepsilon > 0$. Moreover, these attractors tend to the limit attractor $\mathcal{A}^0$ of the homogenized problem as $\varepsilon \to 0$ in the sense of (3.11).

Note also that the equation (4.40) has a gradient form ((4.37) is satisfied) and consequently for generic $g \in L^2(\Omega)$ the limit attractor A_0 is regular and exponential.

Thus, in the case of periodic media the estimate (4.28) is also valid for the external forces g belonging to a certain open and dense set in $L^2(\Omega)$.

Example 4.2. Consider now the following generalization of Lotka-Volterra system:

$$\begin{cases} \partial_t u_i = A^i_\varepsilon u_i - f_i(u_i) - u_i \left(\sum_{j=1}^k b_{ij} u_j^2\right) + g_i(x) \\ u_i\big|_{\partial\Omega} = 0, \quad i = 1, \cdots, k \end{cases} \tag{4.41}$$

where A^i_ε are defined via (0.3), $b_{ij} \geq 0$ are given nonnegative constants, $g_i \in L^2(\Omega)$ and the functions f_i are assumed to satisfy the following assumptions:

$$1.\ f_i \in C^1(\mathbb{R}), \quad 2.\ f_i(v) \cdot \operatorname{sgn} v \geq 0 \text{ for } |v| \geq R, \quad 3.\ |f_i(v)| \leq C(1 + |v|^p) \tag{4.42}$$

for a certain constants $R > 0$, $C > 0$ and $p > 0$. It is not difficult to verify that the system (4.41) satisfies all assumptions of Theorem 3.1 (particularly, the anisotropic dissipativity condition (0.6) is valid with $p_1 = p_2 = \cdots = p_k = q$ for arbitrary $q > 0$), consequently the attractors $\mathcal{A}^\varepsilon$, $\varepsilon > 0$ associated with the problems (4.41) converge as $\varepsilon \to 0$ to the attractor $\mathcal{A}^0$ of the limit homogenized problem.

Note also that in the case where the matrix $\{b_{ij}\}_{i,j=1}^k$ is symmetric, i.e.

$$b_{ij} = b_{ji} \geq 0, \quad i, j = 1, \cdots, k \tag{4.43}$$

the equation (4.41) has a gradient form, therefore in the case of periodic media the estimate (4.28) for the rate of convergence of $\mathcal{A}^\varepsilon$ as $\varepsilon \to 0$ is valid for g_i belonging to a certain open and dense subset of $L^2(\Omega)$ and for b_{ij} satisfying (4.43).

Example 4.3. We conclude our consideration by the so-called FitzHugh-Nagumo system:

$$\begin{cases} \partial_t u_1 = A^1_\varepsilon u_1 - f(u_1) - u_2 + g_1(x), \quad u_1\big|_{\partial\Omega} = 0 \\ \partial_t u_2 = A^2_\varepsilon u_2 + \delta u_1 - \gamma u_2 + g_2(x), \quad u_2\big|_{\partial\Omega} = 0 \end{cases} \tag{4.44}$$

Here $k = 2$, the operators A^i_ε are defined via (0.3), $\delta, \gamma > 0$ are positive constants and the nonlinearity $f(v)$ is assumed to satisfy the fullowing assumptions:

$$1.\ f \in C^1(\Omega), \quad 2.\ f(v).v \geq -C + C_1|v|^{q+1}, \quad 3.\ |f(u)| \leq C(1 + |v|^p) \tag{4.45}$$

for a some positive $C, C_1, q, p > 0$. A simple checking reveals that the equation (4.44) satisfies all assumptions of Theorem 3.1 (particularly the anisotropic dissipativity assumption (0.6) holds with $p_1 = p_2 = r$ and arbitrary $r > 0$), consequently the attractors $\mathcal{A}^\varepsilon$ associated with the nonhomogeneous problem (4.44) tend as $\varepsilon > 0$ to the attractor $\mathcal{A}^0$ of the limit homogenized problem.

Note that the equation (4.44) does not have a gradient structure. Nevertheless, it is shown in [4] that under the additional assumptions

$$1.\ f'(v) \geq -\gamma \quad \text{for all } v \in \mathbb{R}, \tag{4.46}$$

where γ is the same as in the second equation of (4.44), this problem possesses a global Lyapunov function in the form:

$$\begin{aligned} L(u_1, u_2) := \frac{1}{2}\|\partial_t u_1\|^2_{L^2} + \frac{1}{2\delta}\|\partial_t u_2\|^2_{L^2} - \frac{\gamma}{2}\left(A^1_\varepsilon u_1, u_1\right) + \frac{\gamma}{2\delta}\left(A^2_\varepsilon u_2, u_2\right) + \\ + \gamma F(u_1) + \gamma\left(u_1, u_2\right) - \frac{\gamma^2}{2\delta}\|v\|^2_{L^2} - \gamma\left(g_1, u_1\right) + \frac{\gamma}{\delta}\left(g_2, u_2\right) \end{aligned} \tag{4.47}$$

where $(\cdot, \cdot)$ means the inner product in $L^2(\Omega)$, $F(u) = \int_0^u f(u)\,du$ and the terms $\partial_t u_1$ and $\partial_t u_2$ should be expressed from the equations (4.44) (according to Theorem 1.2 all terms in (4.47) are well posed on the attractor $\mathcal{A}^\varepsilon$). Thus, in the case of periodic media and under the additional assumption (4.47) the estimate (4.28) holds for every g_i from a dense and open subset of $L^2(\Omega)$.

Remark 4.3. It is also known that the regular attractors are structurally stable in the sence, that representation (4.39) preserves under the small permutations of the equation (3.10), moreover the uiniform exponential attraction property is also valid if the pertrubation is small enough (see [1]). Note also that the non-homogenized problem (0.1) can be considered as a small pertrubation of the homogenized equation (3.10) because

$$\|(A_\varepsilon)^{-1} - (A_0)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \to 0 \text{ as } \varepsilon \to 0$$

(see [20]). Therefore one can expect that the uniform attraction property (4.35) is valid for our case if the limit homogenized attractor is regular. Then according to Remark 4.2 we will have the lower semicontinuity of attractors $\mathcal{A}^\varepsilon$ as $\varepsilon = 0$ and estimate (4.28) for the *symmetric* Hausdorff distance. We will give the rigorous proof of this assertion in the forthcoming paper.

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