Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities.

Sergey Zelik

Université de Poitiers
Laboratoire d'Applications des Mathématiques - SP2MI
Boulevard Marie et Pierre Curie - Téléport 2
86962 Chasseneuil Futuroscope Cedex - France

(Communicated by R. Temam)

Abstract. We study the asymptotic behavior of weak energy solutions of the following damped hyperbolic equation in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\varepsilon\partial_t^2 u + \gamma\partial_t u - \Delta_x u + f(u) = g, \quad u\big|_{\partial\Omega} = 0,$$

where γ is a positive constant and $\varepsilon > 0$ is a small parameter. We do not make any growth restrictions on the nonlinearity f and, consequently, we do not have the uniqueness of weak solutions for this problem.

We prove that the trajectory dynamical system acting on the space of all properly defined weak energy solutions of this equation possesses a global attractor $\mathcal{A}_\varepsilon^{tr}$ and verify that this attractor consists of global strong regular solutions, if $\varepsilon > 0$ is small enough. Moreover, we also establish that, generically, any weak energy solution converges *exponentially* to the attractor $\mathcal{A}_\varepsilon^{tr}$.

0. Introduction. We consider the following singularly perturbed damped wave equation in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$:

$$\begin{cases} \varepsilon\partial_t^2 u + \gamma\partial_t u - \Delta_x u + f(u) = g, \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \quad \partial_t u\big|_{t=0} = u_0'. \end{cases} \tag{0.1}$$

Here $\varepsilon > 0$ and $\gamma > 0$ are given positive numbers, $u = u(t,x)$ is an unknown function, Δ_x is the Laplacian with respect to the variable $x = (x^1, x^2, x^3)$ and $g = g(x)$ are given external forces which satisfy the following assumption:

$$g \in L^2(\Omega). \tag{0.2}$$

We also assume that the nonlinear interaction function $f(u)$ satisfies the following conditions:

$$\begin{cases} 1. \ \ f \in C^2(\mathbb{R},\mathbb{R}), \quad f(0) = 0, \\ 2. \ \ |f''(v)| \le C(1 + |v|^p), \\ 3. \ \ f'(v) \ge -K + \delta|v|^{p+1}, \end{cases} \tag{0.3}$$

where $p > 0$, $C > 0$, $K > 0$, and $\delta > 0$ are given constants. Equation (0.1) is considered in the standard energy phase space:

$$E = E(\Omega) := \Big[H_0^1(\Omega) \cap L^{p+3}(\Omega)\Big] \times L^2(\Omega). \tag{0.4}$$

AMS 1991 Subject Classification: Primary 35B40, 35B45, 35L30.

Key words and phrases: Singularly perturbed damped wave equation, supercritical nonlinearity, regular attractors.

Consequently, we assume that the solution $\xi_u(t) := (u(t), \partial_t u(t))$ belongs to E, for every $t \geq 0$, and, particularly, the initial data $\xi_u(0) := (u_0, u_0')$ also belongs to E:

$$\xi_u(0) \in E. \tag{0.5}$$

In the subcritical case $p \leq 1$, the behavior of the solutions of (0.1) is now well understood. Indeed, in this case, equation (0.1) generates a differentiable semigroup S_t^ε in the phase space E:

$$S_t^\varepsilon : E \to E, \quad S_t^\varepsilon \xi_u(0) := \xi_u(t), \quad \text{where } u(t) \text{ solves (0.1)}, \tag{0.6}$$

which possesses the compact global attractor $\mathcal{A}_\varepsilon^{gl}$ in E, see e.g. [1], [3], [14–17], [23] and the references therein. These attractors are uniformly (with respect to $\varepsilon \to 0$) bounded in the space

$$E^1 := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \tag{0.7}$$

and (consequently) they converge as $\varepsilon \to 0$ to the limit attractor $\mathcal{A}_0^{gl}$ associated with the limit parabolic equation (at least in the sense of the upper semicontinuity in E^1, see [3], [16] or [11] for the details).

Moreover, since equation (0.1) possesses a global Liapunov function (see [3] or [16]) then, under the additional generic assumption that the set $\mathcal{R}$ of equilibria of equation (0.1) is finite:

$$\#\mathcal{R} = N < \infty \quad \text{and all the equilibria are hyperbolic}, \tag{0.8}$$

the attractor $\mathcal{A}_\varepsilon^{gl}$ consists of a finite collection of finite dimensional unstable C^1-submanifolds in E:

$$\mathcal{A}_\varepsilon^{gl} := \cup_{z_0 \in \mathcal{R}} \mathcal{M}_\varepsilon^+(z_0) \tag{0.9}$$

and the rate of convergence to it is exponential, i.e., for every bounded subset $B \subset E$, the following estimate is valid:

$$\operatorname{dist}_{E,\varepsilon}(S_t B, \mathcal{A}_\varepsilon^{gl}) \leq Q(\|B\|_{E,\varepsilon}) e^{-\alpha t}, \tag{0.10}$$

where the monotonic function Q and the positive constant α are independent of B and $\varepsilon \leq \varepsilon_0$,

$$\|\xi_u\|_{E,\varepsilon}^2 := \varepsilon \|\partial_t u\|_{L^2}^2 + \|u\|_{H^1}^2 + \|u\|_{L^{p+3}}^{p+3} \tag{0.11}$$

and $\operatorname{dist}_{E,\varepsilon}$ denotes the nonsymmetric Hausdorff distance between sets in "metric" (0.11). It is also known that, in this case, we also have the lower semicontinuity of the attractors $\mathcal{A}_\varepsilon^{gl}$ as $\varepsilon \to 0$ and the following estimate on the rate of convergence:

$$\operatorname{dist}_{symm,E^1}(\mathcal{A}_\varepsilon^{gl}, \mathcal{A}_0^{gl}) \leq C\varepsilon^\kappa, \tag{0.12}$$

where $C > 0$ and $\kappa > 0$ are independent of ε (see [3], [11] or [16]).

Moreover, the nonautonomous equations of the form of (0.1) were studied in [4]; exponential attractors for (0.1) were considered in [10] and [11]; the Gevrey regularity of attractors $\mathcal{A}_\varepsilon$ was established in [6]; and attractors for equations (0.1) in unbounded domains Ω were investigated in [12] and [28]. We note however that the proof of the E^1-regularity of the attractor $\mathcal{A}_\varepsilon^{gl}$ in the critical case $p = 1$ essentially uses the finiteness of a suitable dissipation integral (see [3]) so, to the best of our knowledge, the higher (e.g., E^1) regularity of attractors is still an open problem in the case of more general (than (0.1)) semilinear hyperbolic equations and systems (e.g., nonautonomous ones) with critical growth rate on the nonlinearity.

In contrast to this, very few is known about the solutions of (0.1) in the supercritical case $p > 1$. Indeed, although the global existence of weak energy solutions

$$\xi_u \in C(\mathbb{R}_+, E^{\mathrm{w}}) \tag{0.13}$$

(where, as usual, the symbol 'w' denotes the weak topology in E) can be derived in a standard way from the energy estimate (see e.g. [18]), the regularity (0.13) is not enough in order to prove the uniqueness of such solutions and, to the best of our knowledge, only the *local* existence of more regular solutions is known for $p > 1$.

Thus, semigroup (0.6) associated with equation (0.1) can be rigorously defined only as a semigroup of multivalued maps. A (generalized) global attractor for this semigroup has been constructed in [2].

A similar result was reproved in [5], using the concept of trajectory dynamical system and the associated trajectory attractor. We recall that, under this approach, the set K_ε^+ of all properly defined weak energy solutions of (0.1) (for all initial data $\xi_u(0)$ belonging to E, see Definition 1.1) endowed with the appropriate topology is considered as a (trajectory) phase space for the semigroup of positive temporal shifts

$$T_h : K_\varepsilon^+ \to K_\varepsilon^+, \quad (T_h u)(t) := u(t+h), \quad t, h \in \mathbb{R}_+. \tag{0.14}$$

This semigroup (acting on the trajectory phase space K_ε^+) is called a trajectory dynamical system associated with problem (0.1) and its global attractor (if it exists) is called a trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ of problem (0.1). It is worth to note that, in the case where uniqueness holds, the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ is usually equivalent (Lipschitz homeomorphic) to the global one (see Remark 1.1 below).

The trajectory attractors $\mathcal{A}_\varepsilon^{tr}$ for problem (0.1) were constructed in [5] and their weak upper semicontinuity as $\varepsilon \to 0$ was established in [6], see also [5], [13] and [22] for applications of the trajectory approach described above to other classes of ill-posed evolution equations and [19], [25] and [26] for its applications to elliptic boundary value problems in unbounded domains.

In the present paper, we give a systematic study of the attractors associated with problem (0.1) in case $\varepsilon > 0$ is small enough.

In Section 1, we recall the construction of a weak energy solution $\xi_u(t)$ of problem (0.1) using Galerkin approximations. Using this explicit construction, we then define the trajectory phase space K_ε^+ as a space of all weak energy solutions of (0.1) which can be obtained as a weak limit of the corresponding Galerkin approximations and establish that the trajectory dynamical system (0.14) possesses the compact global attractor $\mathcal{A}_\varepsilon^{tr}$ in the following weak-$*$ topology:

$$\Theta^+ := [L^\infty_{loc}(\mathbb{R}_+, E)]^{\mathrm{w}^*} \tag{0.15}$$

(see §1 for the details). Thus, we restrict ourselves to the weak solutions $\xi_u(t)$ of problem (0.1) that can be obtained as a Θ^+-limit of the corresponding Galerkin approximations only (we do not know whether or not every weak solution of (0.1) satisfying (0.13) can be obtained in such way).

As usual (see [2], [5], [6]), the attractor $\mathcal{A}_\varepsilon^{tr}$ possesses the following description:

$$\mathcal{A}_\varepsilon^{tr} = \Pi_+ \mathcal{K}_\varepsilon, \tag{0.16}$$

where $\mathcal{K}_\varepsilon \subset L^\infty(\mathbb{R}, E)$ is the set of all weak solutions of (0.1) that are defined for every $t \in \mathbb{R}$ and can be obtained as a weak limit of the appropriate Galerkin approximations (see Theorem 1.1 below) and $\Pi_+\xi_u := \xi_u\big|_{t\ge0}$ is the restriction of the function $\xi_u \in \mathcal{K}_\varepsilon$ to the semiaxis $\mathbb{R}_+$.

In Section 2, we study the regularity properties of the weak solutions $\xi_u \in \mathcal{K}_\varepsilon$. Particularly, we prove that every such solution is regular if $t \in \mathbb{R}$ is small enough. To be more precise, for every $\xi_u \in \mathcal{K}_\varepsilon$, there exists $T = T_u \in \mathbb{R}$ such that

$$\xi_u(t) \in E^1 \quad \text{if} \quad t \le T. \tag{0.17}$$

Moreover, we obtain some uniqueness result for such solutions. We note that these results are proved without the assumption that $\varepsilon > 0$ is small, but they are essentially based on the finiteness of the dissipation integral

$$\int_{-\infty}^{+\infty} \|\partial_t u(t)\|_{L^2}^2\,dt < \infty, \quad \forall \xi_u \in \mathcal{K}_\varepsilon. \tag{0.18}$$

The main result of Section 3 is the existence of global *strong* solutions of (0.1), if the E^1-energy of the initial data is not very large and $\varepsilon > 0$ is small enough. To be more precise, we prove that there exist $\varepsilon_0 \ll 1$ and a nonincreasing function

$$R : (0, \varepsilon_0] \to \mathbb{R}_+, \quad \lim_{\varepsilon\to0} R(\varepsilon) = +\infty,$$

such that, for every $\varepsilon \le \varepsilon_0$ and every initial data satisfying

$$\|\xi_u(0)\|_{E^1,\varepsilon} \le R(\varepsilon),$$

where

$$\|\xi_u\|_{E^1,\varepsilon}^2 := \varepsilon\|\partial_t u\|_{H^1}^2 + \|u\|_{H^2}^2, \tag{0.19}$$

there exists a unique global strong solution $\xi_u \in C_b(\mathbb{R}, E^1)$ and this solution satisfies the estimate

$$\|\xi_u(t)\|_{E^1,\varepsilon} \le Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t} + Q(\|g\|_{L^2}), \tag{0.20}$$

where the positive constant α and the monotonic function Q are independent of $\varepsilon \le \varepsilon_0$.

In contrast to Section 2, this result is based on the comparison of the strong solution of (0.1) with an appropriate strong solution of the limit ($\varepsilon = 0$) parabolic problem and does not require the dissipation integral (0.18) to be finite.

Combining this result with regularity (0.17) obtained in Section 2, we finally obtain that, for all $\varepsilon \le \varepsilon_0 \ll 1$, the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ consists of the *global strong* solutions:

$$\mathcal{A}_\varepsilon^{tr} \subset C_b(\mathbb{R}_+, E^1). \tag{0.21}$$

Since a strong solution $\xi_u \in C_b(\mathbb{R}_+, E^1)$ is unique, we may define a global attractor $\mathcal{A}_\varepsilon^{gl}$ for equation (0.1) by the standard expression:

$$\mathcal{A}_\varepsilon^{gl} := \Pi_0\mathcal{A}_\varepsilon^{tr} \tag{0.22}$$

where $\Pi_0\xi_u := \xi_u(0)$, and define a classical semigroup associated with (0.1) on this attractor via

$$S_t^\varepsilon\xi_u(0) := \xi_u(t), \quad S_t^\varepsilon : \mathcal{A}_\varepsilon^{gl} \to \mathcal{A}_\varepsilon^{gl}, \tag{0.23}$$

where $\xi_u(t)$ is a unique strong solution of (0.1).

We note that, since $H^2(\Omega)\subset C(\overline{\Omega})$, then estimate (0.20) gives a uniform (with respect to ε) estimate of the C-norm of the trajectories belonging to attractor $\mathcal{A}_\varepsilon^{gl}$. Therefore, the growth rate of the nonlinearity f with respect to u becomes nonessential for further investigations of global attractors $\mathcal{A}_\varepsilon^{gl}$ and we may study them exactly as in the subcritical case $p\le 1$ (see [3] or [11]).

In particular, we indicate in Section 4 that, under the additional assumption (0.8), description (0.9) and estimate (0.10) remain valid for the supercritical case as well.

Moreover, we prove that, not only strong solutions $\xi_u\in C_b(\mathbb{R}_+,E^1)$ converge *exponentially* to the global attractor $\mathcal{A}_\varepsilon^{gl}$ (which can be proved exactly as in the subcritical case), but also that the same is valid for every *weak* solution $\xi_u\in K_\varepsilon^+$. To be more precise, for every $\varepsilon\le\varepsilon_0$ and every bounded subset of weak energy solutions $B\subset K_\varepsilon^+$, the following estimate is valid:

$$\sup_{\xi_u\in B}\operatorname{dist}_{E,\varepsilon}(\xi_u(t),\mathcal{A}_\varepsilon^{gl})\le C_Be^{-\alpha t},\tag{0.24}$$

where C_B and $\alpha>0$ are independent of ε (see Theorem 4.4).

To conclude, we note that our method seems to be applicable for the study of problem (0.1) in $\Omega\subset\subset\mathbb{R}^n$ with an arbitrary $n\ge 1$, but we restrict ourselves to the case $n=3$ only in order to avoid the additional technicalities. Moreover, it is also applicable to other classes of perturbed hyperbolic equations, e.g., to the following problem in a bounded domain $\Omega\subset\mathbb{R}^3$:

$$\partial_t^2u+\gamma\partial_tu-\Delta_xu+\varepsilon u|u|^p+u^3-\beta u=g,\ \ p>2,\ \beta\in\mathbb{R},\ \varepsilon\ll 1.\tag{0.26}$$

We will study these questions in a forthcoming paper.

Acknowledgements. This research was partially supported by the INTAS project no. 00-899 and CRDF grant RM1-2343-MO-02. The author is also grateful to M.I.Vishik and A.Liapin for many stimulating discussions.

1. The trajectory dynamical system and its attractor. In this section, we prove that problem (0.1) possesses at least one weak solution $\xi_u(t)$, for every $\xi_u(0)\in E$. Using the explicit construction of such solutions, we then define a trajectory dynamical system associated with problem (0.1) and verify that this dynamical system possesses the global attractor.

We start with constructing the Galerkin approximations for problem (0.1). Let $\{e_i\}_{i=1}^\infty$ be the orthonormal system of eigenvectors of the Laplacian Δ_x with Dirichlet boundary conditions and let $\{\lambda_i\}_{i=1}^\infty$ be the corresponding eigenvectors:

$$-\Delta_xe_i=\lambda_ie_i,\ \ e_i\big|_{\partial\Omega}=0,\ \ \lambda_{i+1}\ge\lambda_i.\tag{1.1}$$

We denote by $P_N:v\to P_Nv$ the orthoprojector in $L^2(\Omega)$ to the first N eigenvectors of system $\{e_i\}_{i=1}^\infty$ and consider, for every $N\in\mathbb{N}$, the following auxiliary problem in the phase space $E_N:=P_NE$ (e.g., $(u_N,\partial_tu_N)\in E_N$):

$$\begin{cases}\varepsilon\partial_t^2u_N+\gamma\partial_tu_N-\Delta_xu_N+P_Nf(u_N)=g_N:=P_Ng,\\ u_N(t):=\sum_{i=1}^Nu_N^i(t)e_i,\ \ \xi_{u_N}(0)=\xi_N^0\in E_N.\end{cases}\tag{1.2}$$

We note that (1.2) is a second order system of ODE with respect to the functions $\{u_N^i(t)\}_{i=1}^N$ and with the smooth (C^2-smooth) nonlinearity. The following standard lemma gives a uniform with respect to N estimate for the solutions of (1.2) in the energy phase space E.

Lemma 1.1. *Let assumptions* (0.2) *and* (0.3) *hold. Then, there exists a unique solution* $\xi_{u_N}(t)$ *of problem* (1.2) *and the following estimate holds:*

$$\|\xi_{u_N}(t)\|^2_{E,\varepsilon} + \int_t^\infty \|\partial_t u_N(s)\|^2_{L^2}\,ds \le C\|\xi_{u_N}(s)\|^2_{E,\varepsilon}e^{-\alpha(t-s)} + C(1+\|g\|^2_{L^2}), \quad (1.3)$$

where $t \ge s \ge 0$,

$$\|\xi_u(t)\|^2_{E,\varepsilon} := \varepsilon\|\partial_t u(t)\|^2_{L^2} + \|\nabla_x u(t)\|^2_{L^2} + \|u(t)\|^{p+3}_{L^{p+3}}, \quad (1.4)$$

and constants $\alpha > 0$ *and* $C > 0$ *are independent of* N *and* $\varepsilon \in [0, \varepsilon_0]$.

Proof. Multiplying equation (1.2) by $\partial_t u_N(t)$ and integrating over $x \in \Omega$, we have

$$\frac{d}{dt}[\varepsilon\|\partial_t u_N(t)\|^2_{L^2} + \|\nabla_x u_N(t)\|^2_{L^2} + 2(F(u_N(t)),1) - 2(g,u_N(t))] = \\ = -2\gamma\|\partial_t u_N(t)\|^2_{L^2}, \quad (1.5)$$

where $F(v) := \int_0^v f(w)\,dw$. We now recall (see e.g. [27]) that assumption (0.3)(3) implies that

$$\begin{gathered} f(v).v \ge |v|^2\left(-K + \frac{\delta}{p+2}|v|^{p+1}\right), \\ F(v) \ge |v|^2\left(-\frac{K}{2} + \frac{\delta}{(p+2)(p+3)}|v|^{p+1}\right), \\ F(v) \le f(v).v - |v|^2\left(-\frac{K}{2} + \frac{\delta}{p+3}|v|^{p+1}\right). \end{gathered} \quad (1.6)$$

Moreover, assumption (0.3)(2) obviously implies that

$$|f(v)| \le C|v|\left(1+|v|^{p+1}\right), \quad F(v) \le C|v|^2\left(1+|v|^{p+1}\right). \quad (1.7)$$

Integrating identity (1.5) over $t \in (t,T)$ and using estimates (1.6) and (1.7), we have

$$\|\xi_{u_N}(T)\|^2_{E,\varepsilon} + \int_t^T \|\partial_t u_N(s)\|^2_{L^2}\,ds \le \tilde C\left(\|\xi_{u_N}(t)\|^2_{E,\varepsilon} + 1 + \|g\|^2_{L^2}\right), \quad (1.8)$$

where the constant $\tilde C$ is independent of N, t, T and ε. In particular, (1.8) gives the uniform (with respect to T) a priori estimate for the solution $\xi_{u_N}(t)$ of problem (1.2) and, consequently (since (1.2) is an ODE with a smooth nonlinearity), the global solution $\xi_{u_N}(t)$, $t \in \mathbb{R}_+$, of problem (1.2) exists, for every $\xi^0_N \in E_N$, and is unique. Moreover, passing to the limit $T \to +\infty$ in estimate (1.8), we obtain

$$\int_t^\infty \|\partial_t u_N(s)\|^2_{L^2}\,ds \le \tilde C\left(\|\xi_{u_N}(t)\|^2_{E,\varepsilon} + 1 + \|g\|^2_{L^2}\right). \quad (1.9)$$

So, there remains to prove the dissipative estimate for the quantity $\|\xi_{u_N}(t)\|_{E,\varepsilon}$. To this end, as usual, we multiply equation (1.2) by $\alpha u_N(t)$, where $\alpha > 0$ is a small positive parameter which will be fixed below and integrate over $x \in \Omega$. Then, we have

$$\frac{d}{dt}[2\alpha\varepsilon(\partial_t u_N(t), u_N(t)) + \gamma\alpha\|u_N(t)\|^2_{L^2}] - 2\alpha\varepsilon\|\partial_t u_N(t)\|^2_{L^2} + \\ + 2\alpha\|\nabla_x u_N(t)\|^2_{L^2} + 2\alpha(f(u_N(t)), u_N(t)) = 2\alpha(g, u_N(t)). \quad (1.10)$$

Summing identity (1.5) with identity (1.10) and setting

$$E_\varepsilon(t) := \varepsilon\|\partial_t u_N(t)\|^2_{L^2} + \|\nabla_x u_N(t)\|^2_{L^2} + 2(F(u_N(t)),1)+ \\ + 2\alpha\varepsilon(\partial_t u_N(t), u_N(t)) + \alpha\gamma\|u_N(t)\|^2_{L^2} - 2(g, u_N(t)), \quad (1.11)$$

we obtain the following equation:

$$\frac{d}{dt}E_\varepsilon(t) + \alpha E_\varepsilon(t) = h(t) := \\ = -(2\gamma - 3\alpha\varepsilon)\|\partial_t u_N(t)\|^2_{L^2} - \alpha\|\nabla_x u_N(t)\|^2_{L^2}+ \\ + 2\alpha\left(F(u_N(t)) - f(u_N(t)).u_N(t), 1\right) + \alpha^2\gamma\|u_N(t)\|^2_{L^2} + 2\alpha^2\varepsilon(\partial_t u_N(t), u_N(t)). \quad (1.12)$$

It is not difficult to verify, using estimates (1.6) and Schwartz inequality, that it is possible to fix $\alpha > 0$ (which is independent of $\varepsilon \in [0, \varepsilon_0]$ and N) such that

$$h(t) \le C, \quad (1.13)$$

and, consequently, using Gronwall's inequality, we derive from (1.12) that

$$E_\varepsilon(t) \le E_\varepsilon(s)e^{-\alpha(t-s)} + C_1, \quad t \ge s \ge 0, \quad (1.14)$$

where the constant C_1 is independent of ε and N. There only remains to note that, due to (1.6) and (1.7), we have the estimates

$$C_2^{-1}\|\xi_{u_N}(t)\|^2_{E,\varepsilon} - C_3(1 + \|g\|^2_{L^2}) \le E_\varepsilon(t) \le C_2\left(\|\xi_{u_N}(t)\|^2_{E,\varepsilon} + 1 + \|g\|^2_{L^2}\right), \quad (1.15)$$

where the constants $C_i > 0$ are independent of t, ε and N. Indeed, estimate (1.3) is an immediate corollary of (1.14), (1.15) and (1.9). Lemma 1.1 is proven.

We now assume that the initial data ξ^0_N for the Galerkin system (1.2) converge weakly in E to some $\xi^0 \in E$:

$$\xi^0_N \rightharpoonup \xi^0 \quad \text{as} \quad N \to \infty. \quad (1.16)$$

Then, obviously, the sequence ξ^0_N is uniformly bounded in E with respect to N, and consequently, due to estimate (1.3), the sequence of corresponding solutions $\xi_{u_N}(t)$ is uniformly (with respect to N) bounded in the space $L^\infty(\mathbb{R}_+, E)$:

$$\|\xi_{u_N}\|_{L^\infty(\mathbb{R}_+,E)} \le C. \quad (1.17)$$

We recall that bounded subsets in the Frechet space $L^\infty_{loc}(\mathbb{R}_+, E)$ are precompact in the w*-topology (see e.g. [20]) and, consequently, we may extract from the sequence of solutions $\xi_{u_N}(t)$ a subsequence $\xi_{u_{N_k}}(t)$ which w*-converges to some function $\xi_u(t) \in L^\infty(\mathbb{R}_+, E)$:

$$\xi_u = \Theta^+ - \lim_{k\to\infty} \xi_{u_{N_k}}, \quad \text{where } \Theta^+ := \left[L^\infty_{loc}(\mathbb{R}_+, E)\right]^{w^*}. \quad (1.18)$$

We also recall (see [20]) that (1.18) is equivalent to the following: for every $T \in \mathbb{R}_+$

$$\xi_{u_{N_k}} \to \xi_u \ \text{ weakly-}* \text{ in } \quad L^\infty((T, T+1), E).$$

Moreover, since $u_N(t)$ solves (1.2) then, expressing the second derivative $\partial_t^2 u_N(t)$ from equation (1.2) and using estimate (1.17), we have

$$\|\partial_t^2 u_N\|_{L^\infty(\mathbb{R}_+,H^{-1}(\Omega)+L^q(\Omega))} \le C_1, \tag{1.19}$$

where the exponent q is conjugated to $p+3$ (i.e. $\frac{1}{q}+\frac{1}{p+3}=1$) and the constant C_1 is independent of N.

We now note (see e.g. [8] or [18]) that, for every $0<\beta\le 1$, the following embedding is compact:

$$\{(v,\partial_t v)\in L^\infty_{loc}(\mathbb{R}_+,E)\}\cap\{\partial_t^2 v\in L^\infty_{loc}(\mathbb{R}_+,H^{-1}(\Omega)+L^q(\Omega))\}\subset\subset \\ \subset\subset\{(v,\partial_t v)\in C_{loc}(\mathbb{R}_+,[H^{1-\beta}(\Omega)\cap L^{p+3-\beta}(\Omega)]\times H^{-\beta}(\Omega))\}. \tag{1.20}$$

Thus, weak-$*$ convergence (1.18) implies the strong convergence

$$\xi_{u_{N_k}}\to\xi_u \quad \text{strongly in} \quad C_{loc}(\mathbb{R}_+,[H^{1-\beta}(\Omega)\cap L^{p+3-\beta}(\Omega)]\times H^{-\beta}(\Omega)). \tag{1.21}$$

Consequently (see [18])

$$\xi_u\in C(\mathbb{R}_+,E^{\mathrm{w}}) \tag{1.22}$$

and, for every $t\ge 0$, we have the weak convergence

$$\xi_{u_{N_k}}(t)\rightharpoonup\xi_u(t) \text{ in } E. \tag{1.23}$$

Moreover, the strong convergence (1.21) allows to pass in a standard way to the limit $N_k\to\infty$ in equations (1.2) (in the sense of distributions) and verify that the function $\xi_u(t):=(u(t),\partial_t u(t))$ constructed above solves equation (0.1) with

$$\xi_u(0)=E^{\mathrm{w}}-\lim_{k\to\infty}\xi^0_{N_k}. \tag{1.24}$$

Thus, we have proved the following result.

Lemma 1.2. *Let the assumptions of Lemma 1.1 hold. Then, for every $\xi^0\in E$, there exists at least one weak global solution $\xi_u(t)$, $t\in\mathbb{R}_+$, of problem* (0.1) *with*

$$\xi_u(0)=\xi^0, \tag{1.25}$$

which can be obtained as a weak limit (1.18) *of the corresponding solutions $\xi_{u_{N_k}}(t)$ of the Galerkin approximations* (1.2).

Indeed, let $\xi^0\in E$. Then, we can find a sequence $\xi^0_N\in E_N$ such that $\xi^0_N\to\xi^0$ in E (since the orthonormal system $\{e_i\}_{i=1}^\infty$ of the Laplace eigenfunctions is dense in E, see [24]). Thus, the limit process (1.18) gives the desired solution of equation (0.1).

We are now ready to construct the trajectory dynamical system associated with equation (0.1).

Definition 1.1. We define the trajectory phase space K^+_ε of problem (0.1) as the set of all solutions of this problem which can be obtained as a weak-$*$ limit (1.18) of solutions of the Galerkin approximations (1.2):

$$K^+_\varepsilon:=\Big\{\xi_u\in L^\infty(\mathbb{R}_+,E),\ \ \exists\xi_{u_{N_k}}(t) \text{ which solve (1.2)} \\ \text{such that } \xi_u(0)=E^{\mathrm{w}}-\lim_{k\to\infty}\xi_{u_{N_k}}(0) \text{ and } \xi_u=\Theta^+-\lim_{k\to\infty}\xi_{u_{N_k}}\Big\}. \tag{1.26}$$

Obviously, K_ε^+ is a subset of $L^\infty(\mathbb{R}_+, E)$. We endow the trajectory phase space K_ε^+ with the topology induced by the embedding

$$K_\varepsilon^+ \subset \Theta^+ \tag{1.27}$$

(i.e. by the weak-$*$ topology of the space $L^\infty_{loc}(\mathbb{R}_+, E)$).

We now consider the following semigroup of positive temporal translations:

$$T_h : \Theta^+ \to \Theta^+,\ h \geq 0,\quad (T_h u)(t) := u(t+h). \tag{1.28}$$

Then, due to (1.23) and the fact that (0.1) is autonomous, semigroup (1.28) acts on the trajectory phase space K_ε^+:

$$T_h : K_\varepsilon^+ \to K_\varepsilon^+. \tag{1.29}$$

Semigroup (1.29) (acting on the topological space K_ε^+) is called the trajectory dynamical system associated with equation (0.1).

Remark 1.1. It is well known (see e.g. [3]) that, in the subcritical case $p \leq 1$, the solution $u(t)$ of equation (0.1) is unique and, consequently, this equation generates a semigroup in the classical phase space E in a standard way:

$$S_t^\varepsilon : E \to E,\quad t \geq 0,\quad S_t^\varepsilon \xi_u(0) := \xi_u(t). \tag{1.30}$$

Moreover, in this case, the map

$$\Pi_0 : K_\varepsilon^+ \to E,\quad \Pi_0 \xi_u = \xi_u(0) \tag{1.31}$$

is one to one and realizes a (sequential) homeomorphism between K_ε^+ and E^{w}. Thus,

$$S_t^\varepsilon = \Pi_0 \circ T_t \circ (\Pi_0)^{-1}, \tag{1.32}$$

and, therefore (in the subcritical case), the trajectory dynamical system (1.29) is conjugated to the classical dynamical system (1.30) defined on the phase space E endowed with the weak topology.

We note however that, in the supercritical case $p > 1$, the uniqueness problem for (0.1) is not solved yet and classical semigroup (1.30) can be defined as a semigroup of multivalued maps only (see [2] for the details). The use of the trajectory dynamical system (1.29) allows to avoid the multivalued maps and to apply the standard attractors theory in order to study the long time behavior of solutions of (0.1) in the supercritical case.

In order to construct the global attractor for dynamical system (1.29), we need the following generalization of energy functional (1.4).

Definition 1.2. Let $\xi_u \in K_\varepsilon^+$. We define the functional $M_u(t)$, $t \geq 0$, by the following expression:

$$M_u^\varepsilon(t) := \inf\Big\{ \operatorname{liminf}_{k\to\infty} \|\xi_{u_{N_k}}(t)\|_{E,\varepsilon}\ :\ \xi_u = \Theta^+ - \lim_{k\to\infty} \xi_{u_{N_k}},\ \xi_u(0) = E^{\mathrm{w}} - \lim_{k\to\infty} \xi_{u_{N_k}}(0)\Big\}, \tag{1.33}$$

where the external infimum in the right-hand side of (1.33) is taken over all sequences of the Galerkin approximations $\{\xi_{u_{N_k}}(t)\}_{k=1}^\infty$ which weakly-$*$ converge to the given solution ξ_u.

The following corollary gives the simplest properties of the M-energy functional introduced.

Corollary 1.1. *Let the assumptions of Lemma 1.1 hold and let $\xi_u \in K_\varepsilon^+$. Then, the following estimates hold:*

$$M_u^\varepsilon(t) < \infty, \quad \|\xi_u(t)\|_{E,\varepsilon} \le M_u^\varepsilon(t), \quad M_{T_h u}^\varepsilon(t) \le M_u^\varepsilon(t+h) \tag{1.34}$$

and

$$M_u^\varepsilon(t)^2 + \int_t^\infty \|\partial_t u(t)\|_{L^2}^2\,dt \le CM_u^\varepsilon(s)^2 e^{-\alpha(t-s)} + C(1+\|g\|_{L^2}^2), \tag{1.35}$$

where $t \ge s \ge 0$ and constants $\alpha > 0$ and $C > 0$ are the same as in (1.3).

Indeed, estimates (1.34) are immediate corollaries of the definition of K_ε^+ and $M_u^\varepsilon(t)$ and estimates (1.35) follow from estimate (1.3) in which we pass to the limit $N_k \to \infty$.

Remark 1.2. It is known (see [3]) that, in the subcritical case $p \le 1$, we have

$$\|\xi_u(t)\|_{E,\varepsilon} = M_u^\varepsilon(t). \tag{1.36}$$

So, in this case, the M-energy coincides with the classical one. But to the best of our knowledge, neither identity (1.36) nor the fact that any solution $\xi_u \in L^\infty(\mathbb{R}_+, E)$ of (0.1) can be obtained as a limit of the Galerkin approximations (1.2) are known in the supercritical case $p > 1$. Nevertheless, if the solution $\xi_u(t)$ of problem (0.1) is sufficiently regular:

$$\xi_u \in L^\infty(\mathbb{R}_+, E^1), \quad E^1 := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega), \tag{1.37}$$

then it is unique (in class (1.37)) and, consequently, $\xi_u \in K_\varepsilon^+$ and satisfies (1.36). In the sequel, we consider only the solutions of (0.1) which can be approximated by the Galerkin solutions and use the modified energy $M_u^\varepsilon(t)$ instead of the classical one.

Remark 1.3. In contrast to (1.4), the functional $M_u^\varepsilon(t)$ is not a priori local with respect to t, i.e. $M_u^\varepsilon(T)$ depends not only on $\xi_u(T)$, but also on the whole trajectory $\xi_u \in K_\varepsilon^+$.

As usual (see e.g. [2], [3], [21]), in order to define the global attractor of semigroup (1.29), we should define the class of bounded sets which will be attracted by this attractor.

Definition 1.3. A set $B \subset K_\varepsilon^+$ is called M-bounded if the following quantity is finite:

$$\|B\|_M := \sup_{\xi_u \in B} M_u^\varepsilon(0) < \infty. \tag{1.38}$$

In other words, the set $B \subset K_\varepsilon^+$ is M-bounded if the modified energy of all the solutions belonging to B is uniformly bounded.

We are now ready to recall the definition of the global attractor of the trajectory dynamical system (1.29) (=trajectory attractor of equation (0.1)).

Definition 1.4. A set $\mathcal{A}_\varepsilon^{tr}$ is a global attractor of the trajectory dynamical system (1.29) (= the trajectory attractor of equation (0.1)) if the following conditions hold:

1. The set $\mathcal{A}_\varepsilon^{tr}$ is a compact M-bounded set in K_ε^+.
2. This set is strictly invariant, i.e. $T_h\mathcal{A}_\varepsilon^{tr} = \mathcal{A}_\varepsilon^{tr}$, for $h \ge 0$.
3. This set is an attracting set for semigroup (1.29), i.e. for every M-bounded subset $B \subset K_\varepsilon^+$ and every neighborhood $\mathcal{O}(\mathcal{A}_\varepsilon^{tr})$ of $\mathcal{A}_\varepsilon^{tr}$ in K_ε^+, there exists $T = T(B, \mathcal{O})$ such that

$$T_h B \subset \mathcal{O}(\mathcal{A}_\varepsilon^{tr}), \quad \text{for} \quad h \ge T. \tag{1.39}$$

The main result of this section is the following theorem which establishes the existence of the attractor $\mathcal{A}_\varepsilon^{tr}$ for the trajectory dynamical system associated with problem (0.1).

Theorem 1.1. *Let the assumptions of Lemma 1.1 hold. Then, semigroup* (1.29) *possesses the global attractor $\mathcal{A}_\varepsilon^{tr}$ in the sense of Definition 1.4 which can be described in the following way:*

$$\mathcal{A}_\varepsilon^{tr} = \Pi_+\mathcal{K}_\varepsilon. \tag{1.40}$$

Here $\mathcal{K}_\varepsilon \subset L^\infty(\mathbb{R}, E)$ is the set of all the complete solutions of problem (0.1) *which are defined for all $t \in \mathbb{R}$ and can be obtained as a Galerkin limit, i.e. $\xi_u \in \mathcal{K}_\varepsilon$ if and only if there exist a sequence of times $t_k \to -\infty$ and a sequence of solutions $\xi_{u_{N_k}}(t)$ of the problems:*

$$\begin{cases} \varepsilon\partial_t^2 u_{N_k} + \gamma\partial_t u_{N_k} - \Delta_x u_{N_k} + P_{N_k} f(u_{N_k}) = g_{N_k}, \\ \xi_{u_{N_k}}(t_k) = \xi_k^0 \in E_{N_k}, \quad t \geq t_k, \end{cases} \tag{1.41}$$

such that

$$\|\xi_k^0\|_{E,\varepsilon} \leq C, \quad \textit{and} \quad \xi_u = \Theta - \lim_{k\to\infty} \xi_{u_{N_k}}, \tag{1.42}$$

where C is independent of k and

$$\Theta := \left[L^\infty_{loc}(\mathbb{R}, E)\right]^{\mathrm{w}^*}. \tag{1.43}$$

Proof. As usual (see e.g. [3], [23]), in order to prove the attractor's existence, it is sufficient to find a M-bounded and compact absorbing set in the phase space of the semigroup under consideration. We claim that the following set

$$\mathcal{B}_\varepsilon := \{\xi_u \in K_\varepsilon^+, \ \sup_{t\geq 0} M_u^\varepsilon(t)^2 \leq 2C(1 + \|g\|^2_{L^2})\}, \tag{1.44}$$

where the constant C is the same as in (1.35), is the desired compact absorbing set of the trajectory dynamical system (1.29).

Indeed, the fact that $\mathcal{B}_\varepsilon$ absorbs all M-bounded subsets of K_ε^+ is an immediate corollary of estimates (1.34) and (1.35). Moreover, it follows from (1.34) that

$$T_h\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon,$$

and that $\mathcal{B}_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+, E)$ and, consequently, it is precompact in Θ^+. So, there only remains to verify that the set $\mathcal{B}_\varepsilon$ is closed. In order to verify this, we first note that

$$\mathcal{B}_\varepsilon \subset B_0 := \{\xi_u \in L^\infty(\mathbb{R}_+, E), \ \sup_{t\geq 0} \|\xi_u(t)\|^2_{E,\varepsilon} \leq 4C(1 + \|g\|^2_{L^2})\}, \tag{1.45}$$

and that B_0 is a compact and metrizable subspace of the topological space Θ^+ (see [20]). So, there only remains to verify the sequential closedness of $\mathcal{B}_\varepsilon$.

Let $\{\xi_{u^l}\}_{l=1}^\infty \in \mathcal{B}_\varepsilon$ and set

$$\xi_u = \Theta^+ - \lim_{l\to\infty} \xi_{u^l}. \tag{1.46}$$

We need to prove that the limit function $\xi_u(t)$ belongs to $\mathcal{B}_\varepsilon$. Indeed, due to the compactness of embedding (1.20), we derive from (1.46) the strong convergence of ξ_{u_l} in space (1.21) and, consequently, the limit function $\xi_u(t)$ solves equation (0.1). Let us prove that $\xi_u \in K_\varepsilon^+$, i.e. that it can be represented as a Galerkin limit

(1.18). According to the definition of K_ε^+ and the assumption $\xi_{u^l}\in K_\varepsilon^+$, there exist sequences $\{\xi_{u^l_{N_k}}(t)\}_{k=1}^\infty$ of Galerkin solutions such that

$$\xi_{u^l}=\Theta^+-\lim_{k\to\infty}\xi_{u^l_{N_k}}. \tag{1.47}$$

Moreover, since $\xi_{u^l}\in\mathcal{B}_\varepsilon$ then the $M^\varepsilon_{u^l}(t)$ are uniformly bounded with respect to l and, consequently, without loss of generality, we may assume that

$$\xi_{u^l_{N_k}}\in B_0,\quad k,l\in\mathbb{N}. \tag{1.48}$$

We now recall that the topology of Θ^+ restricted to B_0 is metrizable. Let $d_{\Theta^+}(\cdot,\cdot)$ be one of such metrics. Then, due to (1.47), for every l, we may found $N_{k(l)}\in\mathbb{N}$ such that

$$d_{\Theta^+}(\xi_{u_l},\xi_{u^l_{N_{k(l)}}})\le l^{-1}. \tag{1.49}$$

Convergence (1.46), together with (1.49) and with the triangle inequality, imply that

$$d_{\Theta^+}(\xi_u,\xi_{u_{N^l_{k(l)}}})\to 0 \text{ and, consequently, } \xi_u=\Theta^+-\lim_{l\to\infty}\xi_{u^l_{N_{k(l)}}}.$$

Thus, $\xi_u\in K_\varepsilon^+$. Moreover, arguing analogously, we can verify that

$$M_u^\varepsilon(t)\le \operatorname{liminf}_{l\to\infty}M^\varepsilon_{u^l}(t),\quad \forall t\ge 0, \tag{1.50}$$

and, consequently, $\xi_u\in\mathcal{B}_\varepsilon$. Thus, $\mathcal{B}_\varepsilon$ is indeed a compact semiinvariant absorbing set for the trajectory dynamical system (1.29). The desired attractor can now be found in a standard way as the ω-limit set of $\mathcal{B}_\varepsilon$:

$$\mathcal{A}_\varepsilon^{tr}=\omega(\mathcal{B}_\varepsilon):=\bigcap_{h\ge 0}T_h\mathcal{B}_\varepsilon \tag{1.51}$$

(see e.g. [3], [23]). Description (1.40) is also a standard corollary of the explicit formula (1.51) for the attractor and of the diagonal procedure described above in the proof of closedness of $\mathcal{B}_\varepsilon$. Theorem 1.1 is proven.

To conclude this section, we formulate several useful corollaries of the Theorem 1.1.

Corollary 1.2. *Let the assumptions of Theorem 1.1 hold and let $B\subset K_\varepsilon^+$ be an arbitrary M-bounded subset. Then, for every $T\in\mathbb{R}_+$ and every $1\ge\beta>0$, the following convergence holds:*

$$\lim_{h\to\infty}\operatorname{dist}_{\mathbb{L}_\beta((h,T+h))}\left(B\big|_{(h,T+h)},\mathcal{A}_\varepsilon^{tr}\big|_{(h,T+h)}\right)=0, \tag{1.52}$$

where

$$\mathbb{L}_\beta(h,T+h):=C((h,T+h),[H^{1-\beta}(\Omega)\cap L^{p+3-\beta}(\Omega)]\times H^{-\beta}(\Omega)), \tag{1.53}$$

and $dist_{\mathbb{L}}(U,V)$ denotes the nonsymmetric Hausdorff distance between sets in $\mathbb{L}$:

$$\operatorname{dist}_{\mathbb{L}}(U,V)=\sup_{u\in U}\inf_{v\in V}\|u-v\|_{\mathbb{L}}. \tag{1.54}$$

Proof. Indeed, due to embedding (1.20) and the fact that every $\xi_u\in K_\varepsilon^+$ satisfies equation (0.1) (from which we can express and estimate the second derivative $\partial_t^2u(t)$), we have the compact embedding

$$(K_\varepsilon^+,\Theta^+)\subset\subset C_{loc}(\mathbb{R}_+,[H^{1-\beta}(\Omega)\cap L^{p+3-\beta}(\Omega)]\times H^{-\beta}(\Omega)), \tag{1.55}$$

in the sense that every M-bounded subset of K_ε^+ is a precompact set in the space in the right-hand side of (1.55). Convergence (1.52) is an immediate corollary of (1.55) and (1.39) and Corollary 1.2 is proven.

Corollary 1.3. *Let the assumptions of Theorem 1.1 hold and let* $\xi_u \in \mathcal{K}_\varepsilon$*. Then,*

$$\int_{-\infty}^{+\infty} \|\partial_t u(s)\|^2_{L^2}\, ds \le C(1+\|g\|^2_{L^2}), \tag{1.56}$$

where the constant C *is the same as* (1.35)*, and moreover, for every* $1 \ge \beta > 0$,

$$\partial_t u \in C_b(\mathbb{R}, H^{-\beta}(\Omega)) \quad and \quad \lim_{t\to\pm\infty} \|\partial_t u(t)\|_{H^{-\beta}(\Omega)} = 0. \tag{1.57}$$

Proof. Indeed, let $\xi_u \in \mathcal{K}_\varepsilon$ and let $\xi_{u_{N_k}}(t)$, $t \ge t_k$, $t_k \to -\infty$, be the sequence of Galerkin solutions of (1.41) which approximates this trajectory in the sense of (1.42). Then, applying estimate (1.3) to the solutions of (1.41), with $t = t_k/2$ and $s = t_k$, we obtain

$$\int_{t_k/2}^{\infty} \|\partial_t u_{N_k}(s)\|^2_{L^2}\, ds \le C\|\xi^0_k\|^2_{E,\varepsilon} e^{\alpha t_k/2} + C(1+\|g\|^2_{L^2}). \tag{1.58}$$

Passing now to the limit $k \to \infty$ in (1.58) and taking into account the fact that $\alpha > 0$, $t_k \to -\infty$ and that the ξ^0_k are uniformly bounded, we derive the dissipative integral (1.56). In order to obtain convergence (1.57), we first note that, due to the compactness of embedding (1.20), we derive, analogously to (1.55), that, for every $0 < \beta \le 1$,

$$\mathcal{K}_\varepsilon \text{ is bounded in } \quad C_b(\mathbb{R}, [H^{1-\beta}(\Omega) \cap L^{p+3-\beta}(\Omega)] \times H^{-\beta}(\Omega)), \tag{1.59}$$

and, moreover, it is compact in the local topology

$$\mathcal{K}_\varepsilon \subset\subset C_{loc}(\mathbb{R}, [H^{1-\beta}(\Omega) \cap L^{p+3-\beta}(\Omega)] \times H^{-\beta}(\Omega)). \tag{1.60}$$

Convergence (1.57) is a standard corollary of dissipative integral (1.56) and of embedding (1.60) and Corollary 1.3 is proven.

2. The backward regularity of the solutions belonging to the attractor. In this section, we show that every bounded weak solution $\xi_u \in \mathcal{K}_\varepsilon$ of equation (2.1) becomes regular if t is less than the critical value $t \le T_u$. We emphasize that we derive this result without the assumption that ε is small.

The main result of the section is the following theorem.

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold. Then, for every complete solution* $\xi_u \in \mathcal{K}_\varepsilon$ *of equation* (0.1)*, there exists a time* $T = T_u$ *such that*

$$\xi_u \in C_b((-\infty, T], E^1). \tag{2.1}$$

Proof. Let $\xi_u := (u, \partial_t u) \in \mathcal{K}_\varepsilon$ be an arbitrary complete solution of (0.1). Let us rewrite problem (0.1) as follows:

$$\varepsilon\partial_t^2 u + \gamma\partial_t u - \Delta_x u + f(u) + L(-\Delta_x)^{-1}u = h(t) := g + L(-\Delta_x)^{-1}u(t), \tag{2.2}$$

where Δ_x is the Laplacian with Dirichlet boundary conditions and the (large) parameter L will be specified below. It follows from Theorem 1.1 and Lemma 1.1 that

$$\|h(T)\|^2_{L^2} + \int_T^{T+1} \|\partial_t h(t)\|^2_{H^2}\, dt \le C'(1+\|g\|^2_{L^2}), \quad C' = C'(L), \tag{2.3}$$

where the constant C' is independent of ε and T. Moreover, it is important for our method that, according to Corollary 1.3

$$\partial_t h \in C_b(\mathbb{R}, H^{2-\beta}(\Omega)) \quad \text{and} \quad \lim_{t\to-\infty} \|\partial_t h(t)\|_{H^{2-\beta}(\Omega)} = 0, \tag{2.4}$$

for every $0 < \beta \le 1$.

Our strategy is the following: we first show that (2.4) allows to construct a regular backward solution $\xi_v(t)$, $t \le T_u$, for problem (2.2) and then prove the identity $u(t) \equiv v(t)$.

Lemma 2.1. *For a sufficiently large L, there exists time $T = T(\varepsilon, u, L)$ such that the problem*

$$\varepsilon\partial_t^2 v + \gamma\partial_t v - \Delta_x v + f(v) + L(-\Delta_x)^{-1}v = h(t), \quad t \le T \tag{2.5}$$

possesses a unique regular bounded backward solution $\xi_v(t) \in E^1$ which satisfies the following estimate:

$$\|\partial_t v(t)\|_{H^2} + \|v(t)\|_{H^2} \le Q(\|g\|_{L^2}), \quad t \le T, \tag{2.6}$$

where the monotonic function Q depends on L, but is independent of $\varepsilon \le \varepsilon_0$. Moreover, the derivative $\partial_t v(t)$ tends to 0 in the L^∞-norm as $t \to -\infty$:

$$\lim_{t\to-\infty} \|\partial_t v(t)\|_{L^\infty} = 0. \tag{2.7}$$

In order to prove this lemma, we first construct a solution $w(t)$ of the parabolic problem

$$\gamma\partial_t w - \Delta_x w + f(w) + L(-\Delta_x)^{-1}w = h(t), \quad t \in \mathbb{R}. \tag{2.8}$$

Lemma 2.2. *For sufficiently large L, problem (2.8) possesses a unique solution $w(t)$, $t \in \mathbb{R}$, in the class $C_b(\mathbb{R}, H^2(\Omega))$ and the following estimate is valid:*

$$\|w(t)\|_{H^2}^2 \le C_L(1 + \|g\|_{L^2}^2), \tag{2.9}$$

where the constant C_L depends on L, but is independent of ε. Moreover,

$$\partial_t w \in C_b(\mathbb{R}, H^2(\Omega)), \quad \partial_t^2 w \in L^2([T, T+1], H^1(\Omega)), \tag{2.10}$$

for every $T \in \mathbb{R}$, and the following convergence is valid:

$$\lim_{T\to-\infty} \left\{\|\partial_t w(T)\|_{H^2} + \|\partial_t^2 w\|_{L^2([T,T+1],H^1(\Omega))}\right\} = 0. \tag{2.11}$$

Proof of Lemma 2.2. The existence of a solution for problem (2.8) which is bounded in H^2 can be easily derived from estimate (2.3) and from the dissipativity assumption (0.3)(3) using standard parabolic technique (see e.g. [3], [17], [27]). So, there remains to verify (2.9) – (2.11). Differentiating equation (2.8) with respect to t and setting $\theta = \partial_t w$, we have

$$\gamma\partial_t\theta - \Delta_x\theta + f'(w)\theta + L(-\Delta_x)^{-1}\theta = \partial_t h(t), \quad t \in \mathbb{R}. \tag{2.12}$$

Multiplying equation (2.12) by $\theta(t)$ and integrating over $x \in \Omega$, we find

$$\gamma\partial_t\|\theta(t)\|^2 + \|\theta(t)\|_{H^1}^2 + 2L\|\theta(t)\|_{H^{-1}}^2 \le -2(f'(w)\theta(t), \theta(t)) + C\|\partial_t h(t)\|_{L^2}^2. \tag{2.13}$$

We now recall that, due to (0.3) and an appropriate interpolation inequality

$$2(f'(w)\theta, \theta) \le 2K\|\theta\|_{L^2}^2 \le 2CK\|\theta\|_{H^1}\|\theta\|_{H^{-1}} \le \\ \le \frac{1}{2}(\|\theta\|_{H^1}^2 + 4C^2K^2\|\theta\|_{H^{-1}}^2) \le \frac{1}{2}(\|\theta\|_{H^1}^2 + L\|\theta\|_{H^{-1}}^2), \tag{2.14}$$

if $L > 4C^2K^2$. Then, estimate (2.13) reads

$$\gamma\frac{d}{dt}\|\theta(t)\|^2_{L^2} + \frac{1}{2}\|\theta(t)\|^2_{H^1} \le C\|\partial_t h(t)\|^2_{L^2}. \tag{2.15}$$

Therefore, Gronwall's inequality applied to (2.15) gives

$$\|\theta(T)\|^2_{L^2} + \int_T^{T+1} \|\theta(t)\|^2_{H^1}\,dt \le C(1+\|g\|^2_{L^2}). \tag{2.16}$$

Moreover, due to convergence (2.4), we have

$$\lim_{t\to-\infty}\|\theta\|_{L^2((t,t+1),H^1(\Omega))} = 0. \tag{2.17}$$

After obtaining estimate (2.16) for the derivative $\partial_t w(t)$, we may interpret problem (2.8) as an elliptic boundary value problem

$$\Delta_x w(T) - f(w(T)) + L(-\Delta_x)^{-1}w(T) = -h(T) + \gamma\partial_t w(T). \tag{2.18}$$

Multiplying then (2.18) by $\Delta_x w(T)$, integrating over $x\in\Omega$, using estimates (2.3), (2.16) and (0.3)(3) and arguing in a standard way, we derive estimate (2.9). In order to derive (2.10) and (2.11), we note that the function $\theta(t)$ satisfies the heat equation

$$\gamma\partial_t\theta - \Delta_x\theta = h_\theta(t) := \partial_t h(t) - f'(w(t))\theta(t) - L(-\Delta_x)^{-1}\theta(t), \tag{2.19}$$

and, according to (2.3) and (2.17)

$$\lim_{t\to-\infty}\|h_\theta\|_{L^2(t,t+1),H^1(\Omega))} = 0. \tag{2.20}$$

Applying now the standard regularity theorem to the heat equation (2.19), we have

$$\int_T^{T+1}\|\partial_t\theta(t)\|^2_{H^1}\,dt + \|\theta(T)\|^2_{H^2} \le C\int_{-\infty}^{T+1} e^{-\alpha(T-t)}\|h_\theta(t)\|^2_{H^1}\,dt. \tag{2.21}$$

Embedding (2.10) and convergence (2.11) are immediate corollaries of (2.21), (2.4) and (2.20). Lemma 2.2 is proven.

Proof of Lemma 2.1. Let us seek the desired regular solution of problem (2.5) in the form $v(t) = w(t) + W(t)$. Then, the function $W(t)$ solves

$$\varepsilon\partial_t^2 W + \gamma\partial_t W - \Delta_x W + [f(w(t)+W) - f(w(t))] + \\ + L(-\Delta_x)^{-1}W = H(t) := -\varepsilon\partial_t^2 w(t). \tag{2.22}$$

We apply the implicit function theorem in order to solve equation (2.22) in the space

$$\Phi_T := C_b((-\infty,T],E^1), \tag{2.23}$$

where the time T is small enough. Indeed, according to Lemma 2.2, we have $H \in L^2([t,t+1],H^1_0(\Omega))$, for every $t\in\mathbb{R}$, and

$$\lim_{T\to-\infty}\|H\|_{L^2((T,T+1),H^1_0(\Omega))} = 0. \tag{2.24}$$

So, there only remains to verify that the variation equation at $W=0$

$$\varepsilon\partial_t^2 V+\gamma\partial_t V-\Delta_x V+f'(w(t))V+L(-\Delta_x)^{-1}V=G(t),\quad t\le T \tag{2.25}$$

is uniquely solvable for every $G\in L^2_{loc}((-\infty,T],H^1_0(\Omega))$ such that

$$\|G\|_{L^2_b((-\infty,T),H^1_0(\Omega))}:=\sup_{t\in(-\infty,T-1]}\|G\|_{L^2((t,t+1),H^1_0(\Omega))}<\infty, \tag{2.26}$$

if the time T is small enough. Let us verify this fact. Indeed, multiplying equation (2.25) by $\partial_t V(t)+\alpha V(t)$ and integrating over $x\in\Omega$, we have

$$\begin{aligned}\frac{d}{dt}[\varepsilon\|\partial_t V\|^2_{L^2}+\|\nabla_x V\|^2_{L^2}+L\|V\|^2_{H^{-1}}+(f'(w(t))V,V)+2\alpha\varepsilon(V,\partial_t V)+\\ +\alpha\gamma\|V\|^2_{L^2}]+2(\gamma-\alpha\varepsilon)\|\partial_t V\|^2_{L^2}+2\alpha\|\nabla_x V\|^2_{L^2}+\\ +2\alpha L\|V\|^2_{H^{-1}}+2\alpha(f'(w(t))V,V)=2(G,\partial_t V+\alpha V)+(f''(w(t))\partial_t w(t),|V|^2).\end{aligned} \tag{2.27}$$

We denote the expression $[\cdots]$ by $E_V(t)$ and assume that L is large enough so that

$$(f'(w(t))V,V)+\frac{1}{2}(\|V\|^2_{H^1}+L\|V\|^2_{H^{-1}})\ge 0 \tag{2.28}$$

(see (2.14)). Then, analogously to (1.12) and (1.13), there exists a sufficiently small, but independent of ε and L, parameter $\alpha>0$ such that

$$C_1^{-1}(\varepsilon\|\partial_t V(t)\|^2_{L^2}+\|V(t)\|^2_{H^1})\le E_V(t)\le C_1(\varepsilon\|\partial_t V(t)\|^2_{L^2}+\|V(t)\|^2_{H^1}) \tag{2.29}$$

(here we have implicitly used the fact that $\|w(t)\|_{H^2}$ is uniformly bounded with respect to ε) and

$$\frac{d}{dt}E_V(t)+\alpha E_V(t)\le h(t):=C\|G(t)\|^2_{L^2}+(f''(w(t))\partial_t w(t),|V|^2)-\frac{\alpha}{2}\|V\|^2_{H^1}, \tag{2.30}$$

where the constants C and C_1 depend on L, but are independent of ε. Convergence (2.11), together with embedding $H^2(\Omega)\subset C(\overline{\Omega})$ and estimate (2.9), imply that

$$h(t)\le C\|G(t)\|^2_{L^2}, \tag{2.31}$$

if $t\le T$ and T is small enough. Applying Gronwall's inequality to (2.30) and using (2.31) and (2.29), we obtain

$$\varepsilon\|\partial_t V(t)\|^2_{L^2}+\|V(t)\|^2_{H^1}\le C\int_{-\infty}^t e^{-\alpha(t-s)}\|G(t)\|^2_{L^2}\,ds,\quad t\le T, \tag{2.32}$$

where C depends on L, but is independent of ε. Thus, the solution of (2.25) is unique. Moreover, multiplying now equation (2.25) by $-\Delta_x(\partial_t V+\alpha V)$, interpreting the term $f'(w)V$ as an external force, and using estimate (2.32), we have, analogously

$$\varepsilon\|\partial_t V(t)\|^2_{H^1}+\|V(t)\|^2_{H^2}\le C_1\int_{-\infty}^t e^{-\alpha(t-s)}\|G(t)\|^2_{H^1}\,ds,\quad t\le T, \tag{2.33}$$

where C_1 is also independent of ε. Estimate (2.33) implies that

$$\varepsilon\|\partial_t V(t)\|^2_{H^1} + \|V(t)\|^2_{H^2} \le C_2\|G\|^2_{L^2_b((-\infty,T],H^1_0(\Omega))}, \quad t \le T, \tag{2.34}$$

and, consequently, the variation equation (2.25) is indeed uniquely solvable in space (2.23) if T is small enough. Thus, applying the implicit function theorem to equation (2.22), we derive that, for a sufficiently small $T \in \mathbb{R}$, there exists a solution $\xi_W \in \Phi_T$ of problem (2.22). Moreover, since the constant C_1 in (2.33) is independent of ε then,

$$\varepsilon\|\partial_t W(t)\|^2_{H^1} + \|W(t)\|^2_{H^2} \le Q_L(\|g\|_{L^2}), \quad t \le T, \tag{2.35}$$

where the function Q_L depends on L, but is independent of ε and

$$\lim_{t\to-\infty} \|\partial_t W(t)\|_{H^1} = 0 \tag{2.36}$$

(due to convergence (2.24)). Returning to the function $v(t)$ and taking into account the estimates for $w(t)$ obtained in Lemma 2.2, we finally have

$$\varepsilon\|\partial_t v(t)\|^2_{H^1} + \|v(t)\|^2_{H^2} \le Q(\|g\|_{L^2}), \quad t \le T = T(L,\varepsilon,u), \tag{2.37}$$

where the function Q is independent of ε and

$$\lim_{t\to-\infty} \|\partial_t v(t)\|_{H^1} = 0. \tag{2.38}$$

Thus, there only remains to estimate the H^2-norm of $\partial_t v(t)$. To this end, we differentiate equation (2.5) by t and set $\phi(t) = \partial_t v(t)$. Then, we have

$$\varepsilon\partial_t^2\phi + \gamma\partial_t\phi - \Delta_x\phi + L(-\Delta_x)\phi = H_\phi(t) := \partial_t h(t) - f'(v(t))\partial_t v(t). \tag{2.39}$$

It follows from (2.4) and (2.38) that

$$\lim_{t\to-\infty} \|H_\phi(t)\|_{H^1} = 0. \tag{2.40}$$

Equation (2.39) has the form of (2.25) with $f = 0$ and, analogously to (2.33), we derive

$$\varepsilon\|\partial_t\phi(t)\|^2_{H^1} + \|\phi(t)\|^2_{H^2} \le C_1\int_{-\infty}^t e^{-\alpha(t-s)}\|H_\phi(t)\|^2_{H^1}\,ds, \quad t \le T. \tag{2.41}$$

Since $H^2(\Omega) \subset C(\overline{\Omega})$, then (2.40) and (2.41) imply convergence (2.7) and estimate for $\partial_t v(t)$ in (2.6) and Lemma 2.1 is proven.

We are now ready to complete the proof of Theorem 2.1. In order to do so, we need to prove that $u(t) \equiv v(t)$, for $t \le T$. Indeed, let $\xi_{u_{N_k}}(t)$, $t \ge t_k$, be a sequence of Galerkin solutions, which approximates the function $\xi_u \in \mathcal{K}_\varepsilon$. We recall that, due to Theorem 1.1

$$t_k \to -\infty, \quad \xi_u = \Theta - \lim_{k\to\infty} \xi_{u_{N_k}}, \tag{2.42}$$

and the sequence $\xi_{u_{N_k}}(t_k) = \xi^0_k$ is uniformly bounded with respect to k. We also consider a sequence of functions

$$v_{N_k}(t) := P_{N_k} v(t), \quad t \le T, \tag{2.43}$$

where the function $v(t)$ is constructed in Lemma 2.1. According to Lemma 2.1, solution $\xi_v(t)$ is bounded in E^1 as $t\le T$ and, consequently

$$\lim_{k\to\infty}\|\xi_{v_{N_k}}-\xi_v\|_{C_b((-\infty,t],E)}=0 \ \text{ and } \ \lim_{k\to\infty}\|v_{N_k}-v\|_{C_b((-\infty,T]\times\Omega)}=0. \tag{2.44}$$

Moreover, since $\partial_t v(t)$ is also bounded in H^2, then

$$\lim_{k\to\infty}\|\partial_t v_{N_k}-\partial_t v\|_{C_b((-\infty,T]\times\Omega)}=0 \tag{2.45}$$

(these convergences are standard corollaries of the embedding $H^2(\Omega)\subset\subset C(\overline{\Omega})$ and of the fact that the convergence of Fourier series is uniform on compact sets).

We now set $U(t):=u(t)-v(t)$ and $U_{N_k}(t):=u_{N_k}(t)-v_{N_k}(t)$. Then, the last function satisfies the equation

$$\begin{aligned}\varepsilon\partial_t^2U_{N_k}+\gamma\partial_tU_{N_k}-\Delta_xU_{N_k}+&\\ +P_{N_k}(f(v_{N_k}(t)+U_{N_k})-f(v_{N_k}(t)))+L(-\Delta_x)^{-1}U_{N_k}&=\\ =h_{N_k}(t):=P_{N_k}(f(v(t))-f(v_{N_k}(t))),\quad \xi_{U_{N_k}}(t_k):=\xi_k^0-P_{N_k}\xi_v(t_k).&\end{aligned} \tag{2.46}$$

Moreover, due to convergences (2.44), we have

$$\lim_{k\to\infty}\|h_{N_k}\|_{C_b((-\infty,T]\times\Omega)}=0 \quad\text{and}\quad \|\xi_{U_{N_k}}(t_k)\|_{E,\varepsilon}\le C, \tag{2.47}$$

where C is independent of k. Multiplying now equation (2.46) by $\partial_tU_{N_k}(t)+\alpha U_{N_k}(t)$ and setting

$$\begin{aligned}E_{U_{N_k}}(t):=\varepsilon\|\partial_tU_{N_k}\|^2_{L^2}+\|\nabla_xU_{N_k}\|^2_{L^2}+L\|U_{N_k}\|^2_{H^{-1}}+2\alpha\varepsilon(U_{N_k},\partial_tU_{N_k})+&\\ +\alpha\gamma\|U_{N_k}\|^2_{L^2}+2\left(F(v_{N_k}(t)+U_{N_k})-F(v_{N_k}(t))-f(v_{N_k}(t))U_{N_k},1\right),&\end{aligned} \tag{2.48}$$

we derive the identity

$$\frac{d}{dt}E_{U_{N_k}}(t)+\alpha E_{U_{N_k}}(t)=H_{U_{N_k}}(t),\quad t\le T, \tag{2.49}$$

with

$$\begin{aligned}H_{U_{N_k}}(t):=-(2\gamma-3\alpha\varepsilon)\|\partial_tU_{N_k}\|^2_{L^2}-\alpha\|\nabla_xU_{N_k}\|^2_{L^2}-\alpha L\|U_{N_k}\|^2_{H^{-1}}+&\\ +2\alpha\bigg(F(v_{N_k}(t)+U_{N_k})-F(v_{N_k}(t))-f(v_{N_k}(t))U_{N_k}-&\\ -(f(v_{N_k}(t)+U_{N_k})-f(v_{N_k}(t))U_{N_k},1\bigg)+2\alpha^2\varepsilon(U_{N_k},\partial_tU_{N_k})+\alpha^2\gamma\|U_{N_k}\|^2_{L^2}+&\\ +2(h_{N_k}(t),\partial_tU_{n_k}+\alpha U_{N_k})+&\\ +2\bigg(f(v_{N_k}(t)+U_{N_k})-f(v_{N_k}(t))-f'(v_{N_k}(t))U_{N_k},\partial_tv_{N_k}(t)\bigg).&\end{aligned} \tag{2.50}$$

In order to estimate function (2.50), we need the following proposition.

Proposition 2.1. *Let the function f satisfy assumptions* (0.3). *Then,*

$$F(v+w)-F(v)-f(v)w\ge -K|w|^2+\delta_p|w|^2(|v|^{p+1}+|w|^{p+1}),\quad \forall v,w\in\mathbb{R},\tag{2.51}$$

where the constant K is the same as in (0.3) *and δ_p is some positive constant depending only on p. Moreover,*

$$\Phi_v(w):=F(v+w)-F(v)-f(v)w-(f(v+w)-f(v))w\le \\ \le\frac{K}{2}|w|^2-\delta_p'|w|^2(|v|^{p+1}+|w|^{p+1}),\tag{2.52}$$

where δ_p' is positive and depends only on p. And, finally

$$|(f(v+w)-f(v)-f'(v)w|\le C|w|^2(1+|v|^p+|w|^p),\tag{2.53}$$

where the constant C is independent of v and w.

Proof of Proposition 2.1. Estimate (2.53) is an immediate corollary of assumption (0.3)(2). Let us now verify (2.52) using the assumption (0.3)(3). Indeed,

$$\Phi_v(w)=\int_0^1\partial_w\Phi_v(sw)w\,ds=-|w|^2\int_0^1 sf'(v+sw)\,ds\le \\ \le\frac{K}{2}|w|^2-\delta\beta_p(|v|,|w|)|w|^2,\tag{2.54}$$

where

$$\beta_p(x,y):=\int_0^1 s|x-sy|^{p+1}\,ds\ge\left(\int_0^1 s|x-sy|\,ds\right)^{p+1}.$$

The integral in the right-hand side can be computed explicitly:

$$\int_0^1 s|x-sy|\,ds=\begin{cases}\frac{x}{2}-\frac{y}{3} & \text{if } x\ge y,\\ \frac{y}{6}(2(x/y)^3-3(x/y)+2) & \text{if } y>x,\end{cases}\ge\frac{2-\sqrt{2}}{12}(x+y)\tag{2.54$'$}$$

(we recall that $x,y\ge 0$). Estimate (2.52) is an immediate corollary of (2.54) and (2.54$'$). Let us now verify (2.51). Indeed, using assumption (0.3)(3), we have

$$F(v+w)-F(v)-f(v)w=w\int_0^1[f(v+s_1w)-f(v)]\,ds_1= \\ =|w|^2\int_0^1\int_0^1 f'(v+s_1s_2w)\,ds_1\,ds_2\ge -K|w|^2+\delta\tilde\beta_p(|v|,|w|),\tag{2.54$''$}$$

where

$$\tilde\beta_p(x,y):=\int_0^1\int_0^1|x-s_1s_2y|^{p+1}\,ds_1\,ds_2\ge\left(\tilde\beta_1(x,y)\right)^{p+1}\ge \\ \ge\left(\int_0^1\beta_1(x,s_1y)\,ds_1\right)^{p+1}\ge\delta_p'\left(\int_0^1(x+sy)\,ds\right)^{p+1}\ge\delta_p''(x^{p+1}+y^{p+1}).$$

Inserting this estimate into estimate (2.54$''$), we derive (2.51) and finish the proof of Proposition 2.1.

It now follows from estimates (2.52), (2.53) and our choice of L (see (2.14)) that there exist positive constants α_1, C and C_1 (which are independent of U_{N_k}, v_{N_k}, k, L and ε) such that

$$H_{U_{N_k}}(t) \le -\frac{\alpha_1}{2}\|U_{N_k}(t)\|^2_{H^1} - 2\alpha_1(|U_{N_k}(t)|^{p+3},1)- \\ -\frac{\alpha_1 L}{2}\|U_{N_k}(t)\|^2_{H^{-1}} + C\|h_{N_k}(t)\|^2_{L^2}+ \\ + C_1\|\partial_t v_{N_k}(t)\|_{L^\infty}\left(|U_{N_k}(t)|^2(1+|v_{N_k}(t)|^p+|U_{N_k}(t)|^p),1\right). \quad (2.55)$$

According to Lemma 2.1, the derivative $\partial_t v(t)$ tends to zero in $L^\infty(\Omega)$ as $t\to-\infty$ (see (2.7)) and the L^∞-norm of $v(t)$ remains bounded as $t\to-\infty$. Consequently, due to convergences (2.44) and (2.45), there exists time $T'\le T$ such that, for a sufficiently large k, we have

$$H_{u_{N_k}}(t) \le C\|h_{N_k}(t)\|^2_{L^2}, \quad t\le T'. \quad (2.56)$$

Applying now Gronwall's inequality to relation (2.49), we obtain

$$E_{U_{N_k}}(t) \le E_{U_{N_k}}(t_k)e^{-\alpha(t-t_k)} + C\int_{t_k}^t e^{-\alpha(t-s)}\|h_{N_k}(s)\|^2_{L^2}\,ds, \quad (2.57)$$

where $t\le T'$, and the constants C and $\alpha>0$ are independent of k. Using estimate (2.51) and our choice of exponent L (see (2.14)), we derive from (2.57) that

$$\varepsilon\|\partial_t U_{N_k}(t)\|^2_{L^2} + \|U_{N_k}(t)\|^2_{H^1} \le \\ \le C_2\left(1+\|\xi_k^0\|^2_{E,\varepsilon}+\|\xi_{v_{N_k}}(t_k)\|^2_{E,\varepsilon}\right)e^{-\alpha(t-t_k)} + 2C\int_{t_k}^t e^{-\alpha(t-t_k)}\|h_{N_k}(s)\|^2_{L^2}\,ds, \quad (2.58)$$

where the constant C_2 is also independent of k. Passing to the limit $k\to\infty$ in (2.58) and using (2.47), the fact that ξ_k^0 is uniformly bounded in E (due to Theorem 1.1) and $\xi_{v_{N_k}}(t_k)$ is also uniformly bounded in E (due to Lemma 2.1 and convergence (2.44)), we finally derive the estimate

$$\varepsilon\|\partial_t U(t)\|^2_{L^2} + \|U(t)\|^2_{H^1} \le 0, \quad \text{for} \quad t\le T'$$

and, consequently, $u(t)=v(t)$ for $t\le T'$. Theorem 2.1 is proven.

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold and let $\xi_u\in\mathcal{K}_\varepsilon$ be a bounded complete weak solution of problem* (0.1). *Then*

$$\|u(t)\|^2_{H^2} + \|\partial_t u(t)\|^2_{H^2} \le Q(\|g\|_{L^2}), \quad t\le T_u, \quad (2.59)$$

where the monotonic function Q is independent of ε.

Indeed, estimate (2.59) is an immediate corollary of (2.6) and the fact that $u(t)=v(t)$, for $t\le T_u$.

To conclude the section, we prove that the solution $\xi_u(t)\in\mathcal{K}_\varepsilon$ is unique until it is regular.

Theorem 2.2. *Let the assumptions of Lemma 1.1 hold and $\xi_v \in \mathcal{K}_\varepsilon$ be a complete weak solution of* (0.1) *which satisfies* (2.59)*, for $t \le T$. We also assume that $\xi_u \in \mathcal{K}_\varepsilon$ is another complete weak solution which satisfies*

$$\xi_u(t) = \xi_v(t), \quad \textit{for all} \;\; t \le T' < T. \tag{2.60}$$

Then, necessarily

$$\xi_u(t) = \xi_v(t), \quad \textit{for all} \;\; t \le T. \tag{2.61}$$

Proof. The proof of this theorem is very similar to the end of the proof of the previous theorem. Indeed, let $\xi_{u_{N_k}}(t)$ be a sequence of Galerkin solutions which approximates the initial solution $\xi_u(t)$, see Theorem 1.1. Let also

$$\xi_{v_{N_k}}(t) := P_{N_k}\xi_v(t), \;\; U(t) := u(t) - v(t), \;\; U_{N_k}(t) := u_{N_k}(t) - v_{N_k}(t). \tag{2.62}$$

Then, analogously to (2.46), function $U_{N_k}(t)$ solves the equation

$$\begin{aligned}\varepsilon\partial_t^2 U_{N_k} + \gamma\partial_t U_{N_k} - \Delta_x U_{N_k} + \\ + P_{N_k}(f(v_{N_k}(t) + U_{N_k}) - f(v_{N_k}(t))) + L(-\Delta_x)^{-1}U_{N_k} = \\ = h_{N_k}(t) := P_{N_k}(f(v(t)) - f(v_{N_k}(t))) + L(-\Delta_x)^{-1}(u_{N_k}(t) - v_{N_k}(t)), \\ \xi_{U_{N_k}}(t_k) := \xi_k^0 - P_{N_k}\xi_v(t_k),\end{aligned} \tag{2.63}$$

where, in contrast to (2.46), external forces $h_{N_k}(t)$ contain the additional term $L(-\Delta_x)^{-1}(u_{N_k}(t) - v_{N_k}(t))$ and, consequently, instead of (2.47), we have the convergence

$$\begin{aligned}h_{N_k} \to L(-\Delta_x)^{-1}(u - v) \quad \text{strongly in } C_{loc}((-\infty, T], L^2(\Omega)) \\ \text{and} \quad \|h_{N_k}\|_{C_b((-\infty,T],L^2(\Omega))} \le C_1,\end{aligned} \tag{2.64}$$

where the constant C_1 is independent of k (here we have implicitly used embedding (1.20) in order to prove the convergence $u_{N_k} \to u$ in $C_{loc}((-\infty, T], L^2(\Omega))$).

Since equation (2.63) has the form (2.46), then, multiplying it by $\partial_t U_{N_k}(t) + \alpha U_{N_k}(t)$, integrating over $x \in \Omega$ and arguing as in the derivation of (2.49), we obtain the estimate

$$\frac{d}{dt}E_{U_{N_k}}(t) + \alpha E_{U_{N_k}}(t) = H_{U_{N_k}}(t), \quad t \le T, \tag{2.65}$$

where the functions $E_{U_{N_k}}(t)$ and $H_{U_{N_k}}(t)$ are defined by (2.48) and (2.50) respectively. Moreover, analogously to (2.55), there exist positive constants α_1, C and C_1 (which are independent of L) such that, for all $t \le T$

$$\begin{aligned}H_{U_{N_k}}(t) \le -\frac{\alpha_1}{2}\|U_{N_k}(t)\|_{H^1}^2 - 2\alpha_1(|U_{N_k}(t)|^{p+3}, 1) - \\ - \frac{\alpha_1 L}{2}\|U_{N_k}(t)\|_{H^{-1}}^2 + C\|h_{N_k}(t)\|_{L^2}^2 + \\ + C_1\|\partial_t v_{N_k}(t)\|_{L^\infty}\left(|U_{N_k}(t)|^2(1 + |v_{N_k}(t)|^p + |U_{N_k}(t)|^p), 1\right),\end{aligned} \tag{2.66}$$

We note that, in contrast to the case of (2.55), the function $v(t)$ is now independent of the parameter L. That is the reason why it is possible to fix a large L (depending on $Q(\|g\|_{L^2})$ in the right-hand side of (2.66)) such that

$$H_{U_{N_k}}(t) \le C\|h_{U_{N_k}}(t)\|_{L^2}^2, \quad \text{for all } t \le T, \tag{2.67}$$

without decreasing the time interval $t \in (-\infty, T]$ (in contrast to (2.55)). Applying now Gronwall's inequality to identity (2.65) and using (2.67), we derive estimate (2.57), for every $t \le T$. Passing to the limit $k \to \infty$ in estimate (2.57) and using the convergence (2.64) and the fact that $u(t) = v(t)$ for $t \le T'$, we obtain the estimate

$$\varepsilon\|\partial_t u(t) - \partial_t v(t)\|^2_{L^2} + \|u(t) - v(t)\|^2_{H^1} \le \\ \le 2CL^2 \int_{T'}^{t} e^{-\alpha(t-s)}\|(-\Delta_x)^{-1}(u(s) - v(s))\|^2_{L^2}\, ds, \quad (2.68)$$

which is valid for every $t \in [T', T]$. Applying again Gronwall's inequality to relation (2.68) and noting that $u(T') = v(T')$, we finally derive that $v(t) = u(t)$, for all $t \in (-\infty, T]$ and prove Theorem 2.2.

Remark 2.1. Theorems 2.1 and 2.2 show that the only way for a singular weak solution to appear on the attractor $\mathcal{A}^{tr}_\varepsilon$ is by a blow up of the corresponding strong solution belonging to the attractor. In the next section, we will show that this scenario is also impossible if ε is small enough and we thus verify that the attractor $\mathcal{A}^{tr}_\varepsilon$ consists of global strong solutions which satisfy (2.59), for every $t \in \mathbb{R}$.

3. The global existence of strong solutions. In this section, we prove the existence of a global strong solution of problem (0.1) if $\varepsilon > 0$ is small enough and the E^1-energy of the initial data is not very large. Combining this result with the results of the previous section, we prove that the attractor $\mathcal{A}^{tr}_\varepsilon$ consists of strong global solutions if $\varepsilon > 0$ is small enough. The main result of the section is the following theorem.

Theorem 3.1. *Let the assumptions of Lemma 1.1 hold. Then, there exist a small positive ε_0 and a nonincreasing function*

$$R : (0, \varepsilon_0] \to \mathbb{R}_+, \quad \lim_{\varepsilon \to 0} R(\varepsilon) = \infty, \quad (3.1)$$

such that, for every $\varepsilon \le \varepsilon_0$ and every initial data $\xi_u(0) \in E^1$ satisfying

$$\|\xi_u(0)\|_{E^1,\varepsilon} := \left(\varepsilon\|\partial_t u(0)\|^2_{H^1} + \|u(0)\|^2_{H^2}\right)^{1/2} \le R(\varepsilon), \quad (3.2)$$

there exists a unique global strong solution $\xi_u \in L^\infty(\mathbb{R}_+, E^1)$ of problem (0.1) and the following estimate is valid:

$$\|\xi_u(t)\|^2_{E^1,\varepsilon} + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^1}\, ds \le Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t} + Q(\|g\|_{L^2}), \quad (3.3)$$

where the positive constant α and the monotonic function Q are independent of ε.

Proof. We divide the proof of the theorem in two steps. In the first step, we prove that the solution $\xi_u(t)$ of the hyperbolic equation (0.1) is close to the appropriate regular solution of the limit parabolic equation and, in the second step, we deduce from this fact that the strong solution of (0.1) also exists globally if $\varepsilon > 0$ is small enough.

We first note that, due to the embedding $H^2(\Omega) \subset C(\overline{\Omega})$, the strong solution $\xi_u(t) \in E^1$ exists locally (for $t \le T(\xi_u(0))$) and is unique on the existence interval. That is the reason why it is enough to derive a priori estimate (3.3) under the assumption that the strong solution $\xi_u(t)$ exists. We also note that, since the solution $\xi_u(t)$ is assumed to be regular, then we may multiply equation (0.1) by $\partial_t u(t) + \alpha u(t)$ (without using the Galerkin approximations) and derive, arguing as in the proof of Lemma 1.1, that

$$\|\xi_u(t)\|^2_{E,\varepsilon} + \int_t^{t+1} \|\partial_t u(t)\|^2_{L^2}\, ds \le C\|\xi_u(0)\|^2_{E,\varepsilon} e^{-\alpha t} + C(1 + \|g\|^2_{L^2}), \tag{3.4}$$

where the positive constants C and α are independent of ε (as mentioned in Remark 1.2 for the strong solutions, we have equality (1.36) and, consequently, (3.4) can be considered as a corollary of (1.35)).

Moreover, as in the proof of Theorem 2.1, it is convenient to modify the initial equation (0.1) as follows:

$$\begin{gathered}\varepsilon\partial_t^2 u + \gamma\partial_t u - \Delta_x u + f(u) + L(-\Delta_x)^{-1}u = h_u(t) := g + L(-\Delta_x)^{-1}u(t), \\ t \ge 0, \quad \xi_u(0) = \xi^0 := (u_0, u_0'), \quad u\big|_{\partial\Omega} = 0,\end{gathered} \tag{3.5}$$

where the constant L satisfies (2.14). Then, due to (3.4), the external forces $h_u(t)$ satisfy

$$\|h_u(t)\|^2_{L^2} + \int_t^{t+1} \|\partial_t h_u(s)\|^2_{L^2}\, ds \le C\left(1 + \|g\|^2_{L^2} + \|\xi_u(0)\|^2_{E,\varepsilon} e^{-\alpha t}\right), \tag{3.6}$$

where C and α are independent of ε.

We now consider the limit parabolic equation which corresponds to (3.5) as $\varepsilon = 0$

$$\gamma\partial_t v - \Delta_x v + f(v) + L(-\Delta_x)^{-1}v = h_u(t), \quad t \ge 0, \quad v\big|_{t=0} = u_0, \quad v\big|_{\partial\Omega} = 0. \tag{3.7}$$

Equation (3.7) is of the form (2.8). Consequently, using estimate (3.6) and arguing as in the proof of Lemma 2.2, we derive that $v(t) \in H^2(\Omega)$ and

$$\|v(t)\|^2_{H^2} + \|\partial_t v(t)\|^2_{L^2} \le Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t} + C(1 + \|g\|^2_{L^2}), \tag{3.8}$$

where the monotonic function Q and the constants C and α are independent of ε.

The following Lemma shows that the solution $u(t)$ is indeed close to $v(t)$ if ε is small enough.

Lemma 3.1. *Let the assumptions of Theorem 3.1 hold and let $\xi_u(t)$ and $v(t)$ be strong solutions of* (0.1) *and* (3.8) *respectively. Then, the following estimate is valid:*

$$\|u(t) - v(t)\|^2_{L^2} \le \varepsilon\left(Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t} + C(1 + \|g\|^2_{L^2})\right), \tag{3.9}$$

where a monotonic function Q and positive constants C and α are independent of ε.

Proof of Lemma 3.1. We set $w(t) := u(t) - v(t)$. Then, this function satisfies the relation

$$\gamma\partial_t w - \Delta_x w + [f(v(t)+w) - f(v(t))] + L(-\Delta_x)^{-1}w = -\varepsilon\partial_t^2 u(t), \quad w\big|_{t=0} = 0. \tag{3.10}$$

Multiplying equation (3.10) by $w(t)$, integrating over $x \in \Omega$ and using (0.3)(3) and (2.14), we have

$$\begin{aligned}\frac{d}{dt}[\gamma\|w(t)\|^2_{L^2} + 2\varepsilon(\partial_t u(t), w(t))] + \alpha[\gamma\|w(t)\|^2_{L^2} + 2\varepsilon(\partial_t u(t), w(t))] \le \\ \le H(t) := C\varepsilon\left(|\partial_t w(t)| + |w(t)|, |\partial_t u(t)|\right),\end{aligned} \tag{3.11}$$

for some $\alpha > 0$ and $C > 0$ which are independent of ε. Moreover, according to (3.4) and (3.8), we have the estimate

$$\int_t^{t+1} H(s)\,ds \le \varepsilon\left(Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t} + C(1+\|g\|_{L^2}^2)\right), \tag{3.12}$$

where Q, C and α are independent of ε. Applying Gronwall's inequality to (3.11) and taking into account (3.12) and the fact that $w(0) = 0$, we obtain

$$\gamma\|w(t)\|_{L^2}^2 + 2\varepsilon(\partial_t u(t), w(t)) \le \varepsilon\left(Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t} + C(1+\|g\|_{L^2}^2)\right), \tag{3.13}$$

where Q, C and α are independent of ε. There remains to note that

$$-2\varepsilon(\partial_t u(t), w(t)) \le \frac{\gamma}{2}\|w(t)\|_{L^2}^2 + 2\varepsilon\gamma^{-1}\left(\varepsilon\|\partial_t u(t)\|_{L^2}^2\right). \tag{3.14}$$

Indeed, estimate (3.9) is an immediate corollary of (3.13), (3.14) and (3.4) and Lemma 3.1 is proven.

We are now ready to complete the proof of the theorem. To this end, we interpret equation (0.1) as a linear one

$$\varepsilon\partial_t^2 u + \gamma\partial_t u - \Delta_x u = g - f(u(t)), \quad \xi_u(0) = \xi^0. \tag{3.15}$$

Multiplying equation (3.15) by $-\Delta_x(\partial_t u(t) + \alpha u(t))$, integrating over $x \in \Omega$, and arguing in a standard way, we derive the estimate

$$\frac{d}{dt}E_u^1(t) + \alpha\|\nabla_x\partial_t u(t)\|_{L^2}^2 + \alpha E_u^1(t) \le C\left(\|f(u(t))\|_{H^1}^2 + \|g\|_{L^2}^2\right), \tag{3.16}$$

where

$$E_u^1(t) := \varepsilon\|\nabla_x\partial_t u(t)\|_{L^2}^2 + \|\Delta_x u(t)\|_{L^2}^2 + \\ + 2\alpha\varepsilon(\nabla_x u(t), \nabla_x\partial_t u(t)) + 2(g, \Delta_x u(t)) + 2\|g\|_{L^2}^2, \tag{3.17}$$

and the constants $\alpha > 0$ and $C > 0$ are independent of ε. We also note that (3.17) implies the estimates

$$C_1^{-1}\|\xi_u(t)\|_{E^1,\varepsilon}^2 \le E_u^1(t) \le C_1(\|\xi_u(t)\|_{E^1,\varepsilon}^2 + \|g\|_{L^2}^2). \tag{3.18}$$

So, there only remains to estimate $\|f(u(t))\|_{H^1}$. To this end, we use the following trick:

$$\|f(u(t))\|_{H^1}^2 \le \|f(u(t)) - f(v(t))\|_{H^1}^2 + \|f(v(t))\|_{H^1}^2, \tag{3.19}$$

where $v(t)$ is the solution of limit parabolic problem (3.7) constructed in Lemma 3.1. Then, on the one hand, due to estimate (3.8) and embedding $H^2(\Omega) \subset C(\overline{\Omega})$, we have

$$\|f(v(t))\|_{H^1}^2 \le Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t} + Q(\|g\|_{L^2}^2), \tag{3.20}$$

for an appropriate monotonic function Q and positive constant $\alpha > 0$ which are independent of ε and, on the other hand, using assumption (0.3)(2) and embedding $H^2(\Omega) \subset C(\overline{\Omega})$, we obtain

$$\|f(u(t)) - f(v(t))\|_{H^1}^2 \le C\|u(t) - v(t)\|_{H^1}^2\left(1 + \|v\|_{H^2}^{2(p+1)} + \|u\|_{H^2}^{2(p+1)}\right). \tag{3.21}$$

Using the interpolation inequality

$$\|u(t)-v(t)\|_{H^1}^2 \le C\|u(t)-v(t)\|_{L^2}\|u(t)-v(t)\|_{H^2}, \tag{3.22}$$

estimate (3.8) for $v(t)$, estimate (3.9) for the L^2-norm of $u(t)-v(t)$ and estimate (3.18) for the H^2-norm of $u(t)$, we finally obtain

$$\begin{aligned} &C\left(\|f(u(t))\|_{H^1}^2+\|g\|_{L^2}^2\right) \le \\ &\le \varepsilon^{1/2}(Q(\|\xi_u(0)\|_{E^1,\varepsilon})+Q(\|g\|_{L^2}))[E_u^1(t)]^{p+3/2}+Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t}+Q(\|g\|_{L^2}), \end{aligned} \tag{3.23}$$

for an appropriate monotonic function Q and a positive constant α which are independent of ε. Thus, inserting (3.23) to the right-hand side of (3.16), we derive the differential inequality for $E_u^1(t)$:

$$\begin{aligned} &\frac{d}{dt}E_u^1(t)+\alpha E_u^1(t) \le \\ &\varepsilon^{1/2}(Q(\|\xi_u(0)\|_{E^1,\varepsilon})+Q(\|g\|_{L^2}))[E_u^1(t)]^{p+3/2}+Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t}+Q(\|g\|_{L^2}). \end{aligned} \tag{3.24}$$

In order to derive the assertion of the theorem from inequality (3.24), we need the following proposition.

Proposition 3.1. *Let the function $y(t)\ge 0$ satisfy the inequality:*

$$y'(t)+\alpha y(t)-\varepsilon^{1/2}(A+B)[y(t)]^m-Ae^{-\alpha t}-B\le 0,\quad t\ge 0, \tag{3.25}$$

with $0<\alpha\le 1$, $A,B>0$, $m\ge 1$ and $y(0)\le A+B$. We also assume that

$$\varepsilon^{1/2}\left(\frac{3}{\alpha}\right)^{m-1}(A+B)^m\le 1. \tag{3.26}$$

Then, this function satisfies the following inequality

$$y(t)\le y_0(t):=\frac{3}{\alpha}\left(Ae^{-\alpha t/2}+B\right),\quad t\ge 0. \tag{3.27}$$

Indeed, assumption (3.26) guarantees that

$$y_0'(t)+\alpha y_0(t)-\varepsilon^{1/2}(A+B)[y_0(t)]^m-Ae^{-\alpha t}-B\ge 0,\quad t\ge 0$$

and, consequently, applying the comparison principle to (3.25), we derive estimate (3.27).

Applying now Proposition 3.1 with

$$y(t):=E_u^1(t),\quad A:=Q(\|\xi_u(0)\|_{E^1,\varepsilon}),\quad B:=Q(\|g\|_{L^2}),\quad m:=p+3/2,$$

to inequality (3.24), we have

$$E_u^1(t)\le\frac{3}{\alpha}\left(Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t/2}+Q(\|g\|_{L^2})\right), \tag{3.28}$$

and the desired function $R(\varepsilon)$ can be found as a solution of the equation

$$\varepsilon^{1/2}\left(\frac{3}{\alpha}\right)^{p+1/2}(Q(R(\varepsilon))+Q(\|g\|_{L^2}))^{p+3/2}=1.$$

The desired estimate for the integral of $\partial_t u(t)$ follows from (3.16), (3.23) and (3.28) and Theorem 3.1 is proven.

Remark 3.1. We have constructed the global strong solution of equation (0.1) under assumption (3.2) only. Moreover, since the solutions of (3.24) may blow up in finite time, then our method gives no information on the strong solutions of (0.1) whose initial E^1-energy is larger than $R(\varepsilon)$.

We now consider the R-ball in E^1

$$B^\varepsilon(R,E^1):=\{\xi^0\in E^1\ :\ \|\xi^0\|_{E^1,\varepsilon}\le R\}.$$

Then, due to Theorem 3.1, the solving operator

$$S_t^\varepsilon: B^\varepsilon(R,E^1)\to E^1,\quad S_t^\varepsilon\xi_u(0):=\xi_u(t),\quad R\le R(\varepsilon), \tag{3.29}$$

where $\xi_u(t)$ is a unique strong solution of (0.1), is well defined. Moreover, due to estimate (3.3)

$$\|S_t^\varepsilon\left(B^\varepsilon(R,E^1)\right)\|_{E^1,\varepsilon}\le\hat{Q}(R),\quad t\in\mathbb{R}_+, \tag{3.30}$$

for an appropriate monotonic function $\hat{Q}$. We now set

$$\mathbb{B}_R^\varepsilon:=\left[\bigcup_{t\in\mathbb{R}_+}S_t^\varepsilon\left(B^\varepsilon(R,E^1)\right)\right]_{E^1}, \tag{3.31}$$

where $[\cdot]_{E^1}$ denotes the closure in the space E^1. Then, according to (3.30)

$$\|\mathbb{B}_R^\varepsilon\|_{E^1,\varepsilon}\le\hat{Q}(R) \tag{3.32}$$

and, consequently, we have proven the following corollary.

Corollary 3.1. *Let the assumptions of Theorem 3.1 hold. Then, there exist a small positive $\varepsilon_0'\le\varepsilon_0$, a nonincreasing function*

$$R_0:(0,\varepsilon_0']\to\mathbb{R}_+,\quad \lim_{\varepsilon\to 0}R_0(\varepsilon)=\infty \tag{3.33}$$

and a bounded closed subset $\mathbb{B}_{R_0(\varepsilon)}^\varepsilon$ in E^1 which satisfies

$$B^\varepsilon(R_0(\varepsilon),E^1)\subset\mathbb{B}_{R_0(\varepsilon)}^\varepsilon\subset B^\varepsilon(\hat{Q}(R_0(\varepsilon)),E^1) \tag{3.34}$$

such that, for all $\varepsilon\le\varepsilon_0'$, (3.29) *defines a dissipative semigroup in the phase space $\mathbb{B}_{R_0(\varepsilon)}^\varepsilon$:*

$$S_t^\varepsilon:\mathbb{B}_{R_0(\varepsilon)}^\varepsilon\to\mathbb{B}_{R_0(\varepsilon)}^\varepsilon,\quad S_{t+s}^\varepsilon=S_t^\varepsilon\circ S_s^\varepsilon,\quad t,s\ge 0. \tag{3.35}$$

Indeed, the desired function $R_0(\varepsilon)$ can be found from the equation

$$\hat{Q}(R_0(\varepsilon))=R(\varepsilon),\quad \varepsilon\le\varepsilon_0'\le\varepsilon_0,$$

where $R(\varepsilon)$ is the same as in Theorem 3.1.

The following corollary establishes the existence of a global attractor $\mathcal{A}_\varepsilon^{gl}\subset E^1$ for semigroup (3.35).

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. Then, for* $\varepsilon \le \varepsilon_0$*, semigroup* (3.35) *possesses the compact (in* E^1*) global attractor* $\mathcal{A}^{gl}_\varepsilon$*:*

$$S^\varepsilon_t \mathcal{A}^{gl}_\varepsilon = \mathcal{A}^{gl}_\varepsilon \quad \textit{and} \quad \lim_{t\to\infty} \operatorname{dist}_{E^1}(S^\varepsilon_t \mathbb{B}^\varepsilon_{R_0(\varepsilon)}, \mathcal{A}^{gl}_\varepsilon) = 0, \tag{3.36}$$

which satisfies

$$\mathcal{A}^{gl}_\varepsilon = \Pi_0 \hat{\mathcal{K}}_\varepsilon, \tag{3.37}$$

where $\hat{\mathcal{K}}_\varepsilon \subset C_b(\mathbb{R}, E^1)$ *is the set of all the global strong solutions of* (0.1) *belonging to* $\mathbb{B}^\varepsilon_{R_0(\varepsilon)}$*:*

$$\hat{\mathcal{K}}_\varepsilon := \{\xi_u \in C_b(\mathbb{R}, E^1),\ u(t) \textit{ solves } (0.1) \textit{ and } \|\xi_u(t)\|_{E^1,\varepsilon} \le R_0(\varepsilon),\ \ t \in \mathbb{R}\}. \tag{3.38}$$

Moreover, every $\xi_u \in \hat{\mathcal{K}}_\varepsilon$ *satisfies the estimate*

$$\|\partial_t u(t)\|^2_{H^2} + \|u(t)\|^2_{H^2} \le Q(\|g\|_{L^2}), \quad t \in \mathbb{R}, \tag{3.39}$$

where the monotonic function Q *is independent of* ε*.*

Proof. The existence of the global attractor for S^ε_t in E^1 has been proved in [3] for the subcritical case $p \le 1$ (see also [11]). We note however that Theorem 3.1, together with embedding $H^2(\Omega) \subset C(\overline{\Omega})$, give the uniform (with respect to ε) estimate for the C-norm of solution $u(t)$ in the supercritical case as well. Thus, the growth rate of the nonlinearity f becomes nonessential, due to this estimate, and, repeating word by word the proof performed in [3], we establish the existence of the global attractor $\mathcal{A}^{gl}_\varepsilon$. Descriptions (3.37) and (3.38) are the standard corollaries of the attractor's existence theorem. So, there remains to verify estimate (3.39).

Indeed, let $\xi_u \in \hat{\mathcal{K}}_\varepsilon$ Then, due to estimate (3.3)

$$\varepsilon\|\partial_t u(t)\|^2_{H^1} + \|u(t)\|^2_{H^2} + \varepsilon^2\|\partial^2_t u(t)\|^2_{L^2} + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^1}\, ds \le Q(\|g\|_{L^2}), \tag{3.40}$$

where the function Q is independent of ε. Differentiating now equation (0.1) with respect to t and setting $\theta(t) := \partial_t u(t)$, we obtain the linear equation

$$\varepsilon\partial^2_t\theta + \gamma\partial_t\theta - \Delta_x\theta = h_\theta(t) := -f'(u(t))\partial_t u(t), \quad t \in \mathbb{R}. \tag{3.41}$$

Moreover, due to (3.40) and embedding $H^2(\Omega) \subset C(\overline{\Omega})$, we have the estimate

$$\int_t^{t+1} \|h_\theta(s)\|^2_{H^1}\, ds \le Q_1(\|g\|_{L^2}), \quad t \in \mathbb{R}, \tag{3.42}$$

for an appropriate monotonic function Q_1 which is independent of ε. Estimate (3.42) implies in a standard way (multiplying (3.41) by $-\Delta_x(\partial_t\theta(t) + \alpha\theta(t))$ and so on, see e.g. [17]) that

$$\xi_\theta \in C_b(\mathbb{R}, E^1) \quad \text{and} \quad \|\xi_\theta(t)\|_{E^1,\varepsilon} \le CQ_1(\|g\|_{L^2}), \tag{3.43}$$

where C is independent of ε. Estimate (3.39) is an immediate corollary of (3.40) and (3.43) and Corollary 3.2 is proven.

We are now ready to verify that the trajectory attractor $\mathcal{A}^{tr}_\varepsilon$ constructed in Section 1 consists of strong solutions.

Theorem 3.2. *Let the assumptions of Lemma 1.1 hold. Then, there exists a small positive constant* ε_0 *such that, for every* $\varepsilon \le \varepsilon_0$*, the sets* $\mathcal{K}_\varepsilon$ *and* $\hat{\mathcal{K}}_\varepsilon$ *defined in Theorem 1.1 and Corollary 3.2 respectively coincide:*

$$\mathcal{K}_\varepsilon = \hat{\mathcal{K}}_\varepsilon. \tag{3.44}$$

Thus

$$\mathcal{K}_\varepsilon \subset C_b(\mathbb{R}, E^1), \quad \mathcal{A}_\varepsilon^{tr} \subset C_b(\mathbb{R}_+, E^1), \tag{3.45}$$

and every $\xi_u \in \mathcal{K}_\varepsilon$ *satisfies* (3.39)*. Moreover, the attractors* $\mathcal{A}_\varepsilon^{gl}$ *and* $\mathcal{A}_\varepsilon^{tr}$ *satisfy the standard relation*

$$\mathcal{A}_\varepsilon^{gl} = \Pi_0 \mathcal{A}_\varepsilon^{tr}. \tag{3.46}$$

Proof. According to Theorem 1.1 and Corollary 3.2, it is sufficient to verify (3.44). Moreover, since $\hat{\mathcal{K}}_\varepsilon$ consists of strong complete bounded solutions which are unique (see e.g. the proof of Theorem 2.2) (and, consequently, can be approximated by Galerkin solutions), then

$$\hat{\mathcal{K}}_\varepsilon \subset \mathcal{K}_\varepsilon. \tag{3.47}$$

So, there remains to verify the inverse embedding. Indeed, let $\xi_u \in \mathcal{K}_\varepsilon$ be an arbitrary complete weak solution of (0.1). Then, due to Theorem 2.1, there exists a time $T = T_u$ such that $\xi_u(t) \in E^1$, for $t \le T$, and

$$\|\xi_u(t)\|_{E^1,\varepsilon} \le Q(\|g\|_{L^2}), \quad t \le T, \tag{3.48}$$

where the function Q is independent of ε. We now assume that the parameter $0 < \varepsilon \le \varepsilon_0$ is small enough so that

$$Q(\|g\|_{L^2}) \le R_0(\varepsilon_0), \tag{3.49}$$

where $R_0(\varepsilon)$ is the same as in Corollary 3.1 (such ε_0 exists due to (3.1) and the fact that Q is independent of ε). Then, due to Theorem 3.1, there exists a unique strong global solution $v(t)$, $t \ge T$, of problem (0.1) with the initial condition

$$\xi_v\big|_{t=T} = \xi_u(T). \tag{3.50}$$

We now define a new solution $\xi_{\tilde v}(t)$, $t \in \mathbb{R}$, of problem (0.1) via

$$\tilde v(t) = \begin{cases} u(t) & \text{if } t \le T, \\ v(t) & \text{if } t \ge T. \end{cases} \tag{3.51}$$

Then, due to estimates (3.3), (3.48) and (3.49), we have $\xi_{\tilde v} \in \hat{\mathcal{K}}_\varepsilon$ and, consequently, due to (3.47), $\xi_{\tilde v} \in \mathcal{K}_\varepsilon$. Applying now Theorem 2.2 to the solutions $u(t)$ and $\tilde v(t)$, we conclude that $u(t) \equiv \tilde v(t)$, for all $t \in \mathbb{R}$, and Theorem 3.2 is proven.

In the sequel, we need also more regular (than $\xi_u \in E^1$) strong solutions of equation (0.1). We note however that we have the regularity $g \in L^2(\Omega)$ only and, therefore, we cannot expect that u be more regular than $u(t) \in H^2(\Omega)$ even for smoother initial conditions. In order to overcome this difficulty, we fix an arbitrary equilibrium $z_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, the function $z_0 = z_0(x)$ solves

$$-\Delta_x z_0 + f(z_0) = g, \quad z_0\big|_{\partial\Omega} = 0. \tag{3.52}$$

Let us introduce the space

$$E_g^2 := (z_0, 0) + [H^3(\Omega) \cap \{u_0\big|_{\partial\Omega} = \Delta_x u_0\big|_{\partial\Omega} = 0\}] \times [H^2(\Omega) \cap H_0^1(\Omega)]. \tag{3.53}$$

It is not difficult to see that E_g^2 is independent of the concrete choice of the equilibrium z_0 and depends only on g. The following corollary gives the global solvability of problem (0.1) in the phase space E_g^2.

Corolary 3.3. *Let the assumptions of Corollary 3.1 hold. Then, for every*

$$\xi_u(0)\in E^2_g\cap\mathbb{B}^\varepsilon_{R_0(\varepsilon)}, \tag{3.54}$$

there exists a unique strong solution $\xi_u(t)\in E^2_g$ of problem (0.1) *and the following estimate is valid:*

$$\|\xi_u(t)\|^2_{E^2_g,\varepsilon}+\int_t^{t+1}\|\partial_t u(s)\|^2_{H^2}\,ds\le Q(\|\xi_u(0)\|_{E^2_g,\varepsilon})e^{-\alpha t}+Q(\|g\|_{L^2}), \tag{3.55}$$

where

$$\|\xi_u(t)\|^2_{E^2_g,\varepsilon}:=\varepsilon\|\partial_t u(t)\|^2_{H^2}+\|u(t)-z_0\|^2_{H^3} \tag{3.56}$$

and $\alpha>0$ and the monotonic function Q are independent of $\varepsilon\le\varepsilon_0$.

Proof. Let $v(t)=u(t)-z_0$. Then, this function satisfies

$$\begin{cases}\varepsilon\partial_t^2 v+\gamma\partial_t v-\Delta_x v=h(t):=f(z_0)-f(u(t)),\\ v\big|_{\partial\Omega}=\Delta_x v\big|_{\partial\Omega}=0,\ \ \partial_t v\big|_{\partial\Omega}=0,\ \ \xi_v\big|_{t=0}=\xi_u(0)-\xi_{z_0}.\end{cases} \tag{3.57}$$

According to estimate (3.3) and the fact that $H^2(\Omega)\subset C(\overline{\Omega})$, we have $h(t)\in H^2(\Omega)\cap H^1_0(\Omega)$ and the following estimate holds:

$$\|h(t)\|_{H^2}\le Q(\|\xi_u(0)\|_{E^1,\varepsilon})e^{-\alpha t}+Q(\|g\|_{L^2}). \tag{3.58}$$

Multiplying equation (3.57) by $\Delta_x^2(\partial_t u+\alpha u)$ and arguing in a standard way (see e.g. [11]), we derive estimate (3.55) and Corollary 3.3 is proven.

Remark 3.2. Theorem 3.1 and Corollary 3.2 establish the uniqueness of the global strong solution $\xi_u(t):=S^\varepsilon_t\xi_u(0)$, for $\xi_u(0)\in\mathbb{B}^\varepsilon_{R_0(\varepsilon)}$, in the class of *strong* solutions $\xi_u\in C_b(\mathbb{R}_+,E^1)$ only.

In fact, we do not know whether or not this solution is unique in the class of *weak* solutions $\xi_u\in K^+_\varepsilon$ even in the case where $\xi_u(0)\in\mathcal{A}^{gl}_\varepsilon$.

Remark 3.3. The proof of Theorem 3.1 is independent of the results of Section 2 and requires only that $\varepsilon\ll 1$ and the global solvability of the limit parabolic equation at $\varepsilon=0$. In particular, this result does not require the finiteness of the dissipation integral (1.56). Consequently, one may extend Theorem 3.1 and Corollary 3.2, for instance, to the case of systems of hyperbolic equations in the form (0.1) with *nongradient* nonlinearities or for a certain class of nonautonomous equations in the form (0.1) for which the dissipation integral is infinite.

In contrast to this, we have essentially used the dissipation integral in order to prove that there are no any bounded singular weak solution $\xi_u(t)$, $t\in\mathbb{R}$, of problem (0.1) (which does not belong to $\hat{\mathcal{K}}_\varepsilon$) if $\varepsilon>0$ is small enough.

Remark 3.4. The limit value ε_0 of the parameter ε, for which Theorem 3.2 is valid, obviously depends on the other parameter $\gamma>0$ of equation (0.1): $\varepsilon_0=\varepsilon_0(\gamma)$.

Rescaling however the time $t\to\gamma t'$ in equation (0.1), we derive again an equation of the form (0.1) with $\gamma':=1$ and $\varepsilon':=\varepsilon/\gamma^2$. Therefore, Theorem 3.2 remains valid if we replace the assumption $\varepsilon\le\varepsilon_0$ by

$$\frac{\varepsilon}{\gamma^2}\le\varepsilon_0(1), \tag{3.59}$$

with small enough $\varepsilon_0(1)$ which is independent of γ. In particular, the trajectory attractor $\mathcal{A}^{tr}$ of equation (0.1) consists of strong solutions if $\varepsilon > 0$ is fixed and $\gamma \gg 0$ is large enough.

4. The regular attractor and the exponential attraction property. In this concluding section, we give a more detailed study of equation (0.1) in the case where all the equilibria of equation (0.1) are hyperbolic. We extend to the supercritical case the results on the regular structure of $\mathcal{A}^{gl}_\varepsilon$ and on the convergence of $\mathcal{A}^{gl}_\varepsilon$ to the attractor $\mathcal{A}^{gl}_0$ of the limit parabolic problem. Since these results are well known in the subcritical case and the rate of growth of the nonlinearity is nonessential if one already has a-priori estimates in C (which are obtained in Theorem 3.1) then, in order to avoid the technicalities, we give below only the rigorous statements of these results. As the main result of the section, we finally establish that all *weak* solutions of (0.1) are attracted exponentially in the strong topology of E to the global attractor $\mathcal{A}^{gl}_\varepsilon$.

We denote by $\mathcal{R} \subset E$ the set of all the equilibria of equation (0.1):

$$\mathcal{R} := \{(z_0, 0) \in E, \quad \Delta_x z_0 - f(z_0) = g\}. \tag{4.1}$$

Then, obviously, $\mathcal{R}$ is independent of ε. Moreover, since z_0 solves an elliptic boundary value problem, then $z_0 \in H^2(\Omega)$ and

$$\|z_0\|^2_{H^2} \le C(1 + \|g\|^2_{L^2}), \tag{4.2}$$

for every $z_0 \in \mathcal{R}$ (see e.g. [24]).

The main additional assumption of this section is the following:

$$\mathcal{R} := \{z_i\}_{i=1}^N \quad \text{and} \quad \sigma(\Delta_x - f'(z_i)) \cap \{\operatorname{Re}\lambda = 0\} = \varnothing. \tag{4.3}$$

Then, as known (see e.g. [24]), the following value is finite:

$$\operatorname{ind}^+(z_i) := \#\{\lambda \in \sigma(\Delta_x - f'(z_i)) \ : \operatorname{Re}\lambda > 0\} < \infty \tag{4.4}$$

and it is called the instability index of the hyperbolic equilibrium $z_i \in \mathcal{R}$.

The following theorem extends to the supercritical case the well-known description of the structure of $\mathcal{A}^{gl}_\varepsilon$ (see e.g. [3]).

Theorem 4.1. *Let the assumptions of Theorem 3.2 hold and let, in addition, assumption* (4.3) *be valid. Then, the attractor $\mathcal{A}^{gl}_\varepsilon$ of semigroup* (3.35) *possesses the following description:*

$$\mathcal{A}^{gl}_\varepsilon = \cup_{i=1}^N \mathcal{M}^+_\varepsilon(z_i), \tag{4.5}$$

where $\mathcal{M}^+_\varepsilon(z_i)$ are the $\operatorname{ind}^+(z_i)$-dimensional C^1-submanifolds of E^1 which consist of all the strong solutions of (0.1) *defined for $t \in \mathbb{R}$ and converging to $(z_i, 0)$ as $t \to -\infty$:*

$$\mathcal{M}^+_\varepsilon(z_i) := \{\xi_u \in \hat{\mathcal{K}}_\varepsilon \ : \ \lim_{t\to-\infty} \|\xi_u(t) - (z_i, 0)\|_{E^1,\varepsilon} = 0\}. \tag{4.6}$$

Moreover, $\mathcal{M}^+_\varepsilon(z_i)$ is C^1-diffeomorphic to $\mathbb{R}^{\operatorname{ind}^+(z_i)}$ and every solution $\xi_u \in \hat{\mathcal{K}}_\varepsilon$ stabilizes to different equilibria as $t \to \pm\infty$:

$$\lim_{t\to\pm\infty} \|\xi_u(t) - (z_\pm, 0)\|_{E^1} = 0, \quad z_\pm \in \mathcal{R}, \quad z_+ \ne z_-. \tag{4.7}$$

Proof. The proof of Theorem 4.1 is given in [3] in the subcritical case $p \le 1$. We note however that Theorem 3.1 gives the uniform (with respect to ε) estimate for the C-norm of trajectories of semigroup (3.35) and this estimate makes the growth rate of the nonlinearity f unessential for further investigation of the attractor $\mathcal{A}^{gl}_\varepsilon$ of semigroup (3.35). Thus, repeating word by word the proof of Proposition 4.1 in the subcritical case (see [3]) and using this estimate, we extend this theorem to the supercritical case. Theorem 4.1 is proven.

The next theorem establishes that E^1-bounded subsets are attracted exponentially to the attractor $\mathcal{A}^{gl}_\varepsilon$.

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold. Then, for every* $B \subset \mathbb{B}^\varepsilon_{R_0(\varepsilon)}$*, the following estimate is valid:*

$$\operatorname{dist}_{E^1,\varepsilon}(S^\varepsilon_t B, \mathcal{A}^{gl}_\varepsilon) \le Q(\|B\|_{E^1,\varepsilon})e^{-\beta t}, \tag{4.8}$$

where the constant $\beta > 0$ *and the monotonic function* Q *are independent of* $\varepsilon \le \varepsilon_0$ *and* B *and* $\operatorname{dist}_{E^1,\varepsilon}$ *denotes the nonsymmetric Hausdorff distance in metric* (3.2).

As in the previous case, the proof of the uniform exponential attraction property is given in [3] for the subcritical case and the supercritical growth rate of the nonlinearity can be easily overcome, due to the uniform estimate on the C-norms of trajectories of (3.35) which is proven in Theorem 3.1.

Let us establish now the convergence of the global attractors $\mathcal{A}^{gl}_\varepsilon$ to the global attractor $\mathcal{A}^{gl}$ of the limit parabolic equation

$$\gamma\partial_t u - \Delta_x u + f(u) = g, \quad u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0. \tag{4.9}$$

We recall (see e.g. [3] or [27]) that equation (4.9) possesses a compact global attractor $\mathcal{A}^{gl}$ in the phase space $H^2(\Omega)\cap H^1_0(\Omega)$ (even without any growth restriction on the nonlinearity f).

As usual (see [3]), in order to compare the attractors $\mathcal{A}^{gl}_\varepsilon$ and $\mathcal{A}^{gl}$, we introduce the extension of $\mathcal{A}^{gl}$ to the phase space E by

$$\mathcal{A}^{gl}_0 := \{(u_0, v_0) \in E \,:\, u_0 \in \mathcal{A}^{gl}, \quad \gamma v_0 - \Delta_x u_0 + f(u_0) = g\}. \tag{4.10}$$

The following standard theorem gives an estimate of symmetric Hausdorff distance between $\mathcal{A}^{gl}_\varepsilon$ and $\mathcal{A}^{gl}_0$ in the space E^1.

Theorem 4.3. *Let the assumptions of Theorem 4.1 hold. Then,* $\mathcal{A}^{gl}_0 \in E^1$ *and the following estimate is valid:*

$$\operatorname{dist}_{symm,H^2(\Omega)\times H^1(\Omega)}\left(\mathcal{A}^{gl}_\varepsilon, \mathcal{A}^{gl}_0\right) \le C\varepsilon^\kappa, \tag{4.11}$$

where $C > 0$ *and* $0 < \kappa \le 1$ *are independent of* ε.

As before, estimate (4.11) is well known for the subcritical case $p \le 1$ (see e.g. [3] or [11]) and the supercritical growth rate of f is now nonessential, due to the uniform C-estimate of solutions proved in Theorem 3.1.

We are now ready to formulate the main result of the section which establishes the analogue of estimate (4.8) for the *weak* solutions $\xi_u \in K^+_\varepsilon$ of equation (0.1).

Theorem 4.4. *Let assumptions* (0.2)–(0.5) *and* (4.3) *hold. Then, there exists a small positive number* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \le \varepsilon_0$ *and every* M*-bounded (in the sense of Definition 1.3) subset* $B \subset K_\varepsilon^+$*, the following estimate is valid:*

$$\sup_{\xi_u \in B} \operatorname{dist}_{E,\varepsilon}\left(\xi_u(t), \mathcal{A}_\varepsilon^{gl}\right) \le Q(\|B\|_M)e^{-\beta t}, \tag{4.12}$$

where the constant $\beta > 0$ *and the function* Q *are independent of* ε *and* B *and* $\operatorname{dist}_{E,\varepsilon}$ *denotes the nonsymmetric Hausdorff distance with respect to norm* (1.4).

Proof. We divide the proof of the theorem into a number of lemmata which are standard for the proof of exponential attraction property of a regular attractor (see [3]). The first one shows that every trajectory $\xi_u \in K_\varepsilon^+$ stays near the equilibria $\mathcal{R}$ most of the time.

Lemma 4.1. *Let the assumptions of Theorem 4.3 hold. Then, for every small* $\delta > 0$ *and every large* $P > 0$*, there exist*

$$\varepsilon_0 = \varepsilon_0(\delta, P) > 0, \quad T = T(\delta, P) > 0, \quad and \quad 0 < \delta_0 = \delta_0(\delta, P) \le \delta \tag{4.13}$$

such that, for every $\varepsilon \le \varepsilon_0$ *and every trajectory* $\xi_u \in K_\varepsilon^+$ *satisfying* $\|\xi_u\|_M \le P$ *the following condition is satisfied:*

$$\left(\cup_{t\in[0,T]} u(t)\right) \cap \mathcal{O}_\delta(\mathcal{R}, L^2(\Omega)) \ne \varnothing, \tag{4.14}$$

where $O_\delta(V, E)$ *is a* δ*-neighborhood of the set* V *in the space* E.

Moreover, if

$$u(0) \in \mathcal{O}_{\delta_0}(z_i, L^2(\Omega)) \quad and \quad u(t_0) \notin \mathcal{O}_\delta(z_i, L^2(\Omega)), \tag{4.15}$$

for some $i \in \{1, \cdots, N\}$ *and* $t_0 > 0$*, then, necessarily*

$$u(t) \notin \mathcal{O}_{\delta_0}(z_i, L^2(\Omega)), \quad \forall t \ge t_0. \tag{4.16}$$

Proof of Lemma 4.1. We adopt the method of [9] to our situation. Indeed, let us assume that (4.14) is wrong. Then, there exist a sequence $\varepsilon_n > 0$, $\varepsilon_n \to 0$, a sequence $T_n \to \infty$ and a sequence $\xi_{u_N} \in K_{\varepsilon_n}^+$ such that

$$\|\xi_{u_n}\|_M \le C \quad \text{and} \quad \left(\cup_{t\in[0,T_n]} u_n(t)\right) \cap \mathcal{O}_{\delta_0}(\mathcal{R}, L^2(\Omega)) = \varnothing, \tag{4.17}$$

for a fixed $\delta_0 > 0$. Then, due to estimate (1.35)

$$\varepsilon_n\|\partial_t u_n(t)\|_{L^2}^2 + \|u_n(t)\|_{H^1}^2 + \|u_n(t)\|_{L^{p+3}}^{p+3} + \int_0^\infty \|\partial_t u_n(s)\|_{L^2}^2\, ds \le C_1, \tag{4.18}$$

where C_1 is independent of n and ε. Thus, without loss of generality, we may assume that

$$\xi_{u_n} \rightharpoonup \xi_u \quad \text{weakly-}* \text{ in } \ L^\infty_{loc}(\mathbb{R}_+, H_0^1(\Omega) \cap L^{p+3}(\Omega)) \times L^2(\mathbb{R}_+ \times \Omega). \tag{4.19}$$

Using now the compactness of the embedding

$$\begin{aligned}\{u \in L^\infty_{loc}(\mathbb{R}_+, H_0^1(\Omega) \cap L^{p+3}(\Omega))\} \cap \{\partial_t u \in L^2_{loc}(\mathbb{R}_+ \times \Omega)\} \subset\subset \\ \subset\subset \{u \in C_{loc}(\mathbb{R}_+, H^{1-\beta}(\Omega) \cap L^{p+3-\beta}(\Omega))\},\end{aligned} \tag{4.20}$$

for every $\beta \in (0,1)$ (see, e.g. [8]), we derive from (4.19) that

$$u_n \to u \quad \text{strongly in } \ C_{loc}(\mathbb{R}_+, H^{1-\beta}(\Omega) \cap L^{p+3-\beta}(\Omega)). \tag{4.21}$$

The strong convergence (4.21) allows to pass to the limit $n \to \infty$ in equations (0.1) for $u_n(t)$ in a standard way and to establish that the limit function $u(t)$ satisfies the limit parabolic equation (4.9) and satisfies the estimate

$$\|u(t)\|^2_{H^1} + \|u(t)\|^{p+3}_{L^{p+3}} + \int_0^\infty \|\partial_t u(s)\|^2_{L^2}\, ds \le C_1. \tag{4.22}$$

Since $u(t)$ solves the limit parabolic equation (4.9) which possesses a global Liapunov function (see e.g. [3]), then we have the convergence to one of the finite number of equilibria:

$$\lim_{t\to\infty} \|u(t) - z_0\|_{L^2} = 0, \quad \text{for some } z_0 \in \mathcal{R}. \tag{4.23}$$

On the other hand, passing to the limit $n \to \infty$ in (4.17), we have

$$u(t) \notin \mathcal{O}_{\delta_0}(\mathcal{R}, L^2(\Omega)), \quad t \in \mathbb{R}_+. \tag{4.24}$$

This contradiction proves (4.14).

Assuming now that (4.16) is wrong and arguing analogously, we obtain a homoclinic connection

$$z_{i_1} \to z_{i_2} \to \cdots \to z_{i_N} = z_{i_1}, \quad z_{i_k} \in \mathcal{R}, \tag{4.25}$$

which consists of solutions of the limit parabolic equation (4.9), i.e. there exist solutions $u_k(t)$, $t \in \mathbb{R}$, of (4.9) such that

$$\lim_{t\to+\infty} \|u_k(t) - z_{i_{k+1}}\|_{L^2} = \lim_{t\to-\infty} \|u_k(t) - z_{i_k}\|_{L^2} = 0 \tag{4.26}$$

(see [9] for the details). There remains to note that (4.25) clearly contradicts the existence of a global Liapunov function for the parabolic equation (4.9) and Lemma 4.1 is proven.

As usual, Lemma 4.1 implies the following result.

Lemma 4.2. *For every $\delta > 0$, there exist $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and $T = T(\delta)$ such that, for every $\varepsilon \le \varepsilon_0$ and every trajectory $\xi_u \in \mathcal{B}_\varepsilon \subset K^+_\varepsilon$ belonging to the absorbing set $\mathcal{B}_\varepsilon$ defined by* (1.44)*, there exist*

$$K = K_u \in \mathbb{N}, \quad K \le \#\mathcal{R} = N,$$

a sequence of different equilibria $z_k \in \mathcal{R}$, $k \le K$, and two sequences of times $T^+_k := T^+_k(u)$, $T^-_k := T^-_k(u)$, for $k \in \{0, \cdots, K\}$, such that

$$T^-_0 = 0, \quad T^-_K = \infty, \quad T^+_k - T^-_{k-1} \le T, \quad k = 1, \cdots, K \tag{4.27}$$

and

$$u(t) \in O_\delta(z_k, L^2(\Omega)) \quad \text{if} \quad t \in [T^+_k, T^-_k], \quad k = 1, \cdots, K. \tag{4.28}$$

Indeed, Lemma 4.2 is a standard corollary of Lemma 4.1 (see e.g. [3] or [9]).

Particularly, Lemma 4.2 shows that the time which the trajectory $\xi_u \in \mathcal{B}_\varepsilon$ spends outside of $\mathcal{O}_\delta(\mathcal{R}, L^2(\Omega))$ is finite and can be estimated from above in a uniform way by $\#\mathcal{R} \cdot T(\delta)$.

As in Section 1, in order to control distance (4.8), we need the following definition.

Definition 4.1. Let $\xi_u \in K_\varepsilon^+$ and let $\xi_v \in E$ be an arbitrary function. Analogously to (1.33), we introduce the modified distance $M_{u,v}^\varepsilon(t)$ by

$$M_{u,v}^\varepsilon(t) := \inf\Big\{ \liminf_{k\to\infty} \|\xi_{u_{N_k}}(t) - P_{N_k}\xi_v\|_{E,\varepsilon}\ : \\ \xi_u = \Theta^+ - \lim_{k\to\infty} \xi_{u_{N_k}},\ \xi_u(0) = E^{\mathrm{w}} - \lim_{k\to\infty} \xi_{u_{N_k}}(0)\Big\}. \quad (4.29)$$

We also define the M-distance to the set $B \subset E$ by

$$M_{u,B}^\varepsilon(t) := \inf_{\xi_v\in B} M_{u,v}^\varepsilon(t). \quad (4.30)$$

The following Lemma gives the analogues of estimates (1.34) for the M-distance.

Lemma 4.3. *Let $\xi_u \in K_\varepsilon^+$ and $\xi_v \in E^1$. Then,*

$$\|\xi_u(t) - \xi_v\|_{E,\varepsilon} \le M_{u,v}^\varepsilon(t) \le \\ \le C\left(M_u^\varepsilon(t) + \|\xi_v\|_{E,\varepsilon}\right),\quad M_{T_h u,v}^\varepsilon(t) \le M_{u,v}^\varepsilon(t+h), \quad (4.31)$$

where C depends only on p. Moreover, if $\xi_w \in E^1$ is another arbitrary function, then

$$M_{u,v+w}^\varepsilon(t) \le C\left(M_{u,v}^\varepsilon(t) + \|\xi_w\|_{E,\varepsilon}\right). \quad (4.32)$$

Indeed, estimates (4.31) and (4.32) are immediate corollaries of definition (4.29) and of the fact that the Fourier series, associated with $\xi_v, \xi_w \in E^1$, converge *strongly* in E.

The next two lemmata allow to control the M-distance of $\xi_u \in K_\varepsilon^+$ outside $\mathcal{O}_\delta(\mathcal{R}, L^2(\Omega))$.

Lemma 4.4. *Let the assumptions of Theorem 3.2 hold. Then, for every $\xi_u \in K_\varepsilon^+$ and every strong solution $\xi_v(t)$, $t \in [T, T+s]$, of equation* (0.1) *satisfying*

$$\|v(t)\|_{H^2}^2 + \int_t^{t+1} \|\partial_t v(s)\|_{H^2}^2\,ds \le P < \infty, \quad (4.33)$$

we have the following inequality:

$$M_{u,v(T+s)}^\varepsilon(T+s) \le Ce^{Ks} M_{u,v(T)}^\varepsilon(T), \quad (4.34)$$

where the positive constants C and K depend on P, but are independent of $\varepsilon \le \varepsilon_0$ and $\xi_u \in K_\varepsilon^+$.

Proof. Let $\xi_{u_{N_k}}(t)$ be a sequence of Galerkin approximations which converge in Θ^+ to the solution $\xi_u(t)$. Let now $v_{N_k}(t) := P_{N_k}v(t)$, $U_{N_k}(t) := u_{N_k}(t) - v_{N_k}(t)$. Then, analogously to (2.63)

$$\varepsilon\partial_t^2 U_{N_k} + \gamma\partial_t U_{N_k} - \Delta_x U_{N_k} + \\ + P_{N_k}(f(v_{N_k}(t) + U_{N_k}) - f(v_{N_k}(t))) + L(-\Delta_x)^{-1}U_{N_k} = \\ = h_{N_k}(t) := P_{N_k}(f(v(t)) - f(v_{N_k}(t))) + L(-\Delta_x)^{-1}(u_{N_k}(t) - v_{N_k}(t)), \\ \xi_{U_{N_k}}(T) := \xi_{u_{N_k}}(T) - \xi_{v_{N_k}}(T), \quad (4.35)$$

where $L \gg 0$ is a large parameter which will be fixed below. As before, multiplying (4.35) by $\partial_t U_{N_k}(t) + \alpha U_{N_k}(t)$, integrating over $x \in \Omega$ and arguing as in the proof of Theorem 2.1, we derive that there exists a positive $\alpha > 0$ which is independent of ε such that

$$\frac{d}{dt} E_{U_{N_k}}(t) + \alpha E_{U_{N_k}}(t) = H_{U_{N_k}}(t), \quad t \geq T, \tag{4.36}$$

where $E_{U_{N_k}}(t)$ and $H_{U_{N_k}}(t)$ are defined by (2.48) and (2.50) respectively and the function $H_{U_{N_k}}(t)$ satisfies, in addition, inequality (2.66), for every $t \geq T$. Moreover, since the solution $v(t)$ is independent of L and uniformly bounded in C (due to (4.33)) then, there exists $L_0 = L_0(P)$ such that, for every $L > L_0$, we have

$$C_L^{-1} \|\xi_{U_{N_k}}(t)\|^2_{E,\varepsilon} \leq E_{U_{N_k}}(t) \leq C_L \|\xi_{U_{N_k}}(t)\|^2_{E,\varepsilon}, \tag{4.37}$$

where the constant C_L is independent of ε (see Proposition 2.1). Arguing analogously (2.67) (and using again Proposition 2.1), we derive from (2.66) that, for a sufficiently large $L > L_0(P)$ (which can be fixed now), the following estimate holds:

$$H_{U_{N_k}}(t) \leq C\|h_{U_{N_k}}(t)\|^2_{L^2} + C\|\partial_t v_{N_k}(t)\|_{L^\infty} E_{U_{N_k}}(t), \quad t \in [T, T+s], \tag{4.38}$$

where $C = C(P, L)$ is independent of ε. Applying now Gronwall's inequality to (4.36), using (4.37) and (4.38) and noting that, due to (4.33) and embedding $H^2(\Omega) \subset C(\overline{\Omega})$, we have

$$C \int_T^t \|\partial_t v_{N_k}(s)\|_{L^\infty}\, ds \leq C_1(t - T + 1). \tag{4.39}$$

If k is large enough, we derive the inequality

$$\|\xi_{U_{N_k}}(t)\|^2_{E,\varepsilon} \leq C\|\xi_{U_{N_k}}(T)\|^2_{E,\varepsilon} e^{(C_1-\alpha)(t-T)} + \\ + C\int_T^t e^{(C_1-\alpha)(t-l)} \|h_{U_{N_k}}(l)\|^2_{L^2}\, dl, \tag{4.40}$$

where $C = C(P)$ and $C_1 = C_1(P)$ are independent of ε and ξ_u. Passing to the limit $k \to \infty$ in (4.40) and taking into account the fact that

$$h_{N_k} \to L(-\Delta_x)^{-1}(u - v) \text{ strongly in } \quad C([T, T+s], L^2(\Omega)) \tag{4.41}$$

(compare with (2.64)) and that the approximating sequence $\xi_{u_{N_k}}(t)$ was chosen arbitrarily, we derive from (4.40) that

$$M^\varepsilon_{u,v(t)}(t) \leq Ce^{(C_1-\alpha)(t-T)} M_{u,v(T)}(T) + \\ + CL\int_T^t e^{(C_1-\alpha)(t-l)} \|(-\Delta_x)^{-1}(u(l) - v(l))\|^2_{L^2}\, dl. \tag{4.42}$$

Using now estimate (4.31) and applying again Gronwall's inequality to (4.42), we finally derive that

$$M^\varepsilon_{u,v(t)}(t) \leq C_2 e^{K(t-T)} M_{u,v(T)}(T),$$

where the constants $C_2 = C_2(P)$ and $K = K(P)$ are independent of ε and $\xi_u \in K^+_\varepsilon$ and Lemma 4.4 is proven.

Lemma 4.5. *Let the assumptions of Theorem 3.2 hold. Then, for every $\xi_u \in K_\varepsilon^+$ and every $T, s \in \mathbb{R}_+$, we have*

$$M^\varepsilon_{u,\mathcal{A}_\varepsilon^{gl}}(T+s) \le Ce^{Ks} M^\varepsilon_{u,\mathcal{A}_\varepsilon^{gl}}(T), \tag{4.43}$$

where the positive constants C and K are independent of $\varepsilon \le \varepsilon_0$ and $\xi_u \in K_\varepsilon^+$.

Proof. Let $\xi_u \in K_\varepsilon^+$ be an arbitrary weak solution of (0.1) and $\xi^0 \in \mathcal{A}_\varepsilon^{gl}$ be an arbitrary point from the attractor. Then, due to Corollary 3.2, there exists a strong solution $\xi_v(t) \in \mathcal{A}_\varepsilon^{gl}$, $t \in \mathbb{R}$, $\xi_v(T) = \xi^0$ and, according to (3.39), this solution satisfies estimate (4.33) with $P = P(\|g\|_{L^2})$ uniformly with respect to $\xi^0 \in \mathcal{A}_\varepsilon^{gl}$. Then, according to Lemma 4.4, we have

$$M^\varepsilon_{u,v(T+s)}(T+s) \le Ce^{Ks} M^\varepsilon_{u,\xi^0}(T), \quad T, s \in \mathbb{R}_+, \tag{4.44}$$

where the constants C and K are independent of $\xi^0 \in \mathcal{A}_\varepsilon^{gl}$. Since ξ^0 is arbitrary, then (4.44) implies (4.43) and Lemma 4.5 is proven.

The next lemma allows to control the M-distance from $\xi_u(t)$ to the attractor $\mathcal{A}_\varepsilon^{gl}$ in the case where $\xi_u(t)$ remains inside $\mathcal{O}_\delta(\mathcal{R}, L^2(\Omega))$.

Lemma 4.6. *Let the assumptions of Theorem 3.2 and assumption* (4.3) *hold. Then, there exists a small positive constant δ which is independent of ε such that, for every $\xi_u \in \mathcal{B}_\varepsilon \subset K_\varepsilon^+$, the inclusion*

$$u(t) \in \mathcal{O}_\delta(z_0, L^2(\Omega)), \quad \text{for } t \in [T, T+s] \text{ and } \xi_{z_0} \in \mathcal{R}, \tag{4.45}$$

implies that

$$M^\varepsilon_{u,\mathcal{A}_\varepsilon^{gl}}(t) \le Ce^{-\beta(t-T)}[M^\varepsilon_{u,\mathcal{A}_\varepsilon^{gl}}(T)]^\kappa, \quad t \in [T, T+s], \tag{4.46}$$

where the positive constants β, $\kappa < 1$ and C are independent of ε and ξ_u.

Proof. As in the proof of Theorem 2.1, we use the following auxiliary hyperbolic equation

$$\begin{aligned}\varepsilon\partial_t^2 v + \gamma\partial_t v - \Delta_x v + f(v) + L(-\Delta_x)^{-1} v = \\ = h_L(t) := g + L(-\Delta_x)^{-1} u(t), \quad v\big|_{t=T} = z_0, \quad \partial_t v\big|_{t=T} = 0,\end{aligned} \tag{4.47}$$

where L is a large parameter.

The following proposition is an analogue of Lemma 2.1.

Proposition 4.1. *Let the assumptions of Lemma 4.6 hold. Then, for every large $L \ge L_0$ and every small $\nu > 0$, there exists a constant $\delta = \delta(L, \nu)$ (which is independent of ε) such that equation* (4.47) *has a unique strong solution on the interval $t \in [T, T+s]$ and the following estimate is valid:*

$$\varepsilon\|\partial_t v(t)\|^2_{H^2} + \|v(t) - z_0\|^2_{H^3} + \int_t^{t+1} \|\partial_t v(l)\|^2_{H^2}\, dl \le \nu, \tag{4.48}$$

for every $t \in [T, T+s]$.

Proof. Proposition 4.1 is analogous to Lemma 2.1, but its proof is essentially simpler, since we may now set $w(t) \equiv z_0$. Indeed, let us seek the solution of (4.47) in the form $v(t) := W(t) + z_0$. Then, the function $W(t)$ satisfies

$$\begin{aligned}\varepsilon\partial_t^2 W + \gamma\partial_t W - \Delta_x W + [f(W+z_0) - f(z_0)] + L(-\Delta_x)^{-1} W = h_{L,\delta}(t) := \\ = L(-\Delta_x)^{-1}(u(t) - z_0), \quad W\big|_{\partial\Omega} = \Delta_x W\big|_{\partial\Omega} = 0, \quad \xi_W\big|_{t=T} = 0.\end{aligned} \tag{4.49}$$

According to (4.45), we have

$$\|h_{L,\delta}(t)\|^2_{H^2} \le Q(L,\delta), \quad \text{with} \quad \lim_{\delta\to 0} Q(L,\delta) = 0. \tag{4.50}$$

Consequently, applying the implicit function theorem to equation (4.49) (compare with (2.22)), we derive that, for every fixed $L \ge L_0$, there exists $\delta_0 = \delta_0(L) > 0$ such that, for $\delta \le \delta_0$, equation (4.49) has a unique strong solution $\xi_W(t) \in E^1$, $t \in [T, T+s]$, which satisfies the estimate:

$$\|\xi_W(t)\|^2_{E^1,\varepsilon} + \int_t^{t+1} \|\partial_t W(t)\|^2_{H^1}\, ds \le CQ(L,\delta), \tag{4.51}$$

where the constant C is independent of δ and ε. Estimate (4.51), convergence (4.50), together with assumption $f'' \in C(R)$ and with the embedding $H^2(\Omega) \subset C(\overline{\Omega})$, imply that

$$\|f(W(t)+z_0) - f(z_0)\|^2_{H^2} \le Q_1(L,\delta), \tag{4.52}$$

where $\lim_{\delta\to 0} Q_1(L,\delta) = 0$ and the function Q_1 is independent of ε. Multiplying now equation (4.49) by $\Delta_x^2(\partial_t W(t) + \alpha W(t))$, we derive, analogously to Corollary 3.3, that

$$\varepsilon\|\partial_t W(t)\|^2_{H^2} + \|W(t)\|^2_{H^3} + \int_t^{t+1} \|\partial_t W(l)\|^2_{H^2}\, dl \le \\ \le C_1\left(Q(L,\delta) + Q_1(L,\delta)\right), \tag{4.53}$$

where C_1 is independent of δ and ε. Estimates (4.53), together with convergences (4.50) and (4.52), prove Proposition 4.1.

The next proposition shows that, under assumptions of Lemma 4.6, the solution $\xi_u(t)$ converges exponentially to the function $\xi_v(t)$ in E.

Proposition 4.2. *Let the assumptions of Lemma 4.6 hold. Then, there exist a large constant L and a small constant δ, which are independent of ε such that equation* (4.45) *possesses a unique strong solution $v(t) = v_{u,L}(t)$, $t \in [T, T+s]$ which satisfies* (4.48) *(where $\nu = \nu(\delta, L)$ is independent of ε and u) and the following estimate is valid:*

$$M^\varepsilon_{u,v(t)}(t) \le Ce^{-\beta(t-T)} M^\varepsilon_{u,v(T)}(T), \ t \in [T, T+s] \tag{4.54}$$

where constants C and $\beta > 0$ are independent of ε and $\xi_u \in K^+_\varepsilon$ satisfying (4.45).

Proof. Let $\xi_{u_{N_k}}(t)$ be a sequence of Galerkin approximations which converges in Θ^+ to the weak solution $\xi_u \in K^+_\varepsilon$. Let also $\xi_v(t)$, $t \in [T, T+s]$, be a strong solution of equation (4.47), $v_{N_k}(t) := P_{N_k} v(t)$ and let $U_{N_k}(t) := u_{N_k}(t) - v_{N_k}(t)$. Then, the last function satisfies the equation:

$$\varepsilon\partial_t^2 U_{N_k} + \gamma\partial_t U_{N_k} - \Delta_x U_{N_k} + \\ + P_{N_k}(f(v_{N_k}(t) + U_{N_k}) - f(v_{N_k}(t))) + L(-\Delta_x)^{-1} U_{N_k} = \\ = h_{N_k}(t) := P_{N_k}(f(v(t)) - f(v_{N_k}(t))), \\ \xi_{U_{N_k}}(T) := \xi_{u_{N_k}}(T) - \xi_{v_{N_k}}(T). \tag{4.55}$$

Multiplying (4.55) by $\partial_t U_{N_k}(t) + \alpha U_{N_k}(t)$, integrating over $x \in \Omega$ and arguing as before, we derive that there exists a positive constant α which is independent of L, δ and ε such that

$$\frac{d}{dt} E_{U_{N_k}}(t) + \alpha E_{U_{N_k}}(t) = H_{U_{N_k}}(t), \tag{4.56}$$

where the functions $E_{U_{N_k}}(t)$ and $H_{U_{N_k}}(t)$ are defined by (2.48) and (2.50) respectively and the function $H_{U_{N_k}}(t)$ satisfies, in addition, inequality (2.66). According to estimate (4.48), we may fix the constant L such that (4.37) is valid and the function $H_{U_{N_k}}(t)$ satisfies the inequality

$$H_{U_{N_k}}(t) \le C\|h_{U_{N_k}}(t)\|^2_{L^2} + C\|\partial_t v_{N_k}(t)\|_{L^\infty} E_{U_{N_k}}(t), \tag{4.57}$$

where the constant C is independent of ε and δ. Due to Proposition 4.1, we may now fix δ so that

$$C\int_T^t \|\partial_t v_{N_k}(l)\|_{L^\infty}\,dl \le \frac{\alpha}{2}(t-T+1) \tag{4.58}$$

for a sufficiently large k. Applying now Gronwall's inequality to (4.56) and using (4.37), (4.57) and (4.58), we obtain

$$E_{U_{N_k}}(t) \le C_1 e^{-\alpha(t-T)/2} E_{U_{N_k}}(T) + C_1\int_T^t e^{-\alpha(t-l)/2}\|h_{U_{N_k}}(l)\|^2_{L^2}\,dl, \tag{4.59}$$

where $t\in[T,t+s]$ and the constant C_1 is independent of ε and $\xi_u\in K^+_\varepsilon$. Passing to the limit $k\to\infty$ in (4.59) and taking into account the fact that

$$h_{u_{N_k}} \to 0 \ \text{ strongly in } \ C([t,T+s],L^2(\Omega)),$$

we derive estimate (4.54) and Proposition 4.2 is proven.

We are now ready to prove that, under the assumptions of Lemma 4.6, every weak solution $\xi_u\in K^+_\varepsilon$ converges exponentially to the global attractor $\mathcal{A}^{gl}_\varepsilon$.

Proposition 4.3. *Let the assumptions of Lemma 4.6 hold and let δ be the same as in Proposition 4.2. Then, for every $\xi_u\in\mathcal{B}_\varepsilon\subset K^+_\varepsilon$ which satisfies* (4.45)*, the following estimate is valid:*

$$M^\varepsilon_{u,\mathcal{A}^{gl}_\varepsilon}(t) \le Ce^{-\beta_1(t-T)}, \quad t\in[T,T+s], \tag{4.60}$$

where positive constants β_1 and C are independent of ε and $\xi_u\in\mathcal{B}_\varepsilon$.

Proof. Let $\xi_u\in\mathcal{B}_\varepsilon$ and let $\xi_v(t)$ be the corresponding solution of equation (4.47). We also fix an arbitrary $T_1\in[T,T+s]$. Then, according to estimate (4.48), the trajectory $\xi_v(t)$ is uniformly bounded in E^2_g and, consequently (due to Corollary 3.3), there exists a unique strong solution $\xi_{\hat v}(t) := S^\varepsilon_{t-T_1}\xi_v(T_1)$ of equation (0.1) defined for $t\ge T_1$, with $\xi_{\hat v}(T_1)=\xi_v(T_1)$. Moreover, due to (4.48) and (3.55), we have

$$\|\hat v(t)\|^2_{H^2} + \int_t^{t+1}\|\partial_t\hat v(l)\|^2_{H^2}\,dl \le K_1, \tag{4.61}$$

where the constant K_1 is independent of $\varepsilon\le\varepsilon_0$ and of $\xi_u\in\mathcal{B}_\varepsilon$. Consequently, due to Lemma 4.4, we obtain

$$M^\varepsilon_{u,\hat v(t)}(t) \le Ce^{K(t-T_1)}M^\varepsilon_{u,v(T_1)}(T_1), \tag{4.62}$$

where C and K are independent of ε and ξ_u. Inserting estimate (4.54) into the right-hand side of (4.62) and using (4.32), we have

$$M^\varepsilon_{u,\hat v(t)}(t) \le C'e^{K(t-T_1)-\beta(T_1-T)}M^\varepsilon_{u,v(T)}(T) \le C_1e^{K(t-T_1)-\beta(T_1-t)}, \tag{4.63}$$

where the positive constants β, K, C' and C_1 are independent of ε and $\xi_u \in \mathcal{B}_\varepsilon$ (here we have also used the fact that, due to (4.31) and (1.44), the value $M^\varepsilon_{u,v(T)}(T)$ is uniformly bounded with respect to ε and $\xi_u \in \mathcal{B}_\varepsilon$).

On the other hand, due to Theorem 4.2

$$\operatorname{dist}_{E,\varepsilon}\left(\hat{v}(t), \mathcal{A}^{gl}_\varepsilon\right) \le C_1 e^{-\beta(t-T_1)}, \quad t \ge T_1,$$

where the positive constants C and β are independent of ε and K^+_ε. Combining this estimate with (4.63) and taking into account (4.32), we obtain

$$M^\varepsilon_{u,\mathcal{A}^{gl}_\varepsilon}(t) \le C_1\left(e^{-\beta(T_1-T)+K(t-T_1)} + e^{-\beta(t-T_1)}\right), \quad t \in [T_1, T+s]. \tag{4.64}$$

Fixing now the parameter

$$T_1 := \frac{\beta T + (K+\beta)t}{2\beta + K}$$

in an optimal way, we derive estimate (4.60) (with $\beta_1 := \beta^2/(2\beta+K)$) and Proposition 4.3 is proven.

We are now ready to complete the proof of Lemma 4.6. Indeed, it follows from estimates (4.43) and (4.60) that, for every $\kappa \in [0,1]$

$$M^\varepsilon_{u,\mathcal{A}^{gl}_\varepsilon}(t) \le Ce^{(\kappa K-(1-\kappa)\beta_1)(t-T)}[M^\varepsilon_{u,\mathcal{A}^{gl}_\varepsilon}(T)]^\kappa, \quad t \in [T, T+s]. \tag{4.65}$$

Fixing now $\kappa := \beta_1/(2K+2\beta_1)$, we obtain estimate (4.46). Lemma 4.6 is proven.

The assertion of Theorem 4.4 is a standard corollary of Lemmata 4.2, 4.5 and 4.6. Indeed, arguing as in [3] and [9], we derive from these Lemmata that, for a sufficiently small ε and every $\xi_u \in \mathcal{B}_\varepsilon$, the following estimate holds:

$$M^\varepsilon_{u,\mathcal{A}^{gl}_\varepsilon}(t) \le C^N e^{KNT(\delta)} e^{-\beta^N t}[M^\varepsilon_{u,\mathcal{A}^{gl}_\varepsilon}(0)]^{\kappa^N}, \tag{4.66}$$

where $\kappa > 0$, $\beta > 0$, C and δ are the same as in Lemma 4.6, K is the same as in Lemma 4.5 and $T(\delta)$ is defined in Lemma 4.2 (see [9] for the details). Since $\mathcal{B}_\varepsilon$ is a uniform (with respect to ε) absorbing set in K^+_ε, then (4.66) implies (4.12) and Theorem 4.4 is proven.

Remark 4.1. Theorem 4.4 and Lemma 4.5 show that, for a sufficiently small ε and under the additional assumption (4.3), the trajectory attractor $\mathcal{A}^{tr}_\varepsilon$ attracts M-bounded subsets of K^+_ε not only in the weak topology of Θ^+, but also in the strong topology of $L^\infty_{loc}(\mathbb{R}_+, E)$.

References

1. J. Arrieta, A. Carvalho and J. Hale, *A damped hyperbolic equation with critical exponent*, Comm. Partial Diff. Eqns. **17** (1992), 841–866.
2. A. Babin and M. Vishik M, *Maximal Attractors of Semigroups Corresponding to Evolutionary Differential Equations*, Mat. Sb. **126(3)** (1984), 397–419.
3. A. Babin and M. Vishik, *Attractors of Evolutionary Equations*, Nauka, Moscow, 1989.
4. V. Chepyzhov and M. Vishik, *Attractors of Nonautonomous Dynamical Systems and Their Dimension*, J. Math. Pures Appl. **73(3)** (1994), 279–333.
5. V. Chepyzhov and M. Vishik, *Evolution Equations and Their Trajectory Attractors*, J. Math. Pures Appl. **76(10)** (1997), 913–964.
6. V. Chepyzhov and M. Vishik, *Perturbation of Trajectory Attractors for Dissipative Hyperbolic Equations*, The Maz'ya anniversary collection **2** (Rostock 1998), 33–54.

7. I. Chueshov, *Analyticity of Global Attractors and Determining Nodes of Nonlinear Wave Equations with Damping*, Mat. Sb. **191(10)** (2000), 119–136.
8. Yu. Dubinskij, *Weak Convergence for Nonlinear Elliptic and Parabolic Problems*, Mat. Sb. **67(109)** (1965), 609–642.
9. M. Efendiev and S. Zelik, *The Regular Attractor for the Reaction-Diffusion System with a Nonlinearity Rapidly Oscillating in Time and it's Averaging*, Universite de Poitiers, Dept. Mat., preprint **152** (2001), to appear in Adv. Diff. Eqns.
10. P. Fabrie, C. Galusinski and A. Miranville, *Uniform Inertial Sets for Damped Wave Equations*, Disc. Cont. Dyn. Systems **6 (2)** (2000), 393–418.
11. P.Fabrie, C.Galushinski, A.Miranville and S.Zelik, *Uniform Exponential Attractors for a Singular Perturbed Damped Wave Equation*, to appear in Disc. Cont. Dyn. Sys. (2003).
12. E. Feireisl, *Bounded, Locally Compact Global Attractors for Semilinear Damped Wave Equations on* $\mathbb{R}^n$, Diff. Int. Eqns **5** (1996), 1147–1156.
13. E. Feireisl, *Global Attractors for the Navier-Stokes Equations of Three-Dimensional Compressible Flow*, C. R. Acad. Sci. Paris, Ser I Math. **331(5)** (2000), 35–39.
14. J. Ghidaglia and R. Temam, *Attractors for Damped Nonlinear Hyperbolic Equations*, J. Math. Pures Appl **66(9)** (1987), 273–319.
15. J. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys and Mon., Vol. 25, Amer. Math. Soc., Providence, RI, 1987.
16. J. Hale and G. Raugel, *Lower Semicontinuity of the Attractors for a Singular Perturbed Hyperbolic Equation*, J. Dyn. Diff. Eqns **2** (1990), 19–67.
17. O. Ladyzhenskaya, *Attractors for Semigroups of Evolution Equations*, Cambridge University Press, 1991.
18. J. Lions, *Quelques Methodes de Resolution des Problemes aux Limites Non Linearies*, Dunod, Paris, 1969.
19. A. Mielke and S. Zelik, *Infinite Dimensional Trajectory Attractors for Elliptic Problems in Cylinders*, Rus. Math. Survey **54** (2002), no. 4, 119-151.
20. H. Schaefer and M. Wolff, *Topological Vector Spaces, 2 ed.*, Springer-verlag, New-York, 1999.
21. B. Schmalfuss, *Pullback Attractors Under Discretizations*, in proceedings of the conference EQUADIFF 99 (Berlin 1999), 1024–1029.
22. G. Sell, *Global Attractors for the Three-Dimensional Navier-Stokes Equations*, J. Dyn. Diff. Eqns **1** (1996), 1–33.
23. R. Temam, *Infinite Dimensional Dynamical Systems in Physics and Mechanics, 2d ed.*, Springer-Verlag, New-York, 1997.
24. H. Triebel, *Interpolation Theory, Functional Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.
25. M. Vishik and S. Zelik, *The Trajectory Attractor for a Nonlinear Elliptic System in a Cylindrical Domain*, Mat. Sbornik, **187,** (1996), no. 12, 21–56.
26. S. Zelik, *The Trajectory Attractor for a Nonlinear Elliptic System in an Unbounded Domain*, Math. Notes **63** (1998), no. 1, 135–137.
27. S. Zelik, *The Attractor for a Nonlinear Reaction-Diffusion System with a Supercritical Nonlinearity and it's Dimension*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **24** (2000), 1–25.
28. S. Zelik, *The Attractor for a Nonlinear Hyperbolic Equation in an Unbounded Domain*, Disc. Cont. Dyn. Sys. Ser. A **7** (2001), no. 3, 593–641.