INFINITE-DIMENSIONAL EXPONENTIAL ATTRACTORS FOR NONLINEAR REACTION-DIFFUSION SYSTEMS IN UNBOUNDED DOMAINS AND THEIR APPROXIMATION

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Abstract. We study in this article the long time behavior of solutions of reaction-diffusion equations (RDEs) in unbounded domains of $\mathbb{R}^n$. In particular, we prove that, under appropriate assumptions on the nonlinear interaction function and on the external forces, these equations possess infinite-dimensional exponential attractors whose Kolmogorov's ε-entropy satisfies an estimate of the same type as that obtained previously for the ε-entropy of the global attractor. Moreover, we also study the problem of the approximation of these infinite-dimensional exponential attractors by finite-dimensional ones associated with the same RDEs in bounded domains.

Contents

Introduction.

We consider in this article the following reaction-diffusion system:

$$\begin{cases} \partial_t u = a\Delta_x u - (\vec{L}, \nabla_x)u - f(u) - \lambda_0 u + g, \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \end{cases} \tag{0.1}$$

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where Ω is an unbounded domain of $\mathbb{R}^n$ with a sufficiently smooth boundary (see Section 1 for a more precise definition), $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, Δ_x is the Laplacian with respect to x,

$$(\vec{L}, \nabla_x u) := \sum_{i=1}^{n} L_i(x)\partial_{x_i} u,$$

where $\vec{L}(x) := (L_1(x), \cdots, L_n(x))$ is a given vector field in Ω, a is a given constant diffusion matrix with positive symmetric part ($a + a^* > 0$), λ_0 is a given positive constant and $f(u)$ and $g = g(x)$ are given interaction function and external forces respectively which satisfy some natural assumptions (see Section 1).

The long time behavior of the solutions of (0.1) is of a great current interest. It is well-known that, under appropriate assumptions on the nonlinear term $f(u)$, on the external forces g and on the domain Ω, this behavior can be described in terms of the global or/and exponential attractors of the dynamical system generated by (0.1) (see e.g. [1-5], [9-16], [18], [21-25], [27-29] and the references therein). In particular, when Ω is bounded, the global and exponential attractors of problem (0.1) have usually finite Hausdorff and fractal dimensions (see [3], [16] and [24]).

In contrast to this, in case Ω is unbounded, the existence of finite-dimensional global or/and exponential attractors for (0.1) requires very strong additional assumptions on the nonlinear interaction function f and the external forces g which are violated for many interesting (from the physical point of view) examples of equations of the form (0.1), such as the Chafee-Infante equation, the complex Ginzburg-Landau equation, etc. (see [1-2], [4], [9], [12-13], [15] and the references therein for detailed studies of particular cases of equations of the form (0.1) in unbounded domains for which the finite dimensional global or/and exponential attractors exist).

Thus, in case Ω is unbounded, the global attractor associated with problem (0.1) has usually infinite Hausdorff and fractal dimensions, see e.g. [2], [12] and [27]. That is the reason why the concept of Kolmogorov's ε-entropy (see [19]) is exploited in order to obtain some qualitative and quantitative informations on such attractors (the ε-entropy of infinite-dimensional uniform attractors associated with nonautonomous RDEs in bounded domains is studied in [5-6]; the case of autonomous RDEs in $\mathbb{R}^n$ is considered in [7] and [25]; the ε-entropy in the case of general unbounded domains is investigated in [11] and [27-29] for the case of autonomous and nonautonomous RDEs and in [8] and [26] for the case of damped hyperbolic equations).

We recall that, if K is a precompact set in a metric space M, then it can be covered (due to the Hausdorff criteria) by a finite number of ε-balls, for every $\varepsilon > 0$. Let $N_\varepsilon(K, M)$ be the minimal number of such balls. Then, by definition, the Kolmogorov's ε-entropy of K in M is the following number:

$$\mathbb{H}_\varepsilon(K, M) := \ln N_\varepsilon(K, M). \tag{0.2}$$

It is worth emphasizing that, in contrast to the fractal dimension, quantity (0.2) remains finite, for every $\varepsilon > 0$ and every precompact set K in M.

In particular, it is proved in [11] and [25-29] that, for a large class of equations of mathematical physics in unbounded domains (including systems of reaction-diffusion equations of the form (0.1), hyperbolic problems, etc.), the ε-entropy of the restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ of the corresponding global attractor $\mathcal{A}$ to bounded subdomains $\Omega\cap\mathcal{C}^R_{x_0}$, where $\mathcal{C}^R_{x_0}:=x_0+[-R/2,R/2]$ is the R-cube of $\mathbb{R}^n$ centered at x_0, possess the following universal estimates:

$$\mathbb{H}_\varepsilon(\mathcal{A}\big|_{\Omega\cap\mathcal{C}^R_{x_0}},L^\infty(\mathcal{C}^R_{x_0}))\le C\,\mathrm{vol}(\Omega\cap\mathcal{C}^{R+K\ln_+R_0/\varepsilon}_{x_0})\ln_+\frac{R_0}{\varepsilon},\tag{0.3}$$

where $\ln_+ z:=\max\{0,\ln z\}$ and the constants C, K and R_0 depend on the concrete form of the equation, but are independent of ε, R and x_0. Moreover, it is also proved that estimates (0.3) are, in a sense, sharp for all values of the parameters ε, R and x_0. Consequently, since an exponential attractor always contains the global attractor, a finite-dimensional exponential attractor does not exist in general for equations of the form (0.1) in unbounded domains.

In the present article (following [14], where infinite-dimensional exponential attractors were introduced in order to study *nonautonomous* RDEs in bounded domains), we modify the classical definition of exponential attractors (see [16] and Definition 3.1 below) by replacing the condition of finite fractal dimensionality by the assumption that they should satisfy the ε-entropy estimates (0.3) (see Definition 3.2 below). We prove (see Theorem 3.1) that, under natural assumptions on the nonlinear term f, the vector field $\vec{L}$ and the external forces g, problem (0.1) possesses a modified (i.e. an infinite-dimensional) exponential attractor $\mathcal{M}:=\mathcal{M}_\Omega$.

The rest of the article is devoted to the study of the approximation of the infinite dimensional exponential attractor $\mathcal{M}=\mathcal{M}_{\mathbb{R}^n}$ (in the case $\Omega:=\mathbb{R}^n$) by finite-dimensional exponential attractors $\mathcal{M}_r$ corresponding to problem (0.1) in a *bounded* domain Ω_r (for simplicty, we assume that Ω_r is the ball of radius r centered at the origin). We finally construct a uniform (with respect to $r\to\infty$) family of exponential attractors $\mathcal{M}_r$ which satisfies the following estimate:

$$\mathrm{dist}^{sym}_{L^\infty(\Omega_R)}\left(\mathcal{M}_r\big|_{\Omega_R},\mathcal{M}_{\mathbb{R}^n}\big|_{\Omega_R}\right)\le Ce^{-\alpha(r-R)},\tag{0.4}$$

where the constants $C>0$ and $\alpha>0$ are independent of r and $R\le r$ and $\mathrm{dist}^{sym}_V(\cdot,\cdot)$ denotes the symmetric Hausdorff distance between sets in the space V (see Theorem 4.1).

We note that estimate (0.4), which reflects the well-known heuristic principle that the influence of the boundary decays exponentially with respect to the distance to that boundary, is violated, in general, for the global attractor (see Example 4.1 below).

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§1 Analytic properties of solutions of reaction-diffusion systems in unbounded domains.

In this section, we consider the following reaction-diffusion system in an unbounded domain Ω:

$$\begin{cases}\partial_t u=a\Delta_x u-(\vec{L},\nabla_x)u-\lambda_0 u-f(u)+g,\\ u\big|_{\partial\Omega}=0,\quad u\big|_{t=0}=u_0,\end{cases}\tag{1.1}$$

where $u=(u^1,\cdots,u^k)$ is an unknown vector-valued function, $a\in\mathcal{L}(\mathbb{R}^k,\mathbb{R}^k)$ is a given diffusion matrix, $\vec{L}=\vec{L}(x)$ is a given vector field in $\mathbb{R}^n$, $f=f(u)$ and $g=g(x)$

are given functions and λ_0 is a given positive constant. We give below assumptions on f, g and $\vec{L}$ which guarantee the existence and uniqueness of a solution $u(t)$ of (1.1) and formulate several useful estimates for these solutions. We refer the reader to [28] and [29] for their rigorous proof. We start by defining the class of admissible unbounded domains Ω.

Definition 1.1. An unbounded domain $\Omega \subset \mathbb{R}^n$ is called regular if the following assumptions are satisfied:

1. There exists $R_0 > 0$ such that, for every point $x_0 \in \Omega$, there exists a smooth domain $V_{x_0} \subset \Omega$ such that

$$B_{x_0}^{R_0} \cap \Omega \subset V_{x_0} \subset B_{x_0}^{R_0+1} \cap \Omega, \tag{1.2}$$

where B_x^R denotes the ball of radius R centered at $x \in \mathbb{R}^n$.

2. For every $x_0 \in \overline{\Omega}$ there exists a diffeomorphism $\theta_{x_0} : B_0^2 \to B_{x_0}^{R_0+2}$ such that $\theta_{x_0}(x) = x_0 + p_{x_0}(x)$, $\theta_{x_0}(B_0^1) = V_{x_0}$ and

$$\|p_{x_0}\|_{C^N} + \|p_{x_0}^{-1}\|_{C^N} \le K_\Omega, \tag{1.3}$$

where the constant K_Ω is independent of $x_0 \in \Omega$ and N is large enough.

We also recall the definition of uniformly local spaces $W_b^{s,p}(\Omega)$ which are necessary for the study of equation (1.1) in an unbounded domain (see e.g. [12] and [27] for a more detailed study of these spaces).

Definition 1.2. We set, for every $s \in \mathbb{R}$ and $1 \le p \le \infty$

$$W_b^{s,p}(\Omega) := \{u \in D'(\Omega),\ \|u\|_{W_b^{s,p}(\Omega)} := \sup_{x_0 \in \Omega} \|u\|_{W^{s,p}(\Omega \cap B_{x_0}^1)} < \infty\}. \tag{1.4}$$

Here and below, $W^{s,p}(V)$ denotes the classical Sobolev space on V (see e.g. [20]). We write below $L_b^p(\Omega)$ instead of $W_b^{0,p}(\Omega)$.

We are now ready to formulate the assumptions on the various terms of equation (1.1). As usual, we assume that the diffusion matrix a has positive symmetric part

$$a + a^* > 0. \tag{1.5}$$

Futhermore, we assume that the vector field $\vec{L}$ belongs to $W_b^{1,\infty}(\Omega)$ and satisfies

$$\|\operatorname{div}(\vec{L})\|_{L^\infty(\Omega)} \le \frac{\lambda_0}{2}. \tag{1.6}$$

We recall that, in applications, $\vec{L}$ is usually a solution of the stationary Navier-Stokes equations so that (1.6) is not a big restriction.

We then impose the following conditions on the nonlinear term:

$$\begin{cases} 1.\ f \in C^2(\mathbb{R}^k, \mathbb{R}^k), \\ 2.\ f(u).u \ge -C, \quad f'(u) \ge -K, \quad \forall u \in \mathbb{R}^k, \\ 3.\ |f(u)| \le C(1 + |u|^q), \quad q < q_{max} := 1 + \frac{4}{n-4}, \ \forall u \in \mathbb{R}^k, \end{cases} \tag{1.7}$$

where we denote by $u.v$ the standard inner product in $\mathbb{R}^k$ (if $n \le 4$, then the exponent q may be arbitrary).

We also assume that the external forces g satisfy

$$g \in L_b^p(\Omega), \tag{1.8}$$

for some $p > \max\{2, n/2\}$ and the initial data u_0 satisfies

$$u_0 \in \Phi_b = \Phi_b(\Omega) := \{u \in W_b^{2,p}(\Omega),\ \ u\big|_{\partial\Omega} = 0\}. \tag{1.9}$$

The following theorem gives the solvability result for problem (1.1).

Theorem 1.1. *Let Ω be a regular unbounded domain and let assumptions* (1.5)–(1.8) *be satisfied. Then, for every $u_0 \in \Phi_b$, problem* (1.1) *has a unique solution u, $u(t) \in \Phi_b$, for $t \geq 0$, and the following estimate is valid:*

$$\|u(t)\|_{W_b^{2,p}(\Omega)} \leq Q(\|u_0\|_{W_b^{2,p}(\Omega)})e^{-\alpha t} + Q(\|g\|_{L_b^p(\Omega)}), \tag{1.10}$$

where $\alpha > 0$ is a positive constant and Q is an appropriate monotonic function which depend only on a, λ_0, $\vec{L}$, f and on constants R_0 and K_Ω, but are independent of $x_0 \in \Omega$ and of the form of the regular domain Ω.

The proof of this theorem is given in [28-29].

Corollary 1.1. *Let the assumptions of Theorem 1.1 hold. Then, equation* (1.1) *generates a semigroup*

$$S_t : \Phi_b \to \Phi_b \quad via \quad S_t u_0 := u(t), \tag{1.11}$$

where $u(t)$ is solution of (1.1).

The next theorem allows to extend Theorem 1.1 to less regular initial data u_0.

Theorem 1.2. *Let the assumptions of Theorem 1.1 hold. Then, for every $u_0 \in \Phi_b$ and for every $t > 0$, the following estimate holds:*

$$\|u(t)\|_{\Phi_b} + \|\partial_t u(t)\|_{\Phi_b} \leq Q_1(\|u_0\|_{L_b^2(\Omega)})\frac{t^N + 1}{t^N}e^{-\alpha t} + Q_1(\|g\|_{L_b^p(\Omega)}), \ t > 0, \tag{1.12}$$

where N, α are strictly positive constants and Q_1 is a monotonic function. Moroeover, for every solutions $u_1(t)$ and $u_2(t)$, we have

$$\|u_1(t) - u_2(t)\|_{L^2(\Omega\cap B^1_{x_0})} \leq Ce^{Kt} \sup_{x\in\Omega}\left\{e^{-\alpha|x-x_0|}\|u_1(0) - u_2(0)\|_{L^2(\Omega\cap B^1_x)}\right\}, \tag{1.13}$$

where the constants K and $\alpha > 0$ and C are independent of x_0, $u_1(t)$ and $u_2(t)$.

The proof of this theorem can be found e.g. in [27-29].

Corollary 1.2. *Let the assumptions of Theorem 1.1 hold. Then, semigroup* (1.11) *can be extended in a unique way by continuity to the globally Lipschitz continuous semigroup (which we also denote by S_t) acting on the phase space $L_b^2(\Omega)$. Moreover, this semigroup maps $L_b^2(\Omega)$ into Φ_b:*

$$S_t : L_b^2(\Omega) \to \Phi_b, \quad t > 0. \tag{1.14}$$

Proof. Let $u_0 \in L_b^2(\Omega)$ be arbitrary. Then, there exists a sequence $u_0^n \in \Phi_b$ such that

$$\|u_0^n\|_{L_b^2(\Omega)} \leq C\|u_0\|_{L_b^2(\Omega)} \text{ and } u_0^n \to u_0 \text{ in } L^2_{loc}(\overline{\Omega}).$$

We define the solution $u(t) := S_t u_0$ of equation (1.1) which corresponds to the initial data u_0 by the following obvious expression:

$$u(t) := L^2_{loc}(\overline{\Omega}) - \lim_{n\to\infty} S_t u_0^n.$$

We note that, thanks to estimate (1.13), this limit exists and is independent of the choice of the sequence u_0^n and, thanks to estimate (1.12), the function $u(t)$ thus defined belongs to Φ_b, for every $t>0$, and, consequently, satisfies (1.1) in the sense of distributions. Moreover, since $u^n(t):=S_tu_0^n\in C([0,T],L^2_b(\Omega))$, then the limit function $u(t)$ belongs to $C([0,T],L^2_{loc}(\overline{\Omega}))$ and therefore satisfies the initial condition in this sense.

Remark 1.1. We note that, in contrast to the case of bounded domains, the space Φ_b *is not dense* in the space $L^2_b(\Omega)$, but only in the space $L^2_{loc}(\overline{\Omega})$. That is the reason why we have to approximate the initial condition u_0 in the previous proof in the local topology. Moreover, this absence of density leads indeed to the fact that the solution $u(t)$ defined in Corollary 1.2 does not belong to $C([0,T],L^2_b(\Omega))$ for generic $u_0\in L^2_b(\Omega)$ and, analogously, to the fact that the solution $u(t)$ defined in Theorem 1.1 does not belong to $C([0,T],\Phi_b)$ for generic $u_0\in\Phi_b$.

We conclude this section by formulating a result on the *local* Lipschitz continuity of maps S_t in the phase space Φ_b.

Theorem 1.3. *Let the assumptions of Theorem 1.1 hold. Then, every pair of solutions $(u_1(t),u_2(t))$ satisfies*

$$\|u_1(t)-u_2(t)\|_{W^{2,p}(\Omega\cap B^1_{x_0})}\le \le C_1e^{Kt}\sup_{x\in\Omega}\left\{e^{-\alpha|x-x_0|}\|u_1(0)-u_2(0)\|_{W^{2,p}(\Omega\cap B^1_x)}\right\}, \tag{1.15}$$

where K and α are the same as in Theorem 1.2 and the constant C_1 depends on $\|u_i(0)\|_{\Phi_b}$, $i=1,2$, but is independent of x_0. Moreover, the following smoothing estimate is valid:

$$\|u_1(t)-u_2(t)\|_{W^{2,p}(\Omega\cap B^1_{x_0})}\le \le C_2\frac{t^N+1}{t^N}e^{Kt}\sup_{x\in\Omega}\left\{e^{-\alpha|x-x_0|}\|u_1(0)-u_2(0)\|_{L^2(\Omega\cap B^1_x)}\right\},\ t>0, \tag{1.16}$$

where K, N and α are the same as in Theorem 1.2 and the constant C_2 depends on $\|u_i(0)\|_{L^2_b(\Omega)}$, $i=1,2$, but is independent of x_0.

The proof of this theorem can be found in [28] and [29].

§2 The global attractor and its Kolmogorov's ε-entropy.

In this section, we briefly discuss the known results on the description of the long time behaviour of the solutions of (1.1) in terms of global attractors, see e.g. [11] and [27-29] for the rigorous proofs.

We first recall that, in contrast to the case of bounded domains, the existence of the global attractor in the phase space Φ_b for the semigroup S_t associated with equation (1.1) requires strong (and in a sense unnatural) restrictions on the structure of the nonlinear interaction term f and the external forces g in the unbounded case (see [2], [12], [15] and Remark 2.1 below). That is the reason why the following weakened version of the attractor's concept is usually exploited in the case of unbounded domains.

Definition 2.1. A set $\mathcal{A}$ is the *locally compact* attractor for semigroup (1.11) if the following conditions are satisfied:

1. The set $\mathcal{A}$ is bounded in Φ_b and is compact in $\Phi_{loc} := W^{2,p}_{loc}(\overline{\Omega})$.

2. It is strictly invariant, i.e. $S_t\mathcal{A} = \mathcal{A}$.

3. The set $\mathcal{A}$ attracts all bounded subsets of Φ_b in the topology of Φ_{loc}, i.e. for every bounded subset $B \subset \Phi_b$ and for every neighbourhood $\mathcal{O}(\mathcal{A})$ of the set $\mathcal{A}$ in the topology of Φ_{loc}, there exists $T = T(\mathcal{O}, B)$ such that

$$S_tB \subset \mathcal{O}(\mathcal{A}), \quad \text{for } t \geq T. \tag{2.1}$$

We recall that the first assumption of Definition 2.1 means that the restriction $\mathcal{A}\big|_{\Omega_1}$ of the attractor $\mathcal{A}$ to every *bounded* subdomain $\Omega_1 \subset \Omega$ is a compact set of $W^{2,p}(\Omega_1)$. Analogously, the third assumption is equivalent to the following: for every bounded subset $B \subset \Phi_b$ and every *bounded* subdomain $\Omega_1 \subset \Omega$

$$\lim_{t\to\infty} \text{dist}_{W^{2,p}(\Omega_1)} \left((S_tB)\big|_{\Omega_1}, \mathcal{A}\big|_{\Omega_1} \right) = 0, \tag{2.2}$$

where $\text{dist}_V(X, Y)$ denotes the nonsymmetric distance between sets X and Y in the space V:

$$\text{dist}_V(X, Y) := \sup\nolimits_{x\in X} \inf\nolimits_{y\in Y} \|x - y\|_V. \tag{2.3}$$

The following theorem guarantees the existence of such an attractor for the semigroup S_t under consideration.

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold. Then, the semigroup S_t associated with equation* (1.1) *possesses a locally compact attractor $\mathcal{A}$ in the sense of Definition 2.1 which can be described as follows:*

$$\mathcal{A} = \mathcal{K}\big|_{t=0}, \tag{2.4}$$

where $\mathcal{K}$ is the set of all the solutions $u \in L^\infty(\mathbb{R}, \Phi_b)$ of equation (1.1) *which are defined for all $t \in \mathbb{R}$ and bounded.*

For the proof of this theorem, see [28] and [29].

Remark 2.1. We emphasize once more that, without additional essential restrictions on the structure of f and g, the attractor $\mathcal{A}$ is usually not compact in Φ_b and does not attract the bounded subsets of Φ_b in the uniform topology of Φ_b. Namely, these restrictions are usually formulated in the following form:

$$f(v).v \geq 0, \quad \forall v \in \mathbb{R}^k$$

and the external forces $g(x)$ tend in a sense to zero as $|x| \to \infty$, e.g.

$$\lim_{|x|\to\infty} \|g\|_{L^p(\Omega\cap B^1_x)} = 0$$

(see e.g. [12], [13] and [15]). Moreover, it can be easily proved (see [27]) that, for the spatially homogeneous case ($\Omega = \mathbb{R}^n$, $\vec{L} \equiv \text{const} \in \mathbb{R}^n$ and $g \equiv \text{const} \in \mathbb{R}^k$), the existence of the global attractor in $\Phi_b(\mathbb{R}^n)$ for equation (1.1) implies the extremely unnatural condition that all equilibria of this equation are almost periodic with respect to $x \in \mathbb{R}^n$. For instance, this condition is violated even for the following

Chafee-Infante equation in $\Omega := \mathbb{R}^n$ (which is the simplest example of equation of the form (1.1)):

$$\partial_t u = \Delta_x u + u - u^3, \quad x \in \mathbb{R}^n, \tag{2.5}$$

see e.g. [27] for the details.

Moreover, we also recall that, in contrast to the case of bounded domains (see e.g. [2], [24] and the references therein), the locally compact attractor $\mathcal{A}$, and even its restrictions $\mathcal{A}\big|_{\Omega_1}$ to bounded subdomains of Ω, have usually infinite Hausdorff and fractal dimensions, see [2], [12], [27] and Theorem 2.3 below. That is the reason why the concept of Kolmogorov's ε-entropy is usually exploited in order to obtain some quantitative informations on such attractors. For the reader's convenience, we recall below the definition of such an entropy, see [19] for a detailed exposition.

Defition 2.2. Let K be a precompact set in a metric space M. Then, for every $\varepsilon > 0$, it can be covered by a finite number of ε-balls in M. We denote by $N_\varepsilon(K, M)$ the minimal number of such balls. The Kolmogorov's ε-entropy of K in M is the following number:

$$\mathbb{H}_\varepsilon(K, M) := \ln N_\varepsilon(K, M). \tag{2.6}$$

We also recall that, by definition, the fractal dimension of K in M is the following number:

$$\dim_F(K, M) := \limsup_{\varepsilon\to 0} \frac{\mathbb{H}_\varepsilon(K, M)}{\ln \frac{1}{\varepsilon}}. \tag{2.7}$$

The next theorem gives typical upper bounds for the entropy of the restrictions $\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}$ of the attractor $\mathcal{A}$ to bounded subdomains $\Omega \cap B^R_{x_0}$.

Theorem 2.2. *Let the assuptions of Theorem 1.1 hold. Then, the following estimates are valid, for every $R \in \mathbb{R}_+$, $\varepsilon > 0$ and $x_0 \in \Omega$:*

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{\Omega\cap B^R_{x_0}}, W^{2,p}_b(\Omega \cap B^R_{x_0})\right) \le C \operatorname{vol}\left(\Omega \cap B^{R+L\ln_+ \frac{R_0}{\varepsilon}}_{x_0}\right) \ln_+ \frac{R_0}{\varepsilon}, \tag{2.8}$$

where $\ln_+ z := \max\{0, \ln z\}$ *and the positive constants C, R_0 and L are independent of ε, R and x_0.*

Indeed, estimate (2.8) is proved in [28-29].

In the particular case $\Omega = \mathbb{R}^n$, we have $\operatorname{vol}(B^R_{x_0}) = cR^n$ and, consequently, estimate (2.8) reads

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{B^R_{x_0}}, W^{2,p}_b(B^R_{x_0})\right) \le K\left(R + L\ln_+ \frac{R_0}{\varepsilon}\right)^n \ln_+ \frac{R_0}{\varepsilon}. \tag{2.9}$$

Moreover, since $W^{2,p} \subset C$, then the space $W^{2,p}_b$ in (2.8)–(2.9) can be replaced by C or L^∞.

In order to indicate the sharpness of estimates (2.8) and (2.9), we formulate below a theorem on lower bounds on the Kolmogorov's entropy.

Theorem 2.3. *Let the assumptions of Theorem 1.1 hold and let in addition*

$$\Omega = \mathbb{R}^n, \quad \vec{L}(x) \equiv \vec{L} \in \mathbb{R}^n, \quad f(0) = 0, \quad g(x) \equiv 0. \tag{2.10}$$

We also assume that the equilibrium $u \equiv 0$ *of equation* (1.1) *is exponentially unstable, i.e.*

$$\sigma\left(a\Delta_x - (\vec{L}, \nabla_x) - f'(0), L^2(\mathbb{R}^n)\right) \cap \{\operatorname{Re}\lambda > 0\} \neq \varnothing, \tag{2.11}$$

where $\sigma(L, V)$ *denotes the spectrum of the linear operator* L *in the space* V*. Then, the entropy of the restrictions* $\mathcal{A}\big|_{B_0^R}$ *satisfies the following lower bound:*

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{B_0^R}, L^\infty(B_{x_0}^R)\right) \geq C' R^n \ln_+ \frac{R_0'}{\varepsilon}, \tag{2.12}$$

where the positive constants C' *and* R_0' *are independent of* R *and* ε*. Moreover, for every* $\delta > 0$*, there exists a positive constant* C_δ *such that*

$$\mathbb{H}_\varepsilon\left(\mathcal{A}\big|_{B_{x_0}^1}, L^\infty(B_{x_0}^1)\right) \geq C_\delta \left(\ln_+ \frac{R_0'}{\varepsilon}\right)^{n+1-\delta}. \tag{2.13}$$

The proof of this theorem is given in [27-29].

In particular, estimate (2.13) shows that the restrictions $\mathcal{A}\big|_{B_{x_0}^R}$ have usually infinite fractal dimension. Moreover, estimates (2.12) and (2.13) confirm the fact that the upper bounds (2.10) are in a sense sharp for every fixed values of R, ε and x_0.

To conclude this section, we reformulate the result of Theorem 2.2 in a form which is more adequate for our purposes. To this end, we fix a sufficiently large absorbing ball $\mathbb{B}$ in the space Φ_b via

$$\mathbb{B} := \{u_0 \in \Phi_b, \quad \|u_0\|_{\Phi_b} \leq 2Q(\|g\|_{L_b^p(\Omega)})\}, \tag{2.14}$$

where Q is the same as in Theorem 1.1. Then, thanks to estimate (1.10), $\mathbb{B}$ is indeed an absorbing ball for the semigroup S_t. It is also convenient to replace the balls $B_{x_0}^R$ in Theorem 2.2 by the cubes

$$\mathcal{C}_{x_0}^R := x_0 + [-R/2, R/2]^n. \tag{2.15}$$

The next theorem plays a fundamental role in the next sections for our study of exponential attractors.

Theorem 2.4. *Let the assuptions of Theorem 1.1 hold. Then, the following estimates are valid, for every* $R \in \mathbb{R}_+$*,* $\varepsilon > 0$ *and* $x_0 \in \Omega$*:*

$$\mathbb{H}_\varepsilon\left((S_t\mathbb{B})\big|_{\Omega\cap\mathcal{C}_{x_0}^R}, W_b^{2,p}(\Omega \cap \mathcal{C}_{x_0}^R)\right) \leq \\ \leq C \operatorname{vol}\left(\Omega \cap \mathcal{C}_{x_0}^{R + L\ln_+\frac{R_0}{\varepsilon}}\right) \ln_+ \frac{R_0}{\varepsilon}, \quad \text{for } t \geq K \ln_+ \frac{R_0}{\varepsilon}, \tag{2.16}$$

where the positive constants C*,* R_0*,* L *and* K *depend only on equation* (1.1) *and on the constants* N*,* R_0 *and* K_Ω *appearing in Definition 1.1, but are independent of* ε*,* R *and* x_0 *and of the shape of the domain* Ω*.*

The proof of this theorem repeats word by word that of Theorem 2.2, given in [27-29], and is omitted here.

§3 Infinite-dimensional exponential attractors.

In this section, we construct an infinite-dimensional exponential attractor for problem (1.1) based on the entropy estimates (2.16). For the reader's convenience, we first recall the definition of standard (finite-dimensional) exponential attractors (see [16] for the details).

Definition 3.1. A set $\mathcal{M}$ is an exponential attractor for the semigroup S_t in Φ_b if the following assumptions are satisfied:

1. The set $\mathcal{M}$ is compact in Φ_b.
2. This set is invariant with respect to S_t, i.e. $S_t\mathcal{M} \subset \mathcal{M}$, $t \geq 0$.
3. The set $\mathcal{M}$ attracts *exponentially* all the bounded subsets of Φ_b, i.e. there exist a positive constant $\alpha > 0$ and a monotonic function Q such that, for every bounded subset $B \subset \Phi_b$

$$\operatorname{dist}_{\Phi_b}(S_tB, \mathcal{M}) \leq Q(\|B\|_{\Phi_b})e^{-\alpha t}.$$

4. The set $\mathcal{M}$ has finite fractal dimension in Φ_b.

We note that an exponential attractor in the sense of Definition 3.1 (if it exists) always contains the global attractor. However, as shown in the previous section, the global attractor $\mathcal{A}$ of problem (0.1) is usually not compact in Φ_b and is infinite-dimensional. So, in that case, equation (1.1) cannot possess an exponential attractor in the classical sense of Definition 3.1. That is the reason why, keeping in mind the estimates of Theorems 2.2–2.4 for the ε-entropy of the global attractor, we suggest the following adaptation of the concept of exponential attractor for the case of unbounded domains.

Definition 3.2. A set $\mathcal{M}$ is an (infinite-dimensional) exponential attractor for problem (1.1) if the following conditions are satisfied:

1. The set $\mathcal{M}$ is bounded in Φ_b and compact in Φ_{loc}.
2. This set is invariant with respect to S_t, i.e. $S_t\mathcal{M} \subset \mathcal{M}$, $t \geq 0$.
3. The set $\mathcal{M}$ attracts *exponentially* all the bounded subsets of Φ_b, i.e. there exist a positive constant $\alpha > 0$ and a monotonic function Q such that, for every bounded subset $B \subset \Phi_b$

$$\operatorname{dist}_{\Phi_b}(S_tB, \mathcal{M}) \leq Q(\|B\|_{\Phi_b})e^{-\alpha t}. \tag{3.1}$$

4. The ε-entropy of the restrictions $\mathcal{M}\big|_{\Omega\cap\mathcal{C}_{x_0}^R}$ satisfies the following estimate:

$$\mathbb{H}_\varepsilon\left(\mathcal{M}\big|_{\Omega\cap\mathcal{C}_{x_0}^R}, W_b^{2,p}(\Omega\cap\mathcal{C}_{x_0}^R)\right) \leq C'\operatorname{vol}\left(\Omega\cap\mathcal{C}_{x_0}^{R+L'\ln_+\frac{R_0'}{\varepsilon}}\right)\ln_+\frac{R_0'}{\varepsilon}, \tag{3.2}$$

where the constants C', R_0' and L' are independent of ε, R and x_0.

We emphasize that, according to Definition 3.2, such an exponential attractor $\mathcal{M}$ attracts exponentially all the bounded subsets of Φ_b in the *uniform topology* of the space Φ_b, although the global attractor $\mathcal{A}$ attracts them only in a *local topology* of Φ_{loc}. This shows a first advantadge of the exponential attractors' approach.

The main result of this section is the following theorem.

Theorem 3.1. *Let the assumptions of Theorem 1.1 hold. Then, problem* (1.1) *possesses an infinite-dimensional exponential attractor $\mathcal{M}$ in the sense of Definition 3.2.*

Proof. Let $\mathcal{B}_{R_0} \subset L^\infty(\Omega)$ be the R_0-ball centered at the origin in $L^\infty(\Omega)$ and let us fix R_0 so that

$$\mathbb{B} \subset \mathcal{B}_{R_0}, \tag{3.3}$$

where $\mathbb{B}$ is the same as in (2.14). Then, it follows from estimates (1.10) and (1.12) that there exists a time $T_0 > 0$ such that

$$S_{T_0}\mathcal{B}_{R_0} \subset \mathbb{B} \subset \mathcal{B}_{R_0}. \tag{3.4}$$

As usual, we first construct an exponential attractor $\mathcal{M}_d$ for the discrete semigroup $S^{(m)} := S_{mT_0}$, $m \in \mathbb{N}$, acting on the phase space $\mathcal{B}_{R_0}$ and then extend the result to the desired continuous semigroup S_t.

We first note that, according to Theorem 2.4

$$\mathbb{H}_{R_0 2^{-k}}\left((S^{(k)}\mathbb{B})\big|_{\Omega\cap\mathcal{C}^k_{x_0}}, L^\infty(\mathcal{C}^k_{x_0})\right) \le C_1 \operatorname{vol}\left(\Omega \cap \mathcal{C}^{L_1 k}_{x_0}\right) k, \tag{3.5}$$

where the constants C_1 and L_1 are independent of $k \in \mathbb{N}^\star$ and x_0. Indeed, in order to derive (3.5) from (2.16), it suffices to fix $\varepsilon := R_0 2^{-k}$ and $R = k$ in (2.16), use the embedding $W^{2,p}_b \subset L^\infty$ and fix a sufficiently large constant T_0 in (3.4). Let us now introduce a family of special grids $\mathbb{Z}_\Omega(k)$ in Ω, for every $k \in \mathbb{N}^\star$, as follows:

$$\mathbb{Z}_\Omega(k) := \{x_0 \in k\mathbb{Z}^n, \ \ \mathcal{C}^k_{x_0} \cap \Omega \neq \varnothing\} \tag{3.6}$$

and, for every $k \in \mathbb{N}^\star$ and $l \in \mathbb{Z}_\Omega(k)$, we fix an $R_0 2^{-k}$-net $\mathbb{V}^l_k$ in $(S^{(k)}\mathbb{B})\big|_{\Omega\cap\mathcal{C}^k_l}$ (with respect to the L^∞-metric) such that

$$\mathbb{V}^l_k \subset (S^{(k)}\mathbb{B})\big|_{\Omega\cap\mathcal{C}^k_l} \ \text{ and } \ \ln \#\mathbb{V}^l_k \le C_1 \operatorname{vol}\left(\Omega \cap \mathcal{C}^{L_1 k}_l\right) k \tag{3.7}$$

(it is possible to do so thanks to (3.5)). We now define the $R_0 2^{-k}$-net $\tilde{\mathbb{V}}_k$ in $S^{(k)}\mathbb{B}$ by

$$\tilde{\mathbb{V}}_k := \{v \in L^\infty(\Omega), \ \ v\big|_{\Omega\cap\mathcal{C}^k_l} \in \mathbb{V}^l_k, \ \forall l \in \mathbb{Z}_\Omega(k)\}$$

and modify this net by dropping out the unnecessary elements:

$$\mathbb{V}_k := \{v \in \tilde{\mathbb{V}}_k, \ \ \operatorname{dist}_{L^\infty(\Omega)}\left(v, S^{(k)}\mathbb{B}\right) \le R_0 2^{-k}\}.$$

The most important properties of the sets $\mathbb{V}_k$ introduced above are gathered in the following lemma.

Lemma 3.1. *The sets $\mathbb{V}_k$ satisfy the following conditions:*

1. Let $k \in \mathbb{N}^\star$ be arbitrary. Then, we have

$$\operatorname{dist}^{sym}_{L^\infty(\Omega)}\left(\mathbb{V}_k, S^{(k)}\mathbb{B}\right) \le R_0 2^{-k}, \tag{3.8}$$

where $dist^{sym}_V(X,Y) := \max\{\mathrm{dist}_V(X,Y), \mathrm{dist}_V(Y,X)\}$ *denotes the symmetric Hausdorff distance.*

2. Let, in addition, $m \in \mathbb{N}$ *be arbitrary. Then, we have*

$$\mathrm{dist}_{L^\infty(\Omega)}(\mathbb{V}_{k+m}, \mathbb{V}_k) \le R_0 2^{1-k}. \tag{3.9}$$

3. Let $R \in \mathbb{R}_+$*,* $x_0 \in \Omega$ *and* $\varepsilon > 0$ *be arbitrary. Then, we have*

$$\mathbb{H}_\varepsilon\left(\mathbb{V}_k\big|_{\Omega\cap\mathcal{C}^R_{x_0}}, L^\infty(\mathcal{C}^R_{x_0})\right) \le C_2 \operatorname{vol}\left(\Omega \cap \mathcal{C}^{R+L_2k}_{x_0}\right) k, \tag{3.10}$$

where the constants C_2 *and* L_2 *are independent of* R*,* k *and* x_0*.*

Proof. Estimate (3.8) is an immediate corollary of the definition of $\mathbb{V}_k$ which is given above. Let us verify estimate (3.9), using estimate (3.8):

$$\begin{aligned} \mathrm{dist}_{L^\infty(\Omega)}(\mathbb{V}_{k+m}, \mathbb{V}_k) &\le \mathrm{dist}_{L^\infty(\Omega)}\left(\mathbb{V}_{k+m}, S^{(k)}\mathbb{B}\right) + \\ &+ \mathrm{dist}_{L^\infty(\Omega)}\left(S^{(k)}\mathbb{B}, \mathbb{V}_k\right) \le \mathrm{dist}_{L^\infty(\Omega)}\left(\mathbb{V}_{k+m}, S^{(k+m)}\mathbb{B}\right) + R_0 2^{-k} \le 2R_0 2^{-k}. \end{aligned} \tag{3.11}$$

So, there only remains to verify estimate (3.10). The proof of this estimate is essentially based on (3.7) and on the following obvious subadditivity property of the ε-entropy with respect to the domain in the L^∞-metric: let V_1 and V_2 be two subdomains of $\mathbb{R}^n$, $V := V_1 \cup V_2$ and let $B \subset L^\infty(V)$ be an arbitrary precompact set. Then, the following inequality holds:

$$\mathbb{H}_\varepsilon(B, L^\infty(V)) \le \mathbb{H}_\varepsilon(B\big|_{V_1}, L^\infty(V_1)) + \mathbb{H}_\varepsilon(B\big|_{V_2}, L^\infty(V_2)), \tag{3.12}$$

for every $\varepsilon > 0$. According to (3.12) and (3.7), we have, for every $m \in \mathbb{N}^\star$ and every $l \in \mathbb{Z}_\Omega(k)$

$$\begin{aligned} \mathbb{H}_\varepsilon\left(\mathbb{V}_k\big|_{\Omega\cap\mathcal{C}^{mk}_l}\right) &\le \sum_{l'\in\mathbb{Z}_\Omega(k)\,:|l'_i-l_i|\le km, i=1,\cdots,n} \ln\#\left(\mathbb{V}_k\big|_{\Omega\cap\mathcal{C}^k_{l'}}\right) k \le \\ k &\sum_{l'\in\mathbb{Z}_\Omega(k)\,:|l'_i-l_i|\le km, i=1,\cdots,n} \operatorname{vol}\left(\Omega\cap\mathcal{C}^{L'k}_{l'}\right) \le C'(2L'+1)^n \operatorname{vol}\left(\Omega\cap\mathcal{C}^{(L'+m)k}_l\right) k. \end{aligned} \tag{3.13}$$

Fixing now $m := \left[\frac{R}{k}\right] + 1$ in (3.13), we obtain

$$\mathbb{H}_\varepsilon\left(\mathbb{V}_k\big|_{\Omega\cap\mathcal{C}^R_l}, L^\infty(\mathcal{C}^R_l)\right) \le C_2 \operatorname{vol}\left(\Omega\cap\mathcal{C}^{R+L_2k}_l\right) k, \tag{3.14}$$

where $C_2 := C'(2L'+1)^n$ and $L_2 := L'+1$. Thus, we have proved estimate (3.10) for $x_0 \in \mathbb{Z}_\Omega(k)$. Since $\mathrm{dist}^{sym}_{\mathbb{R}^n}(\Omega, \mathbb{Z}_\Omega(k)) \le \sqrt{n}k$, then, adding $\sqrt{n}$ to L_2, we find (3.10) for every $x_0 \in \Omega$, which finishes the proof of Lemma 3.1.

We are now ready to define the desired exponential attractor $\mathcal{M}_d$ for the discrete semigroup $S^{(k)}$ acting on $\mathcal{B}_{R_0}$:

$$\mathcal{M}'_d := \cup_{k=1}^\infty \cup_{l=0}^\infty S^{(l)}\mathbb{V}_k, \quad \mathcal{M}_d := \left[\mathcal{M}'_d\right]_{L^\infty_{loc}(\Omega)}, \tag{3.15}$$

where $\left[X\right]_V$ denotes the closure of the set X in the space V. Indeed, the exponential attraction property for $\mathcal{M}_d$ thus defined is obvious, due to estimate (3.8) and the fact that $\mathbb{V}_k \subset \mathcal{M}_d$, for every $k \in \mathbb{N}^\star$. So, there only remains to verify the locally compactness of $\mathcal{M}_d$ and the validity of the entropy estimate (3.2). To this end, we need the following lemma, which describes the properties of the sets $S^{(m)}\mathbb{V}_k$ with respect to parameters k and m.

Lemma 3.2. *Let*

$$\mathbb{V}_\infty := \cup_{k=0}^{\infty}\mathbb{V}_k. \tag{3.16}$$

Then, there exists a positive constant α such that

$$\operatorname{dist}_{L^\infty(\Omega)}\left(S^{(m)}\mathbb{V}_k, \mathbb{V}_\infty\right) \le \frac{R_0}{2^{\alpha(k+m)-2}}, \tag{3.17}$$

for every $k \in \mathbb{N}^\star$, $m \in \mathbb{N}$. Moreover, there exist positive constants C_3 and L_3 such that

$$\mathbb{H}_\varepsilon\left(S^{(m)}\mathbb{V}_k\big|_{\Omega\cap\mathcal{C}^R_{x_0}}, L^\infty(\mathcal{C}^R_{x_0})\right) \le C_3 \operatorname{vol}\left(\Omega \cap \mathcal{C}_{x_0}^{R+L_3(k+m+\ln_+\frac{R_0}{\varepsilon})}\right) k, \tag{3.18}$$

for every $\varepsilon > 0$, $R \in \mathbb{R}_+$ and $x_0 \in \Omega$.

Proof. Let us first prove estimate (3.17). To this end, we recall that, according to estimates (1.15) and (1.16) and due to the embeding $W^{2,p} \subset L^\infty$, there exist positive constants L and β such that

$$\|S^{(m)}u_1 - S^{(m)}u_2\|_{L^\infty(\Omega\cap\mathcal{C}^1_{x_0})} \le L^m \sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_1-u_2\|_{L^\infty(\Omega\cap\mathcal{C}^1_x)}\right\} \tag{3.19}$$

holds, for every $u_1, u_2 \in \mathcal{B}_{R_0}$, $x_0 \in \Omega$ and $m \in \mathbb{N}$. In particular, (3.19) implies that

$$\|S^{(m)}u_1 - S^{(m)}u_2\|_{L^\infty(\Omega)} \le L^m\|u_1-u_2\|_{L^\infty(\Omega)}, \quad u_1, u_2 \in \mathcal{B}_{R_0}. \tag{3.20}$$

Estimates (3.20) and (3.8), together with the fact that $\mathbb{V}_k \subset \mathcal{B}_{R_0}$, yield

$$\begin{aligned}\operatorname{dist}_{L^\infty(\Omega)}\left(S^{(m)}\mathbb{V}_k, \mathbb{V}_{k+m}\right) &\le \operatorname{dist}_{L^\infty(\Omega)}\left(S^{(m)}\mathbb{V}_k, S^{(m+k)}\mathbb{B}\right) + \\ &+ \operatorname{dist}_{L^\infty(\Omega)}\left(S^{(m+k)}\mathbb{B}, \mathbb{V}_{m+k}\right) \le R_0L^m2^{-k} + R_02^{-(k+m)} \le 2R_0L^m2^{-k}.\end{aligned} \tag{3.21}$$

On the other hand, since $\mathbb{V}_k \subset \mathcal{B}_{R_0}$, then, due to (3.8)

$$\operatorname{dist}_{L^\infty(\Omega)}\left(S^{(m)}\mathbb{V}_k, \mathbb{V}_m\right) \le R_02^{-m}. \tag{3.22}$$

Combining (3.21) and (3.22), we have

$$\begin{aligned}\operatorname{dist}_{L^\infty(\Omega)}\left(S^{(m)}\mathbb{V}_k, \mathbb{V}_\infty\right) &\le \\ &\le 2R_02^{-\alpha(k+m)}\min\{2^{-m+\alpha(k+m)}, L^m2^{-k+\alpha(k+m)}\},\end{aligned} \tag{3.23}$$

for every $\alpha \in \mathbb{R}$. Fixing now $\alpha := \frac{\ln 2}{\ln 2+\ln L}$, we derive from (3.23) that

$$\operatorname{dist}_{L^\infty(\Omega)}\left(S^{(m)}\mathbb{V}_k, \mathbb{V}_\infty\right) \le R_02^{\alpha^2+1-\alpha(k+m)}, \tag{3.24}$$

which proves (3.17). There remains to note that estimate (3.18) is an immediate corollary of (3.10) and (3.19) and Lemma 3.2 is proved.

It is now easy to derive the desired entropy estimate for the attractor $\mathcal{M}_d$. Indeed, let $\varepsilon > 0$ be sufficiently small. Then, it follows from estimates (3.9) and (3.17) that

$$\operatorname{dist}_{L^\infty(\Omega)}\left(\mathcal{M}'_d, \bigcup_{k+m\le\kappa\ln_+\frac{R_0}{\varepsilon}} S^{(m)}\mathbb{V}_k\right) \le \frac{\varepsilon}{2}, \tag{3.25}$$

for some positive constant κ which is independent of ε. Let now $R \in \mathbb{R}_+$ and $x_0 \in \Omega$ be arbitrary. We need to estimate the ε-entropy of the restriction $\mathcal{M}_d\big|_{\Omega\cap\mathcal{C}^R_{x_0}}$. To this end, it is sufficient, due to (3.25), to estimate the $\varepsilon/2$-entropy of $S^{(m)}\mathbb{V}_k\big|_{\Omega\cap\mathcal{C}^R_{x_0}}$, for $k+m \le \kappa\ln_+\frac{R_0}{\varepsilon}$ only. Using estimate (3.18), we finally find

$$\mathbb{H}_\varepsilon\left(\mathcal{M}_d\big|_{\Omega\cap\mathcal{C}^R_{x_0}}, L^\infty(\mathcal{C}^R_{x_0})\right) \le \\ \le C_4\operatorname{vol}\left(\Omega\cap\mathcal{C}^{R+L_4\ln\frac{R'_0}{\varepsilon}}_{x_0}\right)\ln_+\frac{R'_0}{\varepsilon} + n^2\ln_+\ln_+\frac{R'_0}{\varepsilon}, \tag{3.26}$$

where the constants C_4, L_4 and R'_0 are independent of R, x_0 and ε. Since the second term in the right-hand side of (3.26) is obviously subordinated to the first one, we have verified that the set $\mathcal{M}$ is indeed an exponential attractor for the semigroup $S^{(m)}$ acting on $\mathcal{B}_{R_0}$ (in the sense of Definition 3.2).

We are now ready to complete the proof of Theorem 3.1 by introducing the desired exponential attractor for the continuous semigroup S_t via the following standard expression:

$$\mathcal{M} := \cup_{T_0\le t\le 2T_0}\mathcal{M}_d. \tag{3.27}$$

Indeed, it follows from estimate (1.16) that, for every u_1 and u_2 belonging to $\mathcal{B}_{R_0}$ and every $x_0 \in \Omega$, the following estimate is valid:

$$\|S_tu_1 - S_tu_2\|_{W^{2,p}_b(\Omega\cap\mathcal{C}^1_{x_0})} \le L\sup_{x\in\Omega}\left\{e^{-\beta|x-x_0|}\|u_1-u_2\|_{L^\infty(\Omega\cap\mathcal{C}^1_x)}\right\}, \tag{3.28}$$

where $T_0 \le t \le 2T_0$ and the positive constants L and β are independent of x_0, t, u_1 and u_2. Estimate (3.28) implies, in particular, that

$$\|S_tu_1 - S_tu_2\|_{\Phi_b} \le L\|u_1-u_2\|_{L^\infty(\Omega)}, \quad t\in[T_0,2T_0], \quad u_1,u_2\in\mathcal{B}_{R_0}. \tag{3.29}$$

Let us verify that the set $\mathcal{M}$ defined by (3.27) satisfies all the assumptions of Definition 3.2. Indeed, the first and second assumptions follow immediately from definition (3.27) and from analogous properties for the discrete exponential attractor $\mathcal{M}_d$. The exponential attraction property (3.1) follows from the facts that $\mathcal{B}_{R_0}$ is an absorbing set for S_t and $\mathcal{M}_d$ is an exponential attractor for the discrete semigroup $S^{(m)} := S_{T_0m}$ and from estimate (3.29). So, there only remains to verify the entropy estimate (3.2). To this end, we recall that, due to Theorem 1.2, $\|\partial_tS_tu_0\|_{\Phi_b}$ is uniformly bounded with respect to $t\in\mathbb{R}_+$ and $u_0\in S_{T_0}\mathcal{B}_{R_0}$ and, consequently

$$\|S_{t+h}u_0 - S_tu_0\|_{\Phi_b} \le Mh, \quad t\ge T_0, \quad h\ge 0, \quad u_0\in S_{T_0}\mathcal{B}_{R_0}, \tag{3.30}$$

for some positive constant M which is independent of t, h and u_0. Estimates (3.26), (3.28) and (3.30) imply in a standard way that the set $\mathcal{M}$ defined in (3.27) satisfies the entropy estimate (3.3), which completes the proof of Theorem 3.1.

Remark 3.1. Let us apply Theorem 3.1 to the particular case where Ω is a bounded regular domain. Taking the radius R in (3.2) larger that the diameter of Ω, we have

$$\mathbb{H}_\varepsilon\left(\mathcal{M}, W^{2,p}_b(\Omega)\right) \le C' \operatorname{vol}(\Omega) \ln_+ \frac{R'_0}{\varepsilon}, \tag{3.31}$$

which shows that the exponential attractor $\mathcal{M}$ constructed in Theorem 3.1 has finite fractal dimension and, therefore, is an exponential attractor for (1.1) in the classical sense of Definition 3.1. Nevertheless, even in the case of bounded domains Ω, the construction of exponential attractors given in Theorem 3.1 has an additional advantage over the classical construction of exponential attractors in bounded domains (see [16]). Indeed, the usual scheme gives an exponential attractor which only satisfies (3.31), but our scheme allows to obtain (3.2) and to control not only the entropy of the whole attractor $\mathcal{M}$, but also the entropy of all their restrictions $\mathcal{M}\big|_{\mathcal{C}^R_{x_0}}$. In particular, for the restriction $\mathcal{M}\big|_{\mathcal{C}^1_{x_0}}$ of $\mathcal{M}$ to the unit cube we obtain the following estimate:

$$\mathbb{H}_\varepsilon\left(\mathcal{M}\big|_{\mathcal{C}^1_{x_0}}, W^{2,p}(\mathcal{C}^1_{x_0})\right) \le C' \operatorname{vol}\left(\Omega \cap \mathcal{C}^{L' \ln_+ \frac{R'_0}{\varepsilon}}_{x_0}\right) \ln_+ \frac{R'_0}{\varepsilon}. \tag{3.32}$$

We note that the right-hand side of (3.32) is essentially smaller than the right-hand side of (3.31) if the domain Ω and ε satisfy

$$\ln_+ \frac{R'_0}{\varepsilon} \ll \operatorname{diam}(\Omega), \tag{3.33}$$

which is natural in the case of large bounded domains Ω.

§4 Approximation of infinite-dimensional exponential attractors.

The main task of this section is to study the approximation of the infinite-dimensional exponential attractor $\mathcal{M}$ associated with equation (1.1) in an unbounded domain Ω constructed in the previous section by finite-dimensional exponential attractors $\mathcal{M}_{\Omega_r}$ associated with equations (1.1) in *bounded* domains Ω_r such that $\Omega_r \to \Omega$ as $r \to \infty$. For simplicity, we restrict ourselves to the case $\Omega := \mathbb{R}^n$ and $\Omega_r := B^r_0$ only, although similar results hold for an essentially larger class of unbounded domains Ω and their approximations Ω_r.

In order to approximate the dynamics associated with equation (1.1) in an unbounded domain $\Omega = \mathbb{R}^n$, we consider the following problem in a ball $\Omega_r := B^r_0$:

$$\begin{cases} \partial_t u = a\Delta_x u - (\vec{L}, \nabla_x)u - \lambda_0 u - f(u) + g, \\ u\big|_{B^r_0} = 0, \quad u\big|_{t=0} = u_0. \end{cases} \tag{4.1}$$

Then, since the domains Ω_r satisfy the regularity assumptions of Definition 1.1, uniformly with respect to $r \ge 1$, all the estimates mentioned in Section 1 and

2 hold for the solutions of (4.1) uniformly with respect to $r \geq 1$. In particular, equation (4.1) generates a semigroup S_t^r in the phase space $L^\infty(\Omega_r)$ via

$$S_t^r : L^\infty(\Omega_r) \to L^\infty(\Omega_r), \quad S_t^r u_0 := u_r(t), \tag{4.2}$$

where $u_r(t)$ is the unique solution of (4.1). Moreover, this semigroup possesses the compact global attractor $\mathcal{A}_r \subset W^{2,p}(\Omega_r)$ and (thanks to Theorem 2.3) this attractor has a finite fractal dimension satisfying

$$\dim_F\left(\mathcal{A}_r, W^{2,p}(\Omega_r)\right) \leq C \operatorname{vol}(\Omega_r) = C_1 r^n, \tag{4.3}$$

where the constant C_1 is independent of r. Nevertheless, as the following example shows, the problem of approximating the dynamics associated with equation (1.1) in $\Omega = \mathbb{R}^n$ by equations (4.1) in bounded domains Ω_r is usually not solvable in terms of global attractors.

Example 4.1. Let us consider the Chafee-Infante equation perturbed by a sufficiently large transport term:

$$\partial_t u = \Delta_x u - L\partial_{x_1} u + u - u^3, \quad x \in \mathbb{R}^n, \quad |L| > 2. \tag{4.4}$$

Then, on the one hand, equation (4.4) satisfies all the assumptions of Theorem 2.3 (see [29]) and, consequently, its global attractor $\mathcal{A}$ has infinite fractal dimension (see the lower estimates (2.12) and (2.13)) and equation (4.4) generates highly nontrivial dynamics on it (which e.g. has an infinite topological entropy, see [29]). On the other hand, it can be easily shown that, for the analogous equation in any *bounded* domain $\Omega' \subset\subset \mathbb{R}^n$

$$\partial_t u = \Delta_x u - L\partial_{x_1} u - u + u^3, \quad x \in \Omega', \quad u\big|_{\partial\Omega'} = 0, \tag{4.5}$$

its attractor $\mathcal{A}_{\Omega'}$ consists of the unique zero equilibrium:

$$\mathcal{A}_{\Omega'} = \{0\}. \tag{4.6}$$

Thus, attractors $\mathcal{A}_{\Omega_r}$ do not approximate $\mathcal{A}$ as $r \to \infty$.

The main result of this section is the following theorem, which shows that the above approximation problem can be reasonably solved under the exponential attractors' approach.

Theorem 4.1. *Let $\Omega := \mathbb{R}^n$, the assumptions of Theorem 1.1 be satisfied and $\mathcal{M} = \mathcal{M}_\infty$ be the infinite-dimensional exponential attractor for* (1.1) *constructed in Section 3. Then, there exists a family of finite-dimensional exponential attractors $\mathcal{M}(r)$, $r \geq 1$, of problems* (4.1) *which satisfies the following properties:*

1. The sets $\mathcal{M}(r)$ are uniformly (with respect to r) bounded in $W_b^{2,p}(\Omega_r)$.

2. There exist a positive constant $\alpha > 0$ and a monotonic function Q which are independent of r such that

$$\operatorname{dist}_{W_b^{2,p}(\Omega_r)}\left(S_t^r B, \mathcal{M}(r)\right) \leq Q(\|B\|_{W_b^{2,p}(\Omega_r)})e^{-\alpha t}, \tag{4.7}$$

for every bounded subset $B \subset W_b^{2,p}(\Omega_r)$ (uniform exponential attraction property).

3. The entropy of the restrictions $\mathcal{M}(r)\big|_{\Omega_r\cap\mathcal{C}^R_{x_0}}$ satisfies the following analogue of (3.2):

$$\text{(4.8)}\quad \mathbb{H}_\varepsilon\left(\mathcal{M}(r)\big|_{\Omega_r\cap\mathcal{C}^R_{x_0}}, W^{2,p}_b(\Omega\cap\mathcal{C}^R_{x_0})\right)\le C''\operatorname{vol}\left(\Omega_r\cap\mathcal{C}^{R+L''\ln_+\frac{R_0''}{\varepsilon}}_{x_0}\right)\ln_+\frac{R_0''}{\varepsilon},$$

where the constants C'', L'' and R_0'' are independent of r, R, ε and x_0.

4. The attractors $\mathcal{M}(r)$ tend to $\mathcal{M}_\infty$ as $r\to\infty$ in the following sense:

$$\text{(4.9)}\qquad \operatorname{dist}^{sym}_{W^{2,p}_b(B^R_0)}\left(\mathcal{M}(r)\big|_{B^R_0},\mathcal{M}_\infty\big|_{B^R_0}\right)\le\widehat{R}_0e^{-\gamma(r-R)},$$

where the positive constants $\widehat{R}_0$ and γ are independent of $r\ge1$ and $R\le r$.

Proof. Let the set $\mathcal{B}_{R_0}\subset L^\infty(\mathbb{R}^n)$ and $\mathbb{B}\subset\Phi_b:=\Phi_b(\mathbb{R}^n)$ be the same as in the proof of Theorem 3.1. We set

$$\text{(4.10)}\qquad \mathcal{B}_{R_0}(r):=\mathcal{B}_{R_0}\big|_{\Omega_r},\quad \mathbb{B}(r):=\mathbb{B}\big|_{\Omega_r}\cap\{u\big|_{\partial\Omega_r}=0\}.$$

Then, due to the fact that all the estimates of Section 1 hold for the solutions of equation (4.1) uniformly with respect to r, we note that the sets $\mathcal{B}_{R_0}(r)$ are uniform (with respect to r) absorbing sets for semigroups S^r_t and that

$$\text{(4.11)}\qquad \mathbb{B}(r)\subset\mathcal{B}_{R_0}(r),\quad S^r_{T_0}\mathcal{B}_{R_0}(r)\subset\mathbb{B}(r)\subset\mathcal{B}_{R_0}(r),$$

where the time T_0 is the same as in the proof of Theorem 3.1. As above, we introduce the discrete semigroups $S^{(l)}_r:=S^r_{lT_0}$, $l\in\mathbb{N}$, and first construct the family of exponential attractors $\mathcal{M}_d(r)$ for these semigroups acting on $\mathcal{B}_{R_0}(r)$. To this end, we need the following lemma.

Lemma 4.1. *Let $u_0\in\mathbb{B}$ and $u^r_0\in\mathbb{B}(r)$. Let also $u(t):=S_tu_0$ and $u^r(t):=S^r_tu^r_0$ be the corresponding solutions of equations (1.1) and (4.1). Then, the following estimates hold:*

$$\text{(4.12)}\quad \|u(t)-u^r(t)\|_{W^{2,p}(\mathcal{C}^1_{x_0})}\le \\ \le Ce^{Kt}\left(\sup_{x\in\Omega_r}\{e^{-\alpha|x-x_0|}\|u_0-u^r_0\|_{W^{2,p}(\mathcal{C}^1_x)}\}+e^{-\alpha\operatorname{dist}_{\mathbb{R}^n}(x_0,\partial\Omega_r)}\right)$$

and

$$\text{(4.13)}\quad \|u(t)-u^r(t)\|_{L^\infty(\mathcal{C}^1_{x_0})}+\|u(t)-u^r(t)\|_{W^{2,p}(\mathcal{C}^1_{x_0})}\le \\ \le Ce^{Kt}\frac{t^N+1}{t^N}\left(\sup_{x\in\Omega_r}\{e^{-\alpha|x-x_0|}\|u_0-u^r_0\|_{L^\infty(\mathcal{C}^1_x)}\}+e^{-\alpha\operatorname{dist}_{\mathbb{R}^n}(x_0,\partial\Omega_r)}\right),\ t>0,$$

where the positive constants C, K and α are independent of r, x_0, u_0 and u^r_0.

Proof of Lemma 4.1. The difference $v(t):=u(t)-u^r(t)$ satisfies the following linear equation with nonhomogeneous boundary conditions:

$$\text{(4.14)}\quad \partial_tv=a\Delta_xv-(\vec{L},\nabla_x)v-l(t)v,\quad v\big|_{t=0}=u_0-u^r_0,\quad v\big|_{\partial\Omega_r}=v^0:=u\big|_{\partial\Omega_r},$$

where $l(t) := \int_0^1 f'(su(t) + (1-s)u^r(t))\,ds$. Since $u_0 \in \mathbb{B}$ and $u_0^r \in \mathbb{B}(r)$, then (due to Theorem 1.1)

$$\|l(t)\|_{W_b^{1,p}(\Omega_r)\cap C(\Omega_r)} \le C, \tag{4.15}$$

where C is independent of u_0, u_0^r, t and r. Moreover, the trace $v^0 := u(t)\big|_{\partial\Omega_r}$ belongs to the corrresponding trace space $W_b^{(1-1/(2p),2-1/p),p}(\mathbb{R}_+ \times \partial\Omega_r)$ and satisfies

$$\|v^0\|_{W_b^{(1-1/(2p),2-1/p),p}(\mathbb{R}_+\times\partial\Omega_r)} \le C_1,$$

where C_1 is also independent of r, u_0 and u_1, see e.g. [29]. Applying now the standard parabolic regularity estimate to equation (4.14) (see [20]), we have

$$\|v(t)\|_{W^{2(1-1/p),2}(\Omega_r\cap\mathcal{C}_{x_0}^1)} \le$$
$$\le Ce^{Kt}\left(\sup_{x\in\Omega_r}\{e^{-\alpha|x-x_0|}\|v(0)\|_{W^{2(1-1/p),p}(\Omega_r\cap\mathcal{C}_x^1)}\} + e^{-\alpha\,\mathrm{dist}_{\mathbb{R}^n}(x_0,\partial\Omega_r)}\right).$$

Using this estimate, the fact that $u(t)$ and $u^r(t)$ are uniformly bounded in $\Phi_b(\mathbb{R}^n)$ and $\Phi_b(\Omega_r)$ respectively and applying the classical interior parabolic estimates (see e.g. [20]) to the linear equation (4.14), we derive estimates (4.12) and (4.13) and finish the proof of the lemma.

The following important estimate is an immediate corollary of Lemma 4.1:

$$\mathrm{dist}^{sym}_{L^\infty(\Omega')}\left(S^{(k)}\mathbb{B}, S_r^{(k)}\mathbb{B}(r)\right) \le Ce^{Lk-\alpha d_{\mathbb{R}^n}(\Omega',\partial\Omega_r)}, \tag{4.16}$$

where $d_V(X,Y) := \inf_{(x,y)\in X\times Y}\|x-y\|_V$ denotes the distance between sets X and Y with respect to the metric of the space V and the positive constants C, L and α are independent of $r \ge 1$ and $\Omega' \subset \Omega_r$.

We are now ready to construct the discrete exponential attractors $\mathcal{M}_d(r)$. To this end, we introduce, for every $k \in \mathbb{N}^\star$, the grids $\mathbb{Z}_{\mathbb{R}^n}(k)$ and $\mathbb{Z}_{\Omega_r}(k)$ analogously to (3.6). We also fix, for every $k \in \mathbb{N}^\star$ and every $l \in \mathbb{Z}_{\Omega_r}(k)$, an $R_0 2^{-k}$-net $\mathbb{V}_k^l(r)$ in $(S^{(k)}\mathbb{B}(r))\big|_{\Omega_r\cap\mathcal{C}_l^k}$ (with respect to the $L^\infty(\Omega_r)$-metric) such that

$$\mathbb{V}_k^l(r) \subset (S_r^{(k)}\mathbb{B}(r))\big|_{\Omega_r\cap\mathcal{C}_l^k} \quad\text{and}\quad \ln\#\mathbb{V}_k^l \le C_2\,\mathrm{vol}\left(\Omega_r\cap\mathcal{C}_l^{L_2k}\right)k, \tag{4.17}$$

where the constants C_2 and L_2 are independent of r, k and l (it is possible to do so thanks to Theorem 2.4 and to the fact that domains Ω_r satisfy the regularity assumptions of Definition 1.1 uniformly with respect to $r \ge 1$). Let the sets $\mathbb{V}_k$ be the same as in the proof of Theorem 3.1. We also set $M := \frac{L+\ln 2}{\alpha} + \sqrt{n}$ and define the additional grids

$$\mathbb{Z}^{int}_{\Omega_r}(k) := \{l \in \mathbb{Z}_{\Omega_r}(k),\ \mathrm{dist}_{\mathbb{R}^n}(l,\partial\Omega_r) \ge Mk\},\ \mathbb{Z}^{ext}_{\Omega_r}(k) := \mathbb{Z}_{\Omega_r}(k)\backslash\mathbb{Z}^{int}_{\Omega_r}(k) \tag{4.18}$$

and we set $\Omega_r^{int}(k) := \cup_{l\in\mathbb{Z}^{int}_{\Omega_r}(k)}\mathcal{C}_l^k$. Then, according to (3.8) and (4.16), we have

$$\mathrm{dist}^{sym}_{L^\infty(\Omega_r^{int}(k))}\left((S_r^{(k)}\mathbb{B}(r))\big|_{\Omega_r^{int}(k)}, \mathbb{V}_k\big|_{\Omega_r^{int}(k)}\right) \le R_0' 2^{-k}, \tag{4.19}$$

where the constant R_0' is independent of k and r.

We are now ready to define a sequence of sets $\mathbb{V}_k(r)$, which are the analogues of sets $\mathbb{V}_k$ for problems (4.1), as follows:

$$\tilde{\mathbb{V}}_k(r) := \big\{v \in L^\infty(\Omega_r), \quad v\big|_{\Omega_r^{int}(k)} \in \mathbb{V}_k\big|_{\Omega_r^{int}(k)},\ v\big|_{\Omega_r\cap\mathcal{C}_l^k} \in \mathbb{V}_k^l(r),\ \forall l \in \mathbb{Z}_{\Omega_r}^{ext}(k)\big\} \tag{4.20}$$

and

$$\mathbb{V}_k(r) := \big\{v \in \tilde{\mathbb{V}}_k(r),\quad \mathrm{dist}_{L^\infty(\Omega_r)}(v, S_r^{(k)}\mathbb{B}(r)) \le R_0' 2^{-k}\big\}. \tag{4.21}$$

Then, on the one hand, according to (4.19)-(4.21), we have

$$\mathbb{V}_k(r)\big|_{\Omega_r^{int}(k)} = \mathbb{V}_k\big|_{\Omega_r^{int}(k)} \tag{4.22}$$

and, on the other hand, the assertions of Lemmata 3.1 and 3.2 hold for these sets.

Lemma 4.2. *The sets $\mathbb{V}_k(r)$ defined above satisfy the following conditions:*

1. Let $k \in \mathbb{N}^\star$ be arbitrary. Then

$$\mathrm{dist}^{sym}_{L^\infty(\Omega_r)}\Big(\mathbb{V}_k(r), S_r^{(k)}\mathbb{B}(r)\Big) \le R_0' 2^{-k}. \tag{4.23}$$

2. Let, in addition, $m \in \mathbb{N}$ be arbitrary. Then

$$\mathrm{dist}_{L^\infty(\Omega_r)}(\mathbb{V}_{k+m}, \mathbb{V}_k) \le R_0' 2^{-k}. \tag{4.24}$$

3. Let $\mathbb{V}_\infty(r) := \cup_{k=0}^\infty \mathbb{V}_k(r)$. Then

$$\mathrm{dist}_{L^\infty(\Omega_r)}\Big(S_r^{(m)}\mathbb{V}_k(r), \mathbb{V}_\infty(r)\Big) \le R_0' 2^{-\gamma(k+m)}. \tag{4.25}$$

4. Let $R \in \mathbb{R}_+$, $x_0 \in \Omega_r$, $m \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. Then

$$\mathbb{H}_\varepsilon\left(S_r^{(m)}\mathbb{V}_k(r)\big|_{\Omega_r\cap\mathcal{C}^R_{x_0}}, L^\infty(\mathcal{C}^R_{x_0})\right) \le C' \,\mathrm{vol}\left(\Omega_r \cap \mathcal{C}^{R+L'(k+m+\ln_+\frac{R_0'}{\varepsilon})}\right) k. \tag{4.26}$$

Moroeover, all the constants in estimates (4.23)–(4.26) are independent of $r \ge 1$, $k \in \mathbb{N}^\star$, $m \in \mathbb{N}$, $\varepsilon > 0$ and $x_0 \in \Omega_r$.

The proof of this Lemma repeats word by word that of lemmata 3.1 and 3.2 and is omitted here.

After obtaining the uniform (with respect to r) estimates (4.23)–(4.26) for sets $V_k(r)$, we can (analogously to (3.15)) define the desired uniform family $\mathcal{M}_d(r)$ of discrete exponential attractors for semigroups $S_r^{(l)}$ on $\mathcal{B}_{R_0}(r)$:

$$\mathcal{M}_d'(r) := \cup_{k=1}^\infty \cup_{l=0}^\infty S_r^{(l)}\mathbb{V}_k(r),\quad \mathcal{M}_d(r) := \big[\mathcal{M}_d'(r)\big]_{L^\infty(\Omega_r)}. \tag{4.27}$$

Indeed, arguing as in the proof of Theorem 3.1 (see (3.25)–(3.26)), we easily verify that the discrete exponential attractors $\mathcal{M}_d(r)$ satisfy the first three assertions of Theorem 4.1 (in the L^∞-metric instead of the $W_b^{2,p}(\Omega_r)$-metric). So, there only remains to verify the symmetric distance estimate (4.9). To this end, we need the following lemma.

Lemma 4.3. *The following estimate is valid, for every $r \geq 1$, $R \leq r$ and $k \in \mathbb{N}^\star$:*

$$\begin{aligned}(4.28)\quad & \operatorname{dist}_{L^\infty(B_0^R)}\left(\mathbb{V}_k(r)\big|_{B_0^R}, \mathbb{V}_\infty\big|_{B_0^R}\right) + \\ & + \operatorname{dist}_{L^\infty(B_0^R)}\left(\mathbb{V}_k\big|_{B_0^R}, \mathbb{V}_\infty(r)\big|_{B_0^R}\right) \leq R_0'' e^{-\gamma'(r-R)},\end{aligned}$$

where the positive constants R_0'' and γ' are independent of $r \geq 1$, $R \leq r$ and $k \in \mathbb{N}^\star$.

Proof. Let $r \geq 1$ and $R \leq r$ be fixed. We set

$$k_0 := \left[\frac{r-R}{M+\sqrt{n}}\right]. \tag{4.29}$$

Then, for $k \leq k_0$, we have $B_0^R \subset \Omega_r^{int}(k)$ and, consequently, due to (4.22), the left-hand side of (4.28) vanishes. Therefore, there remains to verify (4.28) only for $k \geq k_0$. Let us verify the first part of estimate (4.28), the second one being proved analogously.

For every $l \leq k$, we have

$$\operatorname{dist}_{L^\infty(\Omega_r)}\left(\mathbb{V}_k(r), S_r^{(l)}\mathbb{B}(r)\right) \leq R_0' 2^{-k}. \tag{4.30}$$

On the other hand, estimate (4.16) yields

$$\operatorname{dist}^{sym}_{L^\infty(B_0^R)}\left(S_r^{(l)}\mathbb{B}(r)\big|_{B_0^R}, S^{(l)}\mathbb{B}\big|_{B_0^R}\right) \leq R_0' e^{Ll-\alpha(r-R)}. \tag{4.31}$$

Combining estimates (4.30)–(4.31) and (3.8), we obtain

$$\operatorname{dist}_{L^\infty(B_0^R)}\left(\mathbb{V}_k(r)\big|_{B_0^R}, \mathbb{V}_l\big|_{B_0^R}\right) \leq R_0'\left(2^{-l} + e^{Ll-\alpha(r-R)}\right). \tag{4.32}$$

We now recall that $k \geq k_0$ and that $l \leq k$ is arbitrary. Setting $l = k_0$ and using (4.29) and our choice of constant M, we deduce from (4.32) that

$$\operatorname{dist}_{L^\infty(B_0^R)}\left(\mathbb{V}_k(r)\big|_{B_0^R}, \mathbb{V}_l\big|_{B_0^R}\right) \leq R_0'' e^{-\gamma'(r-R)}, \tag{4.33}$$

where the positive constants R_0'' and γ' are independent of r, k and $R \leq r$, which proves the first part of estimate (4.28). The second one can be proved analogously and Lemma 4.3 is proved.

In order to obtain the desired estimate for the symmetric distance between the exponential attractors $\mathcal{M}_d(r)$ and $\mathcal{M}_d$, there remains, according to (4.27) and (4.28), to estimate the distance between $S_r^{(l)}\mathbb{V}_k(r)$ and $\mathcal{M}_d$ and between $S^{(l)}\mathbb{V}_k$ and $\mathcal{M}_d(r)$. To this end, we note that estimates (4.13) and (4.28) imply

$$\begin{aligned}(4.34)\quad & \operatorname{dist}_{L^\infty(B_0^R)}\left(S_r^{(l)}\mathbb{V}_k(r)\big|_{B_0^R}, \mathcal{M}_d\big|_{B_0^R}\right) + \\ & + \operatorname{dist}_{L^\infty(B_0^R)}\left(S^{(l)}\mathbb{V}_k\big|_{B_0^R}, \mathcal{M}_d(r)\big|_{B_0^R}\right) \leq R_0'' e^{Ll-\gamma''(r-R)},\end{aligned}$$

where the positive constants L, R_0'' and γ'' are independent of l, k and $R\le r$. On the other hand, for every sufficiently small $\delta>0$, estimates (4.25), (4.28) and (3.17) yield

$$\begin{aligned}(4.35)\quad &\operatorname{dist}_{L^\infty(B_0^R)}\left(S_r^{(l)}\mathbb{V}_k(r)\big|_{B_0^R},\mathcal{M}_d\big|_{B_0^R}\right)+\\ &\qquad+\operatorname{dist}_{L^\infty(B_0^R)}\left(S^{(l)}\mathbb{V}_k\big|_{B_0^R},\mathcal{M}_d(r)\big|_{B_0^R}\right)\le R_0''e^{-\delta(r-R))},\end{aligned}$$

if k and l satisfy the inequality

$$k+l\ge\frac{\delta}{\gamma\ln 2}(r-R). \tag{4.36}$$

Combining (4.34)–(4.36) and taking $\delta>0$ small enough, we finally have

$$\operatorname{dist}^{sym}_{L^\infty(B_0^R)}\left(\mathcal{M}_d(r)\big|_{B_0^R},\mathcal{M}_d\big|_{B_0^R}\right)\le R_0''e^{-\gamma''(r-R)}, \tag{4.37}$$

where the positive constants R_0'' and γ'' are independent of $r\ge1$ and $R\le r$.

Thus, the desired uniform family $\mathcal{M}_d(r)$ of discrete exponential attractors for semigroups $S_r^{(l)}$, $l\in\mathbb{N}$, acting on $\mathcal{B}_{R_0}(r)$ is indeed constructed. There now only remains to define, analogously to (3.27), the uniform family $\mathcal{M}(r)$ of exponential attractors for the continuous semigroups S_t^r by the following standard expression:

$$\mathcal{M}(r):=\bigcup_{T_0\le t\le 2T_0}\mathcal{M}_d(r). \tag{4.38}$$

Using estimates (1.12), (1.16) and (4.13) and arguing as in the end of the proof of Theorem 3.1, we can easily verify that this uniform family of exponential attractors satisfies indeed all the assumptions of the theorem and Theorem 4.1 is proved.

Corollary 4.1. *Let the assumptions of Theorem 4.1 hold and let $\mathcal{M}(r)$ be the uniform family of exponential attractors constructed in this theorem. Then, for every $\varepsilon>0$ and every $R>0$, the following estimate holds:*

$$\operatorname{dist}^{sym}_{W_b^{2,p}(B_0^R)}\left(\mathcal{M}(R+K\ln_+\frac{R_0}{\varepsilon})\big|_{B_0^R},\mathcal{M}_\infty\big|_{B_{x_0}^R}\right)\le\varepsilon, \tag{4.39}$$

where the constants R_0 and K are independent of ε and R.

Indeed, (4.39) is an immediate corollary of (4.9).

Remark 4.1. Estimate (4.39) shows that, in order to approximate the attractor $\mathcal{M}_\infty$ with an accuracy ε inside the ball of radius R, one should consider an exponential attractor of equation (4.1) in a ball of radius $R+K\ln_+\frac{R_0}{\varepsilon}$. We now note that one cannot expect that $\mathcal{M}(r)$ approximates $\mathcal{M}_\infty$ in the whole ball Ω_r, since the solutions of (4.1) should satisfy the boundary conditions at $\partial\Omega_r$. Nevertheless, estimates (4.9) and (4.39) show that the influence of the boundary (and of the boundary conditions) decays exponentially with respect to the distance to the boundary (in complete agreement with physical intuition).

Remark 4.2. Although we have only considered the case $\Omega:=\mathbb{R}^n$ and $\Omega_r:=B_0^r$ in this section, a little more accurate analysis of the proof of Theorem 4.1 shows that these assumptions are not essential and can be weakened as follows: Ω is an arbitrary regular unbounded domain in the sense of Definition 1.1 and $\Omega_r \subset \Omega$, $r \in \mathbb{R}_+^\star$, $\Omega_{r_1} \subset \Omega_{r_2}$ if $r_1 \le r_2$, is an arbitrary sequence of (bounded) regular domains which satisfies the assumptions of Definition 1.1 *uniformly* with respect to $r \in \mathbb{R}_+^\star$. Then, the analogue of the main estimate (4.9) reads

$$\operatorname{dist}^{sym}_{W^{2,p}_b(\Omega_R)}\left(\mathcal{M}(r)\big|_{\Omega_R}, \mathcal{M}_\infty\big|_{\Omega_R}\right) \le \widehat{R}_0 e^{-\gamma d_{\mathbb{R}^n}(\partial\Omega_R, \partial\Omega_r \setminus \partial\Omega)}, \tag{4.40}$$

where the positive constants $\widehat{R}_0$ and γ are independent of $r \ge 1$ and $R \le r$.

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