# INFINITE DIMENSIONAL EXPONENTIAL ATTRACTORS FOR A NON-AUTONOMOUS REACTION-DIFFUSION SYSTEM

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Abstract. In this article, we give a construction of exponential attractors that is valid for general translation-compact non-autonomous systems. Since they are generally infinite dimensional, we replace, compared with the standard definition, the condition of finite fractal dimensionality of exponential attractors by requiring that their epsilon-entropy have the same form as that of the uniform attractor. As an example, we prove the existence of an (infinite dimensional) exponential attractor for a reaction-diffusion system.

## Introduction

Many equations arising from physics and mechanics (e.g. the two-dimensional Navier-Stokes equations, reaction-diffusion systems, damped wave equations, ...) possess the finite dimensional (in the sense of the Hausdorff or the fractal dimension) global attractor, which is a compact invariant set which attracts the trajectories as time goes to infinity. Since it is the smallest closed set enjoying these properties, it is a suitable set for the study of the asymptotic behavior of the system. We refer the reader to [3], [21] and [34] for extensive reviews on this subject.

Now, the global attractor may present two major defaults for practical purposes. Indeed, the rate of attraction of the trajectories may be small and (consequently) it may be sensible to perturbations. Also, it is important to compute global bifurcations of the global attractor, which is a very difficult task. Since (when) the global attractor has finite

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dimension, it is natural to expect that it can be covered by solving a (large but finite) system of ordinary differential equations. One way of obtaining such a system would be to embed the attractor into a smooth positively invariant finite dimensional manifold.

In order to overcome these difficulties, Foias, Sell and Temam proposed in [19] the notion of inertial manifold, which is a smooth finite dimensional hyperbolic (and thus robust) positively invariant manifold which contains the global attractor and attracts exponentially the trajectories. Unfortunately, all the known constructions of inertial manifolds are based on a restrictive condition, the so-called spectral gap condition. Consequently, the existence of inertial manifolds is not known for many physically important equations (e.g. the Navier-Stokes equations, even in two space dimensions). A non-existence result has even been obtained by Mallet-Paret and Sell for a reaction-diffusion equation in higher-space dimensions.

Thus, as an intermediate object between the two ideal objects that the global attractor and an inertial manifold are, Eden, Foias, Nicolaenko and Temam proposed in [10] the notion of exponential attractor, which is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. So, compared with the global attractor, an exponential attractor is more robust under perturbations and numerical approximations (see [10], [16] and [20] for discussions on this subject). One can also note that, generally, one can only establish the upper-semicontinuity of (global) attractors for perturbations of semigroups and partial differential equations, whereas approximate and exact exponential attractors are continuous (up to a time shift), at least for classical Galerkin approximations (see [10] for more details). Another motivation for the study of exponential attractors comes from the fact that the global attractor may be trivial (say, reduced to one point) and may thus fail to capture important transient behaviors. We note however that, contrarily to the global attractor, an exponential attractor is not necessarily unique, so that the actual/concrete choice of an exponential attractor is in a sense artificial.

Exponential attractors have been constructed for a large class of equations (see [1], [2], [10], [11], [12], [13], [16], [17], [20], [29], [30] and [31]). The known constructions of exponential attractors (see for instance [2], [10], [11] and [30]) make an essential use of orthogonal projectors with finite rank (in order to prove the so-called squeezing property) and are thus valid in Hilbert spaces only. Recently, Efendiev, Miranville and Zelik gave in [13] a construction of exponential attractors that is no longer based on the squeezing property and that is thus valid in a Banach setting. So, exponential attractors are as general as global attractors.

In the case of non-autonomous systems, Miranville gave in [29] (see also [17]) a definition of exponential attractor for periodic and quasiperiodic time dependences. Unfortunately, since it requires that the exponential attractors have finite fractal dimension, such a definition cannot be extended to more general translation-compact time dependences. Indeed, contrarily to the autonomous case, the uniform attractor has infinite (fractal) dimension in general (see [4], [5] and [7]). It is important to note that such an infinite dimensionality can be obtained for physically relevant systems (e.g. a cascade system constructed on a partial differential equation in $R^n$, see Section 5 below). Since it contains the uniform attractor, an exponential attractor must thus also have infinite fractal dimension. Of course, the condition of finite (fractal) dimensionality should not be dropped completely: otherwise, any compact positively invariant bounded absorbing set would be an exponential attractor, which is not satisfactory. It is thus reasonable to construct the exponential attractor as small as possible and to have a tool that measures this smallness.

In this article, we propose to use the Kolmogorov's epsilon-entropy (see [23]) to obtain a good/suitable definition of (infinite dimensional) exponential attractor; the Kolmogorov's entropy being a useful tool for the comparison of (infinite dimensional) compact sets (see [7], [8], [9], [23], [37], [38], [39] and [40]). More precisely, we propose to construct an exponential attractor $\mathcal{M}$ whose entropy has in some sense the same form as that of the uniform attractor (see Section 4 for more details). We thus give in Theorem 4.1 a construction of (infinite dimensional) exponential attractors. As an example, we prove the existence of an (infinite dimensional) exponential attractor for a non-autonomous reaction-diffusion system.

## §0 Setting of the problem

We consider the following second order parabolic system in a bounded smooth domain $\omega \subset \mathbb{R}^n$:

$$\begin{cases} \partial_t u = a\Delta_x u - f(u) + g(t), \ x \in \omega, \\ u\big|_{\partial\omega} = 0, \quad u\big|_{t=0} = u_0. \end{cases} \tag{0.1}$$

Here, $u = (u^1, \cdots, u^k)$ is an unknown vector function, $\Delta_x$ is the Laplacian with respect to the variables $x = (x^1, \cdots, x^n)$, $a$ is a given $k \times k$-matrix such that $a + a^* > 0$ and $f(u)$ and $g(t) = g(t, x)$ are given functions.

The nonlinear term $f$ is assumed to satisfy the following conditions:

$$\begin{cases} 1. \ f \in C^1(\mathbb{R}^k, \mathbb{R}^k), \\ 2. \ f(u).u \geq -C, \quad f'(u) \geq -KId, \\ 3. \ |f(u)| \leq C(1 + |u|^p), \quad |f'(u)| \leq C(1 + |u|^{p-1}), \ p < p_c = \frac{n+2}{n-2}, \end{cases} \tag{0.2}$$

where the condition $f'(u) \geq -KId$ means that the matrix $f'(u) + KId$ is positive. Here and below, we denote by $u.v$ the standard inner product in $\mathbb{R}^k$. Moreover, we denote by $W^{l,p}(V)$ the Sobolev space of the functions whose derivatives up to order $l$ belong to $L^p(V)$, which we endow with its usual norm denoted by $\|\cdot, V\|_{l,p}$. We will write $\|\cdot\|_{l,p}$ instead of $\|\cdot, \omega\|_{l,p}$.

We furthermore assume that the external force $g \in L^2(\Omega_T)$ for $T \geq 0$, where $\Omega_T := [T, T+1] \times \omega$, and satisfies

$$\|g\|_{L^2_b} := \sup_{T \geq 0} \|g, \Omega_T\|_{0,2} < \infty. \tag{0.3}$$

We denote by $L^2_b(\Omega_+)$ ($\Omega_+ := \mathbb{R}_+ \times \omega$) the B-space of the functions $g$ with finite norm (0.3). The spaces $W^{l,p}_b(\Omega_+)$ and $W^{l,p}(\Omega)$ ($\Omega := \mathbb{R} \times \omega$) can be defined similarly.

As usual, the solution of problem (0.1) is a function $u \in \Lambda(\Omega_T) := W^{(1,2),2}(\Omega_T)$, for every $T \geq 0$, which satisfies (0.1) in the sense of distributions. We recall that the norm in the Sobolev–Slobodetskij space $\Lambda(\Omega_T)$ is given by the following expression:

$$\|u, \Omega_T\|_\Lambda^2 := \|\partial_t u, \Omega_T\|_{0,2}^2 + \|u, \Omega_T\|_{2,2}^2. \tag{0.4}$$

Consequently, we assume that the initial data $u_0$ belongs to the space $V_0 := W_0^{1,2}(\omega)$, which is the trace space at $t = 0$ of the functions $u \in \Lambda(\Omega_0)$ such that $u\big|_{\partial\omega} = 0$ (see [25]).

**Remark 0.1.** We note that $2p_c$ is the limit exponent in the embedding $\Lambda(\Omega_T) \subset L^{2p_c}(\Omega_T)$. Consequently, the growth restrictions (0.2) guarantee that $f(u) \in L^2(\Omega_T)$, for every $u \in \Lambda(\Omega_T)$, and equation (0.1) can be understood as an equality in $L^2(\Omega_T)$.

## §1 The estimates

In this Section, we give several estimates for the solutions of (0.1) and their difference that will be essential in the next sections for the construction of the global and exponential attractors for (0.1) and for the derivation of estimates on their entropy. We start with the following theorem.

**Theorem 1.1.** *Let the assumptions of the introduction hold. Then, for every $u_0 \in V_0$, problem* (0.1) *has a unique (in the class $u \in \Lambda(\Omega_T)$, $T \geq 0$) solution $u(t)$ that possesses the following estimate:*

$$\|u, \Omega_T\|_\Lambda \leq Q(\|u_0\|_{V_0})e^{-\alpha T} + Q(\|g\|_{L_b^2}), \tag{1.1}$$

*where $Q$ is a monotonic function which depends on $f$ but which is independent of $u_0$ and $g$ and where $\alpha$ is a strictly positive number.*

*Proof.* We restrict ourselves to the derivation of estimate (1.1). The existence of solutions can be obtained in a standard way, using e.g. the Galerkin method or the Leray-Schauder fixed point theorem (see [3] and [36]). The uniqueness will be verified in Theorem 1.2.

In order to derive (1.1), we multiply as usual equation (0.1) by $u(t)$ and integrate over $x \in \omega$. Then, integrating by parts and using the second assumption of (0.2) and the positivity of $a$, we obtain the estimate

$$\partial_t \|u(t)\|_{0,2}^2 + \mu\|u(t)\|_{0,2}^2 + \mu\|u(t)\|_{1,2}^2 \leq C(1 + \|g(t)\|_{0,2}^2), \tag{1.2}$$

where $\mu > 0$ is small enough (here, we have used the inequality $\|u(t)\|_{0,2} \leq C\|\nabla_x u(t)\|_{0,2}$ together with Holder's inequality).

Applying Gronwall's inequality to (1.2), we deduce that

$$\|u(T)\|_{0,2}^2 + \int_T^{T+1} \|u(t)\|_{1,2}^2 \, dt \leq C\|u_0\|_{0,2}^2 e^{-\mu T} + C(1 + \|g\|_{L_b^2}^2). \tag{1.3}$$

Multiplying now equation (0.1) by $-\Delta_x u(t)$, integrating over $x \in \omega$ and integrating by parts, we obtain, using again the second assumption of (0.2), the positivity of $a$ and Holder's inequality

$$\partial_t \|u(t)\|_{1,2}^2 + \mu\|u(t)\|_{1,2}^2 + \mu\|u(t)\|_{2,2}^2 \le C\|u(t)\|_{1,2}^2 + C(1 + \|g(t)\|_{0,2}^2). \tag{1.4}$$

Here, we have implicitly used the fact that the domain $\omega$ is smooth. Indeed, for such domains, we have $\|u(t)\|_{2,2} \le C\|\Delta_x u(t)\|_{0,2}$.

Applying Gronwall's inequality to (1.4) and estimating the first term of the right-hand side of (1.4) by (1.3), we obtain, after simple transformations

$$\|u(T)\|_{1,2}^2 + \int_T^{T+1} \|u(t)\|_{2,2}^2\, dt \le C_1\|u_0\|_{1,2}^2 e^{-\mu T} + C_1(1 + \|g\|_{L_b^2}^2). \tag{1.5}$$

Our next task is to estimate $\|f(u), \Omega_T\|_{0,2}$. To this end, we use the following embedding.

**Lemma 1.1.** *Let $\omega \subset \mathbb{R}^n$, $n > 2$, be a smooth domain. Then*

$$L^\infty([T, T+1], W^{1,2}(\omega)) \cap L^2([T, T+1], W^{2,2}(\omega)) \subset L^{2p_c}(\Omega_T). \tag{1.6}$$

*Proof.* The embedding (1.6) can easily be obtained by interpolation between $L^\infty(W^{1,2})$ and $L^2(W^{2,2})$ and by using standard embedding theorems. Indeed, for every $\theta \in [0, 1]$

$$L^\infty([T, T+1], W^{1,2}(\omega)) \cap L^2([T, T+1], W^{2,2}(\omega)) \subset L^{2/\theta}([T, T+1], W^{1+\theta}(\omega)), \tag{1.7}$$

(see e.g. [35]). We note that, due to a standard embedding theorem, we have $W^{1+\theta,2} \subset L^q$, with $\frac{1}{q} = \frac{1}{2} - \frac{1+\theta}{n}$. Finding $\theta = \frac{n-2}{n+2}$ from the equation $2/\theta = q$, we now obtain the embedding (1.6). Lemma 1.1 is proved.

Thus, according to Lemma 1.1, estimate (1.5) and the growth restrictions (0.2)

$$\|f(u), \Omega_T\|_{0,2}^2 \le C\left(1 + \|u, \Omega_T\|_{0,2p_c}^2\right)^{p_c} \le C_1\Bigg(1 + \|u\|_{L^\infty([T,T+1],W^{1,2})}^{2p_c} + \\ + \|u\|_{L^2([T,T+1],W^{2,2})}^{2p_c}\Bigg) \le Q_1(\|u_0\|_{V_0})e^{-\mu T} + Q_1(\|g\|_{L_b^2}), \tag{1.8}$$

for an appropriate monotonic function $Q_1$. Thanks to (1.8), we can rewrite equation (0.1) as a linear equation:

$$\partial_t u - a\Delta_x u = h_u(t) := f(u(t)) + g(t), \tag{1.9}$$

and then apply the parabolic regularity theorem to (1.9) (see e.g. [25]):

$$\|u, \Omega_T\|_\Lambda^2 \le C(\|u(T)\|_{1,2}^2 + \|g, \Omega_T\|_{0,2}^2 + \|f(u), \Omega_T\|_{0,2}^2). \tag{1.10}$$

Inserting estimates (1.5) and (1.8) into the right-hand side of (1.10), we obtain estimate (1.1). This finishes the proof of Theorem 1.1.

The following theorem gives an estimate for the difference of two solutions of (0.1) for different initial values and right-hand sides. This estimate is of fundamental significance in our study of attractors.

**Theorem 1.2.** *Let the assumptions of Theorem 1.1 hold and let $u_1(t)$ and $u_2(t)$ be two solutions of equation* (0.1)*, with right-hand sides $g_1$ and $g_2$ respectively. Then*

$$\|u_1(T)-u_2(T)\|_{1,2}^2 \le CT^{-1}e^{LT}\|u_1(0)-u_2(0)\|_{0,2}^2 + \\ + Ce^{LT}\int_0^T \|g_1(t)-g_2(t)\|_{0,2}^2\,dt, \tag{1.11}$$

*where the constant $C$ depends on the $V_0$-norm of $u_1(0)$ and $u_2(0)$ and where the constant $L$ depends only on $f$ and $a$.*

*Proof.* We restrict ourselves to the case $T \le 1$ in (1.11) (which in fact will be used in the next sections). The general case can be obtained similarly or derived from the case $T \le 1$ by induction.

Let $v(t) = u_1(t) - u_2(t)$, $h(t) = g_1(t) - g_2(t)$. Then

$$\begin{cases} \partial_t v = a\Delta_x v - l(t)v + h, \\ v\big|_{\partial\omega} = 0, \quad v\big|_{t=0} = u_1(0) - u_2(0), \end{cases} \tag{1.12}$$

where $l(t) := \int_0^1 f'(su_1(t) + (1-s)u_2(t))\,ds$.

The following two Lemmata give estimates for the solutions of (1.12).

**Lemma 1.2.** *Let the above assumptions hold and let us assume that $T \le 1$. Then*

$$\int_0^T \|l(t)v(t)\|_{0,2}^2\,dt \le C\|v,\Omega_{0,T}\|_{0,q}^2, \quad q = 2p_c\frac{1}{1+p_c-p} < 2p_c, \tag{1.13}$$

*where $\delta = \delta(p) > 0$, $\Omega_{0,T} := [0,T]\times\omega$ and $C = C(\|u_1(0)\|_{V_0}, \|u_2(0)\|_{V_0})$.*

*Proof.* Using the growth restictions on $f'$ (see (0.2)) and Holder's inequality with exponents $\frac{p_c}{p-1}$ and $\frac{p_c}{1+p_c-p}$, we find

$$\int_0^T \|l(t)v(t)\|_{0,2}^2 dt \le C(1 + \|u_1,\Omega_{0,T}\|_{0,2p_c} + \|u_2,\Omega_{0,T}\|_{0,2p_c})^{2(p-1)}\|v,\Omega_{0,T}\|_{0,q}^2. \tag{1.14}$$

Furthermore, using Theorem 1.1 and the embedding $\Lambda \subset L^{2p_c}$, it follows that

$$\|u_i,\Omega_{0,T}\|_{0,2p_c} \le C\|u_i,\Omega_0\|_\Lambda \le Q(\|u_i(0)\|_{V_0}) + Q(\|g\|_{L_b^2}). \tag{1.15}$$

Inserting (1.15) into (1.14), we then finish the proof of the lemma.

**Lemma 1.3.** *Let the above assumptions hold and let us assume that $T \le 1$. Then, for every $\varepsilon > 0$, there exists $C_\varepsilon$ depending only on $\|u_i(0)\|_{V_0}$ such that*

$$\int_0^T \|l(t)v(t)\|_{0,2}^2\,dt \le C_\varepsilon \int_0^T \|v(t)\|_{0,2}^2\,dt + \\ + \varepsilon \int_0^T \|v(t)\|_{2,2}^2\,dt + \varepsilon \sup_{t\in[0,T]} \|v(t)\|_{1,2}^2. \tag{1.16}$$

*Proof.* Since $q < 2p_c$, then, according to the interpolation inequality between $L^2$ and $L^{2p_c}$ and according to Young's inequality, we have, for every $\varepsilon > 0$

$$\|v, \Omega_{0,T}\|_{0,q}^2 \le C_\varepsilon \|v, \Omega_T\|_{0,2}^2 + \varepsilon \|v, \Omega_{0,T}\|_{0,2p_c}^2. \tag{1.17}$$

Applying now the result of Lemma 1.1 to the last term of the right-hand side of (1.17), we obtain

$$\|v, \Omega_{0,T}\|_{0,2p_c}^2 \le C\left(\int_0^T \|v(t)\|_{2,2}^2\,dt + \sup_{t\in[0,T]} \|v(t)\|_{1,2}^2\right). \tag{1.18}$$

(Moreover, repeating the proof of Lemma 1.1, we can easily check that the constant $C$ in (1.18) is independent of $T$.)

Combining estimates (1.17) and (1.18) with estimate (1.13), we finish the proof of the lemma.

Now, we return to the study of equation (1.12). Let $v(t) = v_1(t) + v_2(t)$, where $v_1(t)$ satisfies the equation

$$\partial_t v_1 = a\Delta_x v_1 - l(t)v_1 + h, \quad v_1(0) = 0, \tag{1.19}$$

and $v_2$ is solution of

$$\partial_t v_2 = a\Delta_x v_2 - l(t)v_2, \quad v_2(0) = v(0). \tag{1.20}$$

We first study equation (1.19). We note that, multiplying this equation by $v_1(t)$, integrating over $x \in \omega$ and using the fact that $l(t) \ge -KId$ (due to assumptions (0.2)), we deduce (as in the proof of Theorem 1.1) that

$$\|v_1(t)\|_{0,2}^2 \le Ce^{Lt} \int_0^T \|h(t)\|_{0,2}^2\,dt. \tag{1.21}$$

Multiplying now equation (1.19) by $\Delta_x v_1(t)$ and integrating over $x \in \omega$ and $t \in [0, T]$, we deduce the estimate

$$\|v_1(T)\|_{1,2}^2 + \int_0^T \|v_1(t)\|_{2,2}^2\,dt \le C\int_0^T \|h(t)\|_{0,2}^2\,dt + C\int_0^T \|l(t)v_1(t)\|_{0,2}^2\,dt. \tag{1.22}$$

Inserting estimates (1.16) and (1.21) into the right-hand side of (1.22) and assuming that $\varepsilon$ is small enough, we obtain the inequality

$$\|v_1(T)\|_{1,2}^2 \le C_1 \int_0^T \|h(t)\|_{0,2}^2\,dt + \varepsilon \sup_{t\in[0,T]} \|v(t)\|_{1,2}^2. \tag{1.23}$$

Taking the supremum over $[0,T]$ of both sides of (1.23) (with $T$ replaced by $t$) and taking $\varepsilon < 1/2$, we finally obtain

$$\|v_1(T)\|_{1,2}^2 \le C_2 \int_0^T \|h(t)\|_{0,2}^2\,dt. \tag{1.24}$$

Let us now estimate the function $v_2(t)$. We first note that, multiplying (1.20) by $v_2(t)$ and arguing as in the proof of (1.21), we have

$$\|v_2(T)\|_{0,2}^2 + \int_0^T \|v_2(t)\|_{1,2}^2\,dt \le C\|v(0)\|_{0,2}^2. \tag{1.25}$$

(We recall that we have assumed that $T \le 1$.)

We then multiply equation (1.20) by $t\Delta_x v_2(t)$, integrate over $x \in \omega$ and $t \in [0,T]$ and set $w(t) := t^{1/2}v_2(t)$. We find, after simple transformations

$$\|w(T)\|_{1,2}^2 + \int_0^T \|w(t)\|_{2,2}^2\,dt \le C\int_0^T \|v_2(t)\|_{1,2}^2\,dt + C\int_0^T \|l(t)w(t)\|_{0,2}^2\,dt. \tag{1.26}$$

Estimating the right-hand side of (1.26) by (1.25) and (1.16) and taking $\varepsilon$ small enough, we obtain the inequality

$$\|w(T)\|_{1,2}^2 \le C\|v(0)\|_{0,2}^2 + \varepsilon \sup_{t\in[0,T]} \|w(t)\|_{1,2}^2. \tag{1.27}$$

Arguing as in (1.23), we deduce from (1.27) that

$$T\|v_2(T)\|_{1,2}^2 = \|w(T)\|_{1,2}^2 \le C\|v(0)\|_{0,2}^2. \tag{1.28}$$

Combining finally estimates (1.24) and (1.28), we obtain estimate (1.11). Theorem 1.2 is proved.

**Corollary 1.1.** *Let the assumptions of Theorem 1.1 hold. Then, for every solutions $u_1(t)$ and $u_2(t)$ with right-hand sides $g_1$ and $g_2$, the following estimate holds:*

$$\|u_1(t) - u_2(t)\|_{1,2}^2 \le Ce^{Lt}\left(\|u_1(0) - u_2(0)\|_{1,2}^2 + \int_0^t \|g_1(s) - g_2(s)\|_{0,2}^2\,ds\right), \tag{1.29}$$

*where the constants $C$ and $L$ depend only on $\|u_i(0)\|_{V_0}$.*

The proof of (1.29) is similar to that of Theorem 1.2, but is essentially simpler, since we do not need to prove the smoothing property.

To conclude this section, we obtain some regularity results for the solutions of (0.1) under the additional assumption

$$g \in L_b^{2+\beta}(\mathbb{R}, L^2(\omega)), \tag{1.30}$$

where $\beta > 0$ is some positive number. These regularity results will be essential for the construction of exponential attractors for problem (0.1).

**Theorem 1.3.** *Let $u_0 \in V_0$ and let assumption* (1.30) *hold. Then, there exists a constant $\beta' = \beta'(\beta) > 0$ such that*

$$\int_0^1 (t\|\partial_t u(t)\|_{0,2})^{2+\beta'}\, dt + \int_0^1 (t\|u(t)\|_{2,2})^{2+\beta'}\, dt \le Q(\|u_0\|_{V_0}) + Q(\|g\|_{L_b^{2+\beta'}(L^2)}), \tag{1.31}$$

*for an appropriate monotonic function $Q$.*

*Proof.* We set $v(t) = tu(t)$. Then, $v$ is solution of

$$\partial_t v - a\Delta_x v = g_u(t) := u(t) - tf(u(t)) + tg(t), \quad v\big|_{t=0} = 0. \tag{1.32}$$

Let us verify that $g_u \in L^{2+\beta'}([0,1], L^2(\omega))$, for an appropriate $\beta' > 0$. Indeed, since $p < p_c$, the embedding $\Lambda(\Omega_0) \subset L^{2p_c}(\Omega_0)$, together with estimate (1.1), imply that

$$\|f(u), \Omega_0\|_{0,2p_c/p} \le Q_1(\|u, \Omega_0\|_{0,2p_c}) \le Q_2(\|u_0\|_{V_0}) + Q_2(\|g\|_{L_b^2}), \tag{1.33}$$

for appropriate monotonic functions $Q_1$ and $Q_2$. Therefore, the function $f(u)$ belongs to $L^{2+\beta_1}([0,1], L^2(\omega))$, with $\beta_1 := 2(\frac{p_c}{p} - 1) > 0$, and $u(t)$ obviously belongs to at least the same space. Thus, we have proved that

$$\|g_u\|_{L^{2+\beta'}([0,1],L^2(\omega))} \le Q_3(\|u_0\|_{V_0}) + Q_3(\|g\|_{L^{2+\beta'}(L^2)}), \tag{1.34}$$

with $\beta' := \min\{\beta, \beta_1\}$.

Applying now the anisotropic parabolic maximal regularity theorem (see [27]) to equation (1.32), we derive (using (1.34)) estimate (1.31) and Theorem 1.3 is proved.

Let $V_0^{\beta'}$ denote the following trace space:

$$V_0^{\beta'} = \big\{u(0) : \partial_t u \in L^{2+\beta'}([0,1], L^2(\omega)), \quad u \in L^{2+\beta'}([0,1], W^{2,2}(\omega)),\ u\big|_{\partial\omega} = 0\big\}. \tag{1.35}$$

It is known (see [26] and [35]) that

$$V_0^{\beta'} \subset W^{1+\delta,2}(\omega), \tag{1.36}$$

where $\delta = \frac{\beta'}{2+\beta'} > 0$. Thus, Theorem 1.3 implies the

**Corollary 1.2.** *Let the assumptions of Theorem 1.3 hold. Then, $u(t)$ belongs to $V_0^{\beta'}$, for every $t > 0$, and*

$$\|u(t)\|_{V_0^{\beta'}} \le (1 + \frac{1}{t}) \left(Q(\|u_0\|_{V_0}) + Q(\|g\|_{L_b^{2+\beta'}(L^2)})\right), \tag{1.37}$$

*for an appropriate monotonic function $Q$. In particular, due to* (1.36)*, the $V_0^{\beta'}$-norm in the left-hand side of* (1.37) *can be replaced by the $W^{1+\delta,2}$-norm.*

**Corollary 1.3.** *Let the assumptions of Theorem 1.3 hold. We assume in addition that* $u_0 \in V_0^{\beta'}$. *Then*

$$(1.38)\quad \|u(T)\|^{2+\beta'}_{V_0^{\beta'}} + \int_T^{T+1} \|\partial_t u(t)\|^{2+\beta'}_{0,2}\, dt + \int_T^{T+1} \|u(t)\|^{2+\beta'}_{2,2}\, dt \le$$
$$\le Q(\|u_0\|_{V_0^{\beta'}}) + Q(\|g\|_{L_b^{2+\beta'}(L^2)}),$$

*for an appropriate monotonic function* $Q$.

Estimate (1.38) can be derived as in the proof of Theorem 1.3, but is simpler, since we do not need to prove the smoothing property.

The following result, which establishes the Holder continuity with respect to $t$ of the trajectories, is essential for our construction of exponential attractors.

**Theorem 1.4.** *Let the assumptions of Theorem 1.3 hold and let in addition* $u_0$ *belong to* $V_0^{\beta'}$. *Then, there exists* $\gamma > 0$ *such that, for every* $t \ge 0$ *and* $0 < s < 1$

$$(1.39)\quad \frac{\|u(t+s) - u(t)\|_{1,2}}{s^\gamma} \le Q(\|u_0\|_{V_0^{\beta'}}) + Q(\|g\|_{L_b^{2+\beta'}(L^2)}),$$

*for an appropriate monotonic function* $Q$.

*Proof.* We write

$$(1.40)\quad \|u(t+s) - u(t)\|^2_{1,2} =$$
$$= -\int_t^{t+s} (\partial_t u(l), \Delta_x u(l))\, dl = -\int_t^{t+s} (a\Delta_x u(l), \Delta_x u(l))\, dl +$$
$$+ \int_t^{t+s} (f(u(l)), \Delta_x u(l))\, dl - \int_t^{t+s} (g(l), \Delta_x u(l))\, dl := I_1 + I_2 + I_3.$$

Let us estimate $I_1$. Using Holder's inequality with exponents $p_1 = 1+\beta'/2$ and $p_2 = 1+2/\beta'$ and (1.38), we have

$$(1.41)\quad |I_1| \le C\int_t^{t+s} \|u(l)\|^2_{2,2}\, dl \le Cs^{1/p_2}\left(\int_t^{t+s} \|u(l)\|^{2+\beta'}_{2,2}\, dl\right)^{1/p_1} \le C_1 s^{1/p_2}.$$

The second integral $I_2$ can be easily estimated by integrating by parts and by using the fact that $\|u(t)\|_{1,2}$ is uniformly bounded together with the condition $f' \ge -KId$:

$$(1.42)\quad I_2 \le K\int_t^{t+s} \|\nabla_x u(l)\|_{1,2}\, dl \le Cs.$$

So, there remains to estimate the third integral $I_3$. To this end, we apply Holder's inequality with exponents $p_1 = p_2 = 2 + \beta'$ and $p_3 = 1 + 2/\beta$ and use (1.30) and (1.39) to obtain

$$(1.43)\quad |I_3| \le C\int_t^{t+s} \|g(l)\|_{0,2}\|u(l)\|_{2,2}\,dl \le \\ \le Cs^{1/p_3}\left(\int_t^{t+s}\|g(l)\|_{0,2}^{2+\beta'}\,dl\right)^{1/p_1}\left(\int_t^{t+s}\|u(l)\|_{2,2}^{2+\beta'}\,dl\right)^{1/p_2} \le C_1 s^{1/p_3}.$$

Inserting estimates (1.41)–(1.43) into estimate (1.40), we find (1.39). Theorem 1.4 is proved.

## §2 The global (uniform) attractor

In this Section, we prove that equation (0.1) possesses the global attractor $\mathbb{A}$ in the appropriate extended phase space.

We start with recalling the skew product construction of the global attractor for the non-autonomous equation (0.1) (see e.g. [4], [5], [22], [32] and [33] for details). To this end, we further assume that the right-hand side $g(t)$ is defined not only for $t \ge 0$, but also for all $t \in \mathbb{R}$, and is bounded (i.e. it belongs to the space $L^2_b(\Omega)$). Moreover, we assume that the right-hand side is translation-compact (see [4] and [5]). By definition, this means that the hull

$$(2.1)\qquad \mathcal{H}(g) := [T_h g, h \in \mathbb{R}]_{L^2_{loc}(\Omega)}, \quad (T_h g)(t) := g(t+h),$$

is compact in $L^2_{loc}(\Omega)$ (which, by definition, is a F-space generated by the semi-norms $\|\cdot, \Omega_T\|_{0,2}$, $T \in \mathbb{R}$), where $[\cdot]_V$ denotes the closure in $V$.

It is obvious that any periodic, quasiperiodic or almost periodic function (in Bochner-Amerio sense) with values in $L^2(\omega)$ belongs to the class $TC_2(\Omega)$ of translation-compact functions in $L^2_{loc}(\Omega)$. We also note that, in contrast with the definition of almost periodic functions, we take the closure in (2.1) not for a uniform topology of $L^2(\Omega)$ or $C_b(\mathbb{R}, L^2(\omega))$, but only for a local topology of $L^2_{loc}(\Omega)$, so that the class $TC_2(\Omega)$ is essentially larger than the class $AP(\mathbb{R}, L^2(\omega))$ of almost periodic functions. For instance, every function $g \in W^{\alpha,2}_b(\Omega)$, with $\alpha > 0$, is automatically translation-compact:

$$W^{\alpha,2}_b(\Omega) \subset TC_2(\Omega), \quad \alpha > 0.$$

Moreover, the class $TC_2(\Omega)$ enjoys the following description (see [38]):

$$(2.2)\qquad TC_2(\Omega) = [C^1_b(\Omega)]_{L^2_b(\Omega)}.$$

Formula (2.2) clarifies the relationship between smoothness and translation-compactness.

We now consider the family of problems

$$(2.3)\qquad \partial_t u = a\Delta_x u - f(u) + \xi(t), \quad \xi \in \mathcal{H}(g).$$

We note that $\|\xi\|_{L^2_b} \le \|g\|_{L^2_b}$, for every $\xi \in \mathcal{H}(g)$. Consequently, the estimates derived in Section 1 hold *uniformly* with respect to $\xi \in \mathcal{H}(g)$. In particular, the family of solving operators

$$U_\xi(t,\tau) : V_0 \to V_0, \quad u_\xi(t) = U_\xi(t,\tau)u_\xi(\tau), \quad t \ge \tau,$$

where $u_\xi(t)$ is a solution of (2.3) with initial value $u_\xi\big|_{t=\tau} = u_\xi(\tau)$, is well defined.

Let us define a semigroup $\mathbb{S}_t$ acting on the extended phase space $\Phi := V_0 \times \mathcal{H}(g)$ by the formula

$$\mathbb{S}_t(u_0,\xi) := (U_\xi(t,0)u_0, T_t\xi)\,, \quad \mathbb{S}_t : V_0 \times \mathcal{H}(g) \to V_0 \times \mathcal{H}(g). \tag{2.4}$$

(It is easy to check (see e.g. [4], [5], [22], [32] and [33]) that (2.4) is indeed a semigroup.)

Our task now is to verify the existence of the global attractor for this semigroup. For the reader's convenience, we recall the definition of the global attractor (see [3], [21] and [34] for details).

**Definition 2.1.** The set $\mathbb{A} \subset \Phi$ is the global attractor for the semigroup $\mathbb{S}_t$ if the following conditions hold:

1. $\mathbb{A}$ is compact in $\Phi$;
2. $\mathbb{A}$ is strictly invariant with respect to $\mathbb{S}_t$: $\mathbb{S}_t\mathbb{A} = \mathbb{A}$, $\forall t \ge 0$;
3. $\mathbb{A}$ is an attracting set for the semigroup $\mathbb{S}_t$ in the following sense: for every bounded set $\mathbb{B} \subset \Phi$ and for every neighborhood $\mathcal{O}(\mathbb{A})$ of $\mathbb{A}$, there exists a time $T = T(B,\mathcal{O})$ such that

$$\mathbb{S}_t\mathbb{B} \subset \mathbb{A}, \text{ if } t \ge T. \tag{2.5}$$

**Remark 2.1.** The last condition of Definition 2.1 is equivalent to the following: for every bounded set $\mathbb{B} \subset \Phi$

$$\lim\nolimits_{t\to\infty} \operatorname{dist}_\Phi\{\mathbb{S}_t\mathbb{B}, \mathbb{A}\} = 0, \tag{2.6}$$

where

$$\operatorname{dist}_V\{X,Y\} := \sup\nolimits_{x\in X} \inf\nolimits_{y\in Y} d_V(x,y), \tag{2.7}$$

($d_V(\cdot,\cdot)$ denotes a metric in the space $V$) is the non-symmetric Hausdorff distance between the subsets $X$ and $Y$ of the metric space $V$.

**Theorem 2.1.** *Let the previous assumptions hold. Then, the semigroup* (2.4) *acting on the extended phase space* $\Phi$ *possesses the global attractor* $\mathbb{A}$.

*Proof.* According to the attractor's existence theorem for abstract semigroups (see [3], [21] and [34]), it suffices to verify the following assumptions:

1. the semigroup $\mathbb{S}_t : \Phi \to \Phi$ is continuous, for every fixed $t \ge 0$;
2. the semigroup $\mathbb{S}_t$ possesses a compact attracting set $\mathbb{K} \subset \Phi$.

The continuity of $\mathbb{S}_t$ is a straightforward consequence of (1.29). So, there remains to prove the existence of a compact attracting set. To this end, we note that estimate (1.1) implies that the set $\mathbb{B}_0 := B_R \times \mathcal{H}(g)$, where $B_R := \{v \in V_0; \|v\|_{V_0} \le R\}$ is the $R$-ball in $V_0$, is an attracting (and even absorbing) set for the semigroup $\mathbb{S}_t$ if $R$ is large enough (more precisely, for $R \ge 2Q(\|g\|_{L^2_b(\Omega)})$, where $Q$ is the same as in (1.1)). Now, this set is obviously non-compact in $\Phi$.

We claim that the set

$$\mathbb{B} := \mathbb{S}_1 \mathbb{B}_0, \tag{2.8}$$

is a compact attracting set for the semigroup $\mathbb{S}_t$. Indeed, since $\mathbb{B}_0$ is an attracting set, then so is $\mathbb{B}$. So, there only remains to prove the compactness.

We note that, by definition

$$\mathbb{B} = \{(u_\xi(1), T_1\xi) : u_\xi(0) \in B_R, \quad \xi \in \mathcal{H}(g)\}, \tag{2.9}$$

where $u_\xi(t)$ is solution of equation (2.3). We set $v_\xi(t) := tu_\xi(t)$. Then

$$\begin{cases} \partial_t v_\xi - a\Delta_x v_\xi = h_{u_\xi}(t) := u_\xi(t) + tf(u_\xi(t)) + t\xi(t), \quad \xi \in \mathcal{H}(g), \\ v_\xi\big|_{t=0} = 0, \quad u_\xi\big|_{t=0} \in B_R. \end{cases} \tag{2.10}$$

According to Theorem 1.1, the set $\{u_\xi(t) : (u_\xi(0), \xi) \in \mathbb{B}_0\}$ is bounded (and obviously closed) in $\Lambda(\Omega_0)$. We recall that, since the exponent $2p$ is strictly less than the critical exponent $2p_c$, the embedding $\Lambda(\Omega_0) \subset L^{2p}(\Omega_0)$ is compact (see [35]). Thus, we have proved that

$$\{u_\xi(t) : (u_\xi(0), \xi) \in \mathbb{B}_0\} \subset\subset L^{2p}(\Omega_0). \tag{2.11}$$

We note that, according to the growth restrictions (0.2) and to Krasnoselski's theorem (see [24]), the Nemitskij operator generated by $f$ is continuous from $L^{2p}(\Omega_0)$ into $L^2(\Omega_0)$. Consequently, (2.11) implies that the set

$$\{u_\xi(t) + tf(u_\xi(t)) : (u_\xi(0), \xi) \in \mathbb{B}_0\}, \tag{2.12}$$

is compact in $L^2(\Omega_0)$.

We now recall that the right-hand side $g$ is assumed to be translation-compact. Consequently, $\mathcal{H}(g)\big|_{\Omega_0}$ is also compact in $L^2(\Omega_0)$. Thus, we have established that the set $G$ of right-hand sides in (2.10):

$$G := \{h_{u_\xi}(t) : (u_\xi(0), \xi) \in \mathbb{B}_0\}, \tag{2.13}$$

is compact in $L^2(\Omega_0)$. Applying now the parabolic maximal regularity theorem (see [25]) to equation (2.10) and taking into account (2.13) and the fact that $v_\xi(0) = 0$, we have

$$G_1 := \{v_\xi(t) : (u_\xi(0), \xi) \in \mathbb{B}_0\} \subset\subset \Lambda(\Omega_0). \tag{2.14}$$

Noting that $\mathbb{B} \subset G_1\big|_{t=1} \times \mathcal{H}(g) \subset\subset V_0 \times \mathcal{H}(g)$ (since $V_0$ is the trace space at $t = 1$ of the functions belonging to $\Lambda(\Omega_0)$), we finally deduce that $\mathbb{B}$ is a compact attracting set for the semigroup $\mathbb{S}_t$ acting on $\Phi$ and, consequently (due to the attractor's existence theorem mentioned in the beginning of the proof), this semigroup possesses the global attractor $\mathbb{A}$ in $\Phi$. Theorem 2.1 is proved.

**Definition 2.2.** The projection $\mathcal{A} := \Pi_1\mathbb{A}$ of the global attractor $\mathbb{A}$ constructed in the previous theorem onto the first component (i.e. $V_0$) is called the uniform attractor for the initial non-autonomous equation (0.1).

**Corollary 2.1.** *Let the assumptions of Theorem 2.1 hold. Then, equation* (0.1) *possesses the uniform attractor* $\mathcal{A} \subset\subset V_0$.

**Remark 2.2.** The uniform attractor $\mathcal{A}$ possesses an internal definition (i.e. that does not make reference to the semigroup $\mathbb{S}_t$). More precisely, the set $\mathcal{A}$ is called the uniform attractor for equation (0.1) if:

1. $\mathcal{A}$ is compact in $V_0$;
2. for every bounded subset $B$ of $V_0$

$$\lim_{t\to\infty} \sup_{\xi\in\mathcal{H}(g)} \operatorname{dist}_{V_0}\{U_\xi(t,0)B, \mathcal{A}\} = 0; \tag{2.15}$$

3. the set $\mathcal{A}$ is minimal among the closed sets that satisfy 1 and 2.

The equivalence of this definition and Definition 2.2 has been proved in [4]. We also note that (2.15) justifies the name 'uniform' for the attractor $\mathcal{A}$.

To conclude this section, we formulate the non-autonomous analogue of the well-known fact that the global attractor is generated by all complete bounded trajectories of the corresponding semigroup.

**Theorem 2.2.** *Let the assumptions of Theorem 2.1 hold. Then, the uniform attractor* $\mathcal{A}$ *enjoys the following description:*

$$\mathcal{A} = \cup_{\xi\in\mathcal{H}(g)} K_\xi\big|_{t=0}, \tag{2.16}$$

*where* $K_\xi$ *is the set of all bounded solutions* $u \in \Lambda_b(\Omega)$ *of equation* (2.3) *with right-hand sides* $\xi \in \mathcal{H}(g)$ *defined for all* $t \in \mathbb{R}$.

Indeed, it is well known (see [3], [21] and [34]) that the global attractor $\mathbb{A}$ for the semigroup $\mathbb{S}_t$ enjoys the following description:

$$\mathbb{A} = \{\phi_0 \in \Phi : \exists\phi(s) \in L^\infty_b(\mathbb{R}, \Phi), \ \ \mathbb{S}_t\phi(s) = \phi(t+s), \ \forall t \geq 0, \ s \in \mathbb{R}, \ \phi(0) = \phi_0\}.$$

Assertion (2.16) can easily be deduced from this description (see e.g. [4] and [5] for details).

## §3 The entropy of the uniform attractor

This Section is devoted to the study of quantitative characteristics of the uniform attractor $\mathcal{A}$ constructed in the previous Section. Such a characteristic, which is widespread in the attractor's industry, is the fractal (box-counting, entropy) dimension of the attractor. We note however that, in contrast with the autonomous case, without additional assumptions on the 'finite dimensionality' (for the time dependence) of the right-hand side $g$ (e.g. $g$ is quasiperiodic), there is no reason to expect the (uniform) attractor to be finite dimensional in the non-autonomous case (see [4] for examples of infinite dimensional non-autonomous attractors or [8], [14] and [38] for examples of infinite dimensional autonomous attractors in unbounded domains). That is the reason why (following [7], [15], [37], [38] and [39]), we will study the Kolmogorov's entropy of the attractor $\mathcal{A}$, which is finite under the assumptions of Theorem 2.1 (since $\mathcal{A}$ is compact).

We briefly recall the definition of Kolmogorov's $\varepsilon$-entropy and give some classical examples (see [23] and [35] for a detailed study).

**Definition 3.1.** Let $K$ be a (pre)compact set in a metric space $M$. Then, due to Hausdorff's criteria, it can be covered by a finite number of $\varepsilon$-balls in $M$. Let $N_\varepsilon(K,M)$ be the minimal number of $\varepsilon$-balls that cover $K$. Then, we call Kolmogorov's $\varepsilon$-entropy of $K$ the logarithm of this number:

$$\mathbb{H}_\varepsilon(K,M) := \log_2 N_\varepsilon(K,M). \tag{3.1}$$

We now give several examples of typical asymptotics for the $\varepsilon$-entropy.

**Example 3.1.** We assume that $K = [0,1]^n$ and $M = \mathbb{R}^n$ (more generally, $K$ is a $n$-dimensional compact Lipschitz manifold of a metric space $M$). Then

$$\mathbb{H}_\varepsilon(K,M) = (n + \bar{\bar{o}}(1)) \log_2 \frac{1}{\varepsilon}, \text{ as } \varepsilon \to 0. \tag{3.2}$$

This example justifies the definition of the fractal dimension.

**Definition 3.2.** The fractal dimension $\dim_F(K,M)$ is defined as

$$\dim_F(K,M) := \operatorname{limsup}_{\varepsilon\to 0} \frac{\mathbb{H}_\varepsilon(K,M)}{\log_2 1/\varepsilon}. \tag{3.3}$$

The following example shows that, for sets that are not manifolds, the fractal dimension may be non-integer.

**Example 3.2.** Let $K$ be a standard ternary Cantor set in $M = [0,1]$. Then

$$\dim_F(K,M) = \frac{\ln 2}{\ln 3} < 1.$$

The next example gives the typical behavior of the entropy in classes of functions with finite smoothness.

**Example 3.3.** Let $V$ be a smooth bounded domain of $\mathbb{R}^n$ and let $K$ be the unit ball in the Sobolev space $W^{l_1,p_1}(V)$ and $M$ be another Sobolev space $W^{l_2,p_2}(V)$ such that the embedding $W^{l_1,p_1} \subset W^{l_2,p_2}$ is compact, i.e.

$$l_1 > l_2 \geq 0, \ \ \frac{l_1}{n} - \frac{1}{p_1} > \frac{l_2}{n} - \frac{1}{p_2}.$$

Then, the entropy $\mathbb{H}_\varepsilon(K, M)$ has the following asymptotics (see [35]):

$$C_1 \left(\frac{1}{\varepsilon}\right)^{n/(l_1-l_2)} \leq \mathbb{H}_\varepsilon(K, M) \leq C_2 \left(\frac{1}{\varepsilon}\right)^{n/(l_1-l_2)}. \tag{3.4}$$

Finally, the last example shows the typical behavior of the entropy in classes of analytic functions.

**Example 3.4.** Let $V_1 \subset V_2$ be two bounded domains of $\mathbb{C}^n$. We assume that $K$ is the set of all analytic functions $\phi$ in $V_2$ such that $\|\phi\|_{C(V_2)} \leq 1$ and that $M = C(V_1)$. Then

$$C_1 (\log_2 1/\varepsilon)^{n+1} \leq \mathbb{H}_\varepsilon(K\big|_{V_1}, M) \leq C_2 (\log_2 1/\varepsilon)^{n+1}, \tag{3.5}$$

(see [23]).

We now state the main result of this Section.

**Theorem 3.1.** *Let the assumptions of Theorem 2.1 hold. Then, the entropy of the attractor $\mathcal{A}$ satisfies (for $\varepsilon > 0$ small enough):*

$$\mathbb{H}_\varepsilon(\mathcal{A}, V_0) \leq C \log_2 \frac{1}{\varepsilon} + \mathbb{H}_{\varepsilon/K}\left(\mathcal{H}(g)\big|_{[0,L\log_2 1/\varepsilon]\times\omega}, L^2_b([0, L\log_2 1/\varepsilon] \times \omega)\right), \tag{3.6}$$

*where the constants $C$, $L$ and $K$ depend only on the left-hand side of equation* (0.1) *and can be computed explicitly.*

*Proof.* We adapt the proof given in [38] to our situation. Let $B = B(u_0, R, V_0)$ be a $V_0$-ball of radius $R$ centered at $u_0$ such that $\mathcal{A} \subset B$ (such a ball exists, since $\mathcal{A}$ is bounded in $V_0$).

According to Theorem 1.2, there exists a constant $L' \geq 1$ such that, for every solutions $u_1(t)$ and $u_2(t)$ with right-hand sides $g_1$ and $g_2$ such that $u_i(0) \in B(u_0, 2R, V_0)$, then

$$\|u_1(1) - u_2(1)\|_{1,2} \leq L' \left(\|u_1(0) - u_2(0)\|_{0,2} + \|g_1 - g_2, \Omega_0\|_{0,2}\right). \tag{3.7}$$

Our task now is to construct, starting with the initial ball $B$, the $R/2^k$-covering of the attractor $\mathcal{A}$ for every fixed $k$. Estimating then the number of balls in these coverings, we will deduce estimate (3.6).

Let $k \in \mathbb{N}$ be fixed. We fix the minimal $\frac{R}{4L'}\frac{1}{2^k}$-covering of the set $\mathcal{H}(g)\big|_{\Omega_{0,k}}$ for the metric of $L^2_b(\Omega_{0,k})$ (it is possible to do so, since $g$ is translation-compact). Let $G := \{\xi_1, \cdots, \xi_N\} \subset \mathcal{H}(g)$ be the centers of this covering. Then, obviously

$$\log_2 N = \log_2 N(k) := \mathbb{H}_{R/(2^{k+2}L')}(H(g)\big|_{\Omega_k}, L^2_b(\Omega_k)).$$

Let us also fix the minimal covering of the unit ball $B(0,1,V_0)$ for the $V_0$-norm by a finite number $P$ of $1/(8L')$-balls for the metric of $H := L^2(\omega)$ (such a covering exists, since $V_0 \subset\subset H$). Moreover, increasing the radiuses of the balls twice (from $1/(8L')$ to $1/(4L')$), we may assume that all the centers of these balls belong to $V_0$. It is very important that, for every $r > 0$ and $v \in V_0$, the ball $B(v, r, V_0)$ of radius $r$ in $V_0$ centered at $v$ can be covered by *the same number* $P$ of balls of radius $r/(4L')$ for the $H$-norm (this covering can be constructed from the initial covering of the unit ball $B(0,1,V_0)$ by shifting and homotetion). We also note that the centers of the balls in this covering belong to $V_0$.

We now set $\mathcal{U}^j_0 := \{u_0\}$, $j = 1, \cdots, N$, where $u_0$ is the center of our initial $R$-ball and we define the sets $\mathcal{U}^j_l$ by induction.

Let us assume that the sets $\mathcal{U}^j_l \subset V_0$, $j = 1, \cdots, N$, are already defined for some $l < k$. We consider, for every $j \in [1, N]$, the system of $R/2^l$-balls for the $V_0$-norm centered at the points of $\mathcal{U}^j_l$. We cover each of these balls by a finite number $P$ of $R/(2^l 4L')$-balls for the metric of $H$ and we denote by $\mathcal{V}^j_l$ the set of all centers of these new balls. We note that $\mathcal{U}^j_l \subset V_0$ implies, due to our construction of the covering, that $\mathcal{V}^j_l \subset V_0$ and, consequently, we can define the set $\mathcal{U}^j_{l+1}$ by the formula

$$\mathcal{U}^j_{l+1} := U_{\xi_j}(l+1, l)\mathcal{V}^j_l. \tag{3.8}$$

Thus, we have defined by induction the sets $\mathcal{U}^j_l$ and $\mathcal{V}^j_l$, for every $0 \le l \le k$ and $j \in [1, \cdots, k]$. We also note that the number of points in $\mathcal{U}^j_l$ is given by

$$\#\mathcal{U}^j_l = P^l, \quad l \le k. \tag{3.9}$$

**Lemma 3.1.** *The $R/2^k$-balls in $V_0$ centered at the points of the set $\mathcal{U}_k := \cup_{j=1}^N \mathcal{U}^j_k$ cover the attractor $\mathcal{A}$.*

*Proof.* Let $w$ be an arbitrary point of $\mathcal{A}$. Then, according to Theorem 2.2, there exist $\xi \in \mathcal{H}(g)$ and a complete bounded solution $\widehat{u}(t)$ of (2.3) such that $\widehat{u}(k) = w$.

According to our construction, there exist $j^* \in [1, \cdots, N]$ and $u^*_0 := u_0 \in \mathcal{U}^{j^*}_0$ such that

$$\|\xi - \xi_{j^*}, \Omega_{l,l-1}\|_{0,2} \le \frac{R}{4L'}\frac{1}{2^k}, \quad l = 1, \cdots, k, \;\; \|\widehat{u}(0) - u^*_0\|_{0,2} \le R. \tag{3.10}$$

Let us assume that we have already proved that, for some $l < k$, there exists $u^*_l \in \mathcal{U}^{j^*}_l$ such that

$$\|\widehat{u}(l) - u^*_l\|_{1,2} \le \frac{R}{2^l}. \tag{3.11}$$

Estimate (3.11) implies that $\widehat{u}(l)$ belongs to the $R/(2^l)$-ball for the $V_0$-metric centered at $u_l^* \in \mathcal{U}_l^{j^*}$. Therefore, by definition of the sets $\mathcal{V}_l^j$, there exists a point $v_l^* \in \mathcal{V}_l^{j^*}$ such that

$$\|\widehat{u}(l) - v_l^*\|_{0,2} \le \frac{R}{2^l}\frac{1}{4L'}. \tag{3.12}$$

We now consider the point $u_{l+1}^* := U_{\xi_{j^*}}(l+1,l)v_l^* \in \mathcal{U}_{l+1}^{j^*}$. Then, according to (3.7), (3.10) and (3.12)

$$\|\widehat{u}(l+1) - u_{l+1}^*\|_{1,2} \le L'\left(\frac{R}{4L'2^l} + \frac{R}{4L'2^k}\right) < \frac{R}{2^{l+1}}. \tag{3.13}$$

(We note that (3.7) is indeed applicable because $\widehat{u}(l) \in \mathcal{A} \subset B(u_0, R, V_0)$ and, consequently, (3.12) implies that $u_l^* \in B(u_0, 2R, V_0)$.)

Thus, by induction, we conclude that there exists $u_k^{j^*} \in \mathcal{U}_k^{j^*}$ such that

$$\|\widehat{u}(k) - u_k^{j^*}\|_{1,2} \le \frac{R}{2^k}. \tag{3.14}$$

Since $w = \widehat{u}(k)$ is arbitrary, we finish the proof of Lemma 3.1.

Therefore, we have constructed the $R/2^k$-covering of the attractor $\mathcal{A}$ (centered at the points of $\mathcal{U}_k := \cup_{j=1}^N \mathcal{U}_k^j$). Consequently, (3.9) implies that

$$N_{R/2^k}(\mathcal{A}, V_0) \le NP^k. \tag{3.15}$$

Taking the logarithm of both sides of (3.15) and writing the explicit expression of $N = N(k)$, we obtain that, for every $k \in \mathbb{N}$

$$\mathbb{H}_{R/2^k}(\mathcal{A}, V_0) \le k\log_2 P + \mathbb{H}_{R/2^{k+2}L'}(\mathcal{H}(g)\big|_{\Omega_{0,k}}, L_b^2(\Omega_{0,k})). \tag{3.16}$$

Estimate (3.6) follows immediately from (3.16). Indeed, let $R > \varepsilon > 0$ be fixed and let $k$ be such that

$$\frac{R}{2^{k-1}} > \varepsilon > \frac{R}{2^k}. \tag{3.17}$$

Then, noting that the $\varepsilon$-entropy is a non-increasing function of $\varepsilon$, we obtain from (3.16) that

$$\begin{aligned} \mathbb{H}_\varepsilon(\mathcal{A}, V_0) \le (1 + \log_2\frac{R}{\varepsilon})\log_2 P + \\ + \mathbb{H}_{\varepsilon/(8L')}\left(\mathcal{H}(g)\big|_{\Omega_{0,(1+\log_2\frac{R}{\varepsilon})}}, L_b^2(\Omega_{0,(1+\log_2\frac{R}{\varepsilon})})\right). \end{aligned} \tag{3.18}$$

Estimate (3.18) is of the form (3.6) (for appropriate constants $C$, $L$, $K$ and $\varepsilon$). Theorem 3.2 is proved.

## §4 The exponential attractor

The main aim of this Section is to construct an (infinite dimensional) exponential attractor for the non-autonomous problem (0.1). For the reader's convenience, we start our considerations from the standard definition of exponential attractors for semigroups (see [10]).

**Definition 4.1.** Let $\mathbb{S}_t : \Phi \to \Phi$, $t \geq 0$, be a semigroup. Then, a set $\mathbb{M}$ is an exponential attractor for $\mathbb{S}_t$ if

1. $\mathbb{M}$ is compact in $\Phi$;
2. $\mathbb{M}$ is *semi* $-$ *invariant* with respect to $\mathbb{S}_t$, i.e. $\mathbb{S}_t\mathbb{M} \subset \mathbb{M}$, $\forall t \geq 0$;
3. $\mathbb{M}$ attracts the bounded subsets of $\Phi$ *exponentially*, i.e. there exists $\gamma > 0$ such that, for every bounded subset $\mathbb{B} \subset \Phi$, there exists a constant $C = C(\mathbb{B})$ such that

$$\operatorname{dist}_\Phi(\mathbb{S}_t\mathbb{B}, \mathbb{M}) \leq Ce^{-\nu t}, \quad \nu > 0, \quad t \geq 0; \tag{4.1}$$

4. the set $\mathbb{M}$ has finite fractal dimension, i.e.

$$\dim_F(\mathbb{M}, \Phi) < \infty. \tag{4.2}$$

For non-autonomous equations, a definition of exponential attractor can be obtained from Definition 4.1 by using the skew product technique and by projecting the exponential attractor for the extended semigroup onto the first component (see [17] and [29]). We give this definition in the case of equation (0.1).

**Definition 4.2.** Let $U_\xi(t, \tau) : V_0 \to V_0$ be the solving operators for (2.3). Then, a set $\mathcal{M}$ is an exponential attractor for this equation if

1. $\mathcal{M}$ is compact in $V_0$;
2. $\mathcal{M}$ attracts exponentially the trajectories of the family (2.3), i.e. for every bounded subset $B \subset V_0$, there exists $C = C(B)$ such that

$$\sup_{\xi \in \mathcal{H}(g)} \operatorname{dist}_{V_0}(U_\xi(t, 0)B, \mathcal{A}) \leq Ce^{-\nu t}, \quad \nu > 0; \tag{4.3}$$

3. for every $u_0 \in \mathcal{M}$, there exists $\xi \in \mathcal{H}(g)$ such that $U_\xi(t, 0)u_0 \in \mathcal{M}$, for every $t \geq 0$;
4. The set $\mathcal{M}$ has finite fractal dimension: $\dim_F(\mathcal{M}, V_0) < \infty$.

As in the autonomous case, it follows immediately from the definition that the uniform attractor is a subset of any exponential attractor:

$$\mathcal{A} \subset \mathcal{M}. \tag{4.4}$$

We note that, although this definition of non-autonomous exponential attractors is well adapted to the study of non-autonomous equations with periodic or quasiperiodic external forces (see [17], [29] and [31]), it is not convenient for more general translation-compact time dependences. Indeed, as already mentioned, there is no reason to expect the uniform attractor to have finite dimension in general and point 4 of Definition 4.2, together with the embedding (4.4), imply that $\dim_F \mathcal{A} < \infty$. Consequently, there is also no reason to expect the existence of an exponential attractor in the sense of Definition 4.2 in general!

Thus, condition 4 of Definition 4.2 should be modified. We note however that this condition cannot be dropped completely. Indeed, in that case, every compact (semi-invariant) absorbing set of (0.1) would be an exponential attractor, which does not make sense.

We note that, although an exponential attractor must have infinite dimension (if the same is true for the uniform attractor), it is reasonable to construct it as 'small' as possible. Using Kolmogorov's entropy in order to compare the 'size' of infinite dimensional sets, we come to the following problem: construct an exponential attractor $\mathcal{M}$ whose entropy $\mathbb{H}_\varepsilon(\mathcal{M})$ has in some sense the same type of asymptotics, as $\varepsilon \to 0$, as the entropy $\mathbb{H}_\varepsilon(\mathcal{A})$ of the uniform attractor:

$$\mathbb{H}_\varepsilon(\mathcal{M}, V_0) \sim \mathbb{H}_\varepsilon(\mathcal{A}, V_0). \tag{4.5}$$

Having estimate (3.11) for the entropy of the right-hand side of (4.5), it looks reasonable to give the following definition of (infinite dimensional) exponential attractors for equation (0.1).

**Definition 4.3.** A set $\mathcal{M}$ is an (infinite dimensional) exponential attractor for equation (0.1) if conditions 1–3 of Definition 4.2 are satisfied and if, in addition

$$\mathbb{H}_\varepsilon(\mathcal{M}, V_0) \le C_1 \log_2 \frac{1}{\varepsilon} + \mathbb{H}_{\varepsilon/L_1}\left(\mathcal{H}(g)\big|_{\Omega_{0,K_1\log_2 1/\varepsilon}}, L^2_b(\Omega_{0,K_1\log_2 1/\varepsilon})\right), \tag{4.6}$$

for appropriate constants $C_1$, $L_1$ and $K_1$.

**Remark 4.1.** We note that, if the external force $g$ is in some proper sense finite dimensional (e.g. if it is quasiperiodic, see also the examples in Section 5 below), then the second term in the right-hand side of (4.6) has the asymptotics $L''\log_2 1/\varepsilon$ and, consequently, (4.6) implies that $\mathcal{M}$ is finite dimensional (that is the reason why we put into parentheses the words 'infinite dimensional' in Definition 4.3). In this situation, Definition 4.3 for exponential attractors coincides with the standard definition (i.e. Definition 4.2).

The main result of this section is the following theorem.

**Theorem 4.1.** *Let the assumptions of Theorem 3.1 hold and let in addition assumption* (1.30) *be valid. Then, equation* (0.1) *possesses an (infinite dimensional) exponential attractor* $\mathcal{M}$ *in the sense of Definition 4.3.*

*Proof.* We first note that it is sufficient to verify the exponential attraction for an absorbing set for equation (0.1) (indeed, the image of every bounded subset of $V_0$ enters the absorbing set after a finite time).

Let us fix an absorbing set $\mathbb{B}_0$ for the extended semigroup $\mathbb{S}_t : \Phi \to \Phi$ (say, the same as in the proof of Theorem 2.1). We note that this set may not be semi-invariant with respect to $\mathbb{S}_t$, so that we define (as usual) a new absorbing set

$$\mathbb{B} := \left[\cup_{t\ge 0}\mathbb{S}_t\mathbb{B}_0\right]_\Phi, \tag{4.7}$$

which is also bounded in $V_0$ (due to Theorem 1.1) and is obviously semi-invariant:

$$\mathbb{S}_t\mathbb{B} \subset \mathbb{B},\ \ \forall t \geq 0, \tag{4.8}$$

and we set $\mathcal{B} := \Pi_1\mathbb{B}$.

As usual, we first construct an exponential attractor for a discrete time $t_n = n$ and for the discrete semigroup $\mathbb{S}(n) := \mathbb{S}_n$, $n \in \mathbb{N}$. Then, we extend this result to the continuous case.

We are now going to construct the analogues of the sets $\mathcal{U}_l^j$ introduced in the proof of Theorem 3.1. To this end, we fix $u_0 \in V_0$ and $R \in \mathbb{R}$ such that $\mathcal{B} \subset B(u_0, R, V_0)$ and we choose the constant $L'$ such that (due to (1.11))

$$\|u_1(1) - u_2(1)\|_{1,2} \leq L'\left(\|u_1(0) - u_2(0)\|_{0,2} + \|g_1 - g_2, \Omega_0\|_{0,2}\right), \tag{4.9}$$

for every solutions $u_1$ and $u_2$ with right-hand sides $g_1$ and $g_2 \in \mathcal{H}(g)$ such that $u_1(0)$ and $u_2(0) \in B(u_0, 2R, V_0)$ (compare with (3.12)).

Let us now fix an arbitrary $k \in \mathbb{N}$ and let us construct the minimal $R/(2^k 4L')$-covering of the hull $\mathcal{H}(g)\big|_{\Omega_{0,k+1}}$ in the space $L^2_b(\Omega_{0,k+1})$. Let $G(k) = \{\xi_1, \cdots, \xi_{N(k)}\}$ be the set of the centers of this covering. We set

$$\log_2 N(k) = \mathbb{H}_{R/(2^{l+2})}(\mathcal{H}(g)\big|_{\Omega_{0,k+1}}, L^2_b(\Omega_{0,k+1})). \tag{4.10}$$

(We note that, in general, the constants $R$ and $L'$ are not the same as in the proof of Theorem 3.1; we keep here the same notations to indicate the analogy. We also note that we slightly change the definition of the number $N$ and that we consider the time interval $[0, k+1]$ instead of $[0, k]$. The reason for doing this will become clearer in the proof of Lemma 4.1 below.)

We now define the number $P$ (which is independent of $k$!) and we construct the sets $\mathcal{U}_l^j(k)$, $j \in [1, \cdots, N(k)]$, $l \in [0, k]$, by induction, following the proof of Theorem 3.1 (the only difference is that we now write $\mathcal{U}_l^j(k)$ instead of $\mathcal{U}_l^j$ to indicate the dependence on $k$).

Then, it can be proved (repeating word by word the proof of Lemma 3.1) that the system of $R/(2^k)$-balls for the $V_0$-metric centered at the points of $\mathcal{U}_k(k) := \cup_{j=1}^{N(k)}\mathcal{U}_k^j(k)$ covers the set $\Pi_1\mathbb{S}_k\mathbb{B}$. We reformulate this property in the following equivalent way:

$$\operatorname{dist}_{V_0}(\Pi_1(\mathbb{S}_k\mathbb{B}),\, \mathcal{U}_k(k)) \leq \frac{R}{2^k}. \tag{4.11}$$

We also note that, according to our construction

$$\#\mathcal{U}_k^j(k) = P^k. \tag{4.12}$$

Roughly speaking, the main idea of our construction of exponential attractor is to take this attractor as the union $\left[\cup_{k\in\mathbb{N}}\mathcal{U}_k(k)\right]_{V_0}$. Then, (4.11) gives the exponential attraction, whereas (4.12) gives the proper entropy estimate. However, in order to satisfy the other conditions of Definition 4.3, we have to be a little more accurate.

Let us first define the lifting of the sets $\mathcal{U}_k^j(k)$ (which will be denoted below by $\mathcal{U}^j(k)$ in order to simplify the notations) to the extended phase space $\Phi$. To this end, for every $v \in \mathcal{U}^j(k)$, we take a point $(u_v, \xi_v) \in \mathbb{B}$ such that

$$\|v - U_{\xi_v}(k,0)u_v\|_{V_0} \le R/2^k, \quad \|\xi_j - \xi_v\|_{L^2_b(\Omega_{0,k+1})} \le R/(2^{k+2}L'). \tag{4.13}$$

(It is clear from the proof of Lemma 3.1 that the points $v \in \mathcal{U}^j(k)$ for which such a point does not exist can be dropped out from $\mathcal{U}_k^j(k)$, thus preserving property (4.11)).

We set $\mathbb{U}^j(k) := \{\mathbb{S}_k(u_v, \xi_v) \in \Phi : v \in \mathcal{U}^j(k)\}$ and $\mathbb{U}(k) := \cup_{j=1}^N \mathbb{U}^j(k)$. Then (due to (4.11)–(4.13))

$$\operatorname{dist}_{V_0}(\Pi_1(\mathbb{S}_k\mathbb{B}), \Pi_1\mathbb{U}(k)) \le R/2^{k-1}, \ \mathbb{U}(k) \subset \mathbb{S}_k\mathbb{B}, \ \#\mathbb{U}(k) \le N(k)P^k. \tag{4.14}$$

We are now in a position to construct the discrete exponential attractors $\mathbb{M}^d$ and $\mathcal{M}^d := \Pi_1\mathbb{M}^d$. To this end, we first define a sequence of sets $\mathbb{E}(k)$ by induction:

$$\mathbb{E}(0) := \mathbb{U}(0), \quad \mathbb{E}(k+1) := \mathbb{S}_1\mathbb{E}(k) \cup \mathbb{U}(k+1), \tag{4.15}$$

and we then define the exponential attractor $\mathbb{M}^d$ as follows:

$$\mathbb{M}^d := \left[\cup_{k\in\mathbb{N}} \mathbb{E}(k)\right]_\Phi. \tag{4.16}$$

Indeed, the semi-invariance ($\mathbb{S}_1\mathbb{M}^d \subset \mathbb{M}^d$) is an immediate consequence of (4.15) and (4.16). Furthermore, the exponential attraction follows from the first formula of (4.14). Thus, there only remains to verify the entropy estimate. We recall that the entropy of a set coincides with that of its closure. Consequently, we will estimate the entropy of $\mathcal{M}_1^d := \cup_{k\in\mathbb{N}}\Pi_1\mathbb{E}(k)$.

Let us fix an arbitrary $\varepsilon$, $R > \varepsilon > 0$, and compute $k = k(\varepsilon)$ from the inequality

$$\frac{R}{2^k} < \varepsilon < \frac{R}{2^{k-1}}. \tag{4.17}$$

We then decompose $\mathcal{M}_1^d$ as follows:

$$\mathcal{M}_1^d = \cup_{l\le k}\Pi_1\mathbb{E}(l) \cup (\cup_{l>k}\Pi_1\mathbb{E}(k)). \tag{4.18}$$

We note that, according to our construction, $\mathbb{E}(k) \subset \mathbb{S}_k\mathbb{B}$. Consequently (since $\mathbb{B}$ is semi-invariant), the second set in the decomposition (4.18) is a subset of $\mathbb{S}_{k+1}\mathbb{B}$. We recall that (due to (4.14)) the system of $R/2^k(<\varepsilon)$-balls centered at the points of $\Pi_1\mathbb{U}(k+1)$ covers $\Pi_1(\mathbb{S}_{k+1}\mathbb{B})$ and, consequently, covers the second set in (4.18). Thus, the minimal number $N_\varepsilon(\mathcal{M}^d, V_0)$ of $\varepsilon$-balls which cover the attractor satisfies

$$N_\varepsilon(\mathcal{M}^d, V_0) \le \sum_{l\le k} \#\mathbb{E}(k) + \#\mathbb{U}(k+1) \le \sum_{l\le k+1} \#\mathbb{E}(k). \tag{4.19}$$

It follows immediately from the inductive definition of the sets $\mathbb{E}(l)$ that

$$\#\mathbb{E}(l) \le \sum_{m\le l} \#\mathbb{U}(m) \le l\#\mathbb{U}(l) \le (k+1)\#\mathbb{U}(k+1) \le (k+1)N(k+1)P^k. \tag{4.20}$$

Inserting this estimate into (4.19) and applying the $\log_2$, it follows that

$$\mathbb{H}_\varepsilon(\mathcal{M}^d, V_0) \le 2\log_2(k+1) + \log_2 N(k+1) + (k+1)\log_2 P. \tag{4.21}$$

Expressing $k = k(\varepsilon)$ from (4.17) and inserting this expression into (4.21), we obtain (as in the end of the proof of Theorem 3.1) estimate (4.6). Thus, the set $\mathcal{M}^d$ is indeed a discrete exponential attractor.

To complete the proof of the theorem, there remains to extend $\mathcal{M}^d$ to a continuous exponential attractor.

**Lemma 4.1.** *We set*

$$\mathbb{M} = \mathbb{M}^c := \{(v,\xi) \in \Phi :\ \exists t \in [0,1],\ \exists \phi \in \mathbb{M}^d,\quad (v,\xi) = \mathbb{S}_t\phi\}. \tag{4.22}$$

*Then,* $\mathcal{M} := \Pi_1\mathbb{M}$ *is an (infinite dimensional) exponential attractor for equation* (0.1).

*Proof.* Let us verify the conditions of Definition 4.3. The semi-invariance follows immediately from the discrete semi-invariance of $\mathbb{M}^d$ and from (4.22).

Let us now verify the exponential attraction. To this end, we fix an arbitrary $(v,\xi) \in \mathbb{B}$ with corresponding trajectory $u(T) := U_\xi(T,0)v$ and we consider the time $T = k+t$, where $k \in \mathbb{N}$ and $0 \le t < 1$. Then, according to the construction of the sets $\mathbb{U}(k)$, there exists $(v^*,\xi^*) \in \mathbb{B}$ such that $\mathbb{S}_k(v^*,\xi^*) \in \mathbb{U}(k)$ and

$$\|u(k) - U_\xi(k,0)v^*\|_{1,2} \le R/2^{k-1},\ \ \|\xi - \xi^*\|_{L^2_b(\Omega_{0,k+1})} \le R/(2^{k+1}L'). \tag{4.23}$$

The second inequality follows from the second inequality in (4.13) and from the fact that $G(k) = \{\xi_j,\ j = 1,\cdots,N\}$ is an $R/2^{k+2}L'$-net. In particular, we note that this inequality implies that

$$\|T_k\xi - T_k\xi^*, \Omega_0\|_{0,2} \le R/(2^{k+1}L'). \tag{4.24}$$

(That was the reason why we have considered the time interval $[0,k+1]$ instead of $[0,k]$ from the very beginning.)

We note that, by definition of $\mathbb{M}$, the point $u^*(T) := U_{T_k\xi^*}(k+t,k)U_\xi(0,k)v^*$ belongs to $\mathcal{M}$. Moreover, estimate (1.29) implies that

$$\|u(T) - u^*(T)\|_{1,2} \le L''(\|u(k) - u^*(k)\|_{1,2} + \|T_k\xi - T_k\xi^*, \Omega_0\|_{0,2}). \tag{4.25}$$

(We note that the constant $L'' \ge 1$ is chosen such that, for every $u(k), u^*(k) \in B(u_0, 2R, V_0)$ and every $0 < T-k < 1$, (4.25) holds. It is possible to do so thanks to Corollary 1.1.)

Inserting estimates (4.23) and (4.24) into the right-hand side of (4.25), we have

$$\|u(T) - u^*(T)\|_{1,2} \le L_1 2^{-k} \le L_1 2^{-T+1}. \tag{4.26}$$

Since $u^*(T) \in \mathcal{M}$, we obtain the attraction property.

So, there only remains to verify the entropy estimate. To this end, it is convenient to introduce the operator

$$\mathbb{S} : \Phi \to C([0,1], V_0), \quad \mathbb{S}(v,\xi) := U_\xi(t,0)v, \tag{4.27}$$

and to consider the set $\hat{\mathcal{M}} := \mathbb{S}\mathbb{M} \subset C([0,1], V_0)$. (Roughly speaking, we replace every point of $\mathcal{M}$ by the piece of the corresponding trajectory of length one.) To this end, we need the following Proposition.

**Proposition 4.1.** *The $L''R/2^{k-2}$-balls for the topology of $C([0,1], V_0)$ centered at the points of $\mathbb{S}(\mathbb{U}(k))$ cover $\mathbb{S}(\mathbb{S}_k\mathbb{B})$.*

The proof of this proposition is similar to our proof of the attraction property. We take an arbitrary point $(v,\xi) \in \mathbb{B}$ and we set $u(t) := U_\xi(t,0)v$. Then, there exists a point $(v^*,\xi^*) \in \mathbb{B}$, $u^*(t) = U_{\xi^*}(t,0)v^*$, such that $(u^*(k), T_k\xi^*) \in \mathbb{U}(k)$ and

$$\|u(k) - u^*(k)\|_{1,2} \le R/2^{k-1}, \quad \|\xi - \xi^*\|_{L^2_b(Omega_{0,k+1})} \le R/(2^{k+1}L') < R/2^{k+1}. \tag{4.28}$$

Thus, according to estimate (4.25)

$$\|u(t+k) - u^*(t+k)\|_{1,2} \le L''(R/2^{k-1} + R/2^{k+1}) \le L''R/2^{k-2}, \ 0 < t < 1, \tag{4.29}$$

and Proposition 4.1 is proved.

Let us now fix $\varepsilon$, $R > \varepsilon > 0$, and let us construct the $\varepsilon$-covering of $\hat{\mathcal{M}}$. As in the discrete case, we compute $k = k(\varepsilon)$ from the inequality

$$L''R/2^k < \varepsilon < L''R/2^{k-1}. \tag{4.30}$$

We then decompose the set $\hat{\mathcal{M}}$ as follows:

$$\hat{\mathcal{M}} = \cup_{l \le k}\mathbb{S}(\mathbb{E}(l)) \cup \mathbb{S}\left(\cup_{l \ge k}\mathbb{E}(l)\right) := \hat{\mathcal{M}}_1 \cup \hat{\mathcal{M}}_2. \tag{4.31}$$

We note that, as in the discrete case, $\hat{\mathcal{M}}_2 \subset \mathbb{S}(\mathbb{S}_{k+1}\mathbb{B})$ and, due to Proposition 4.1, it can be covered by the $\varepsilon$-balls centered at the points of $\mathbb{S}(\mathbb{U}(k+1)) \subset \mathbb{S}(\mathbb{E}(k+1))$. Thus, we have proved that the system of $\varepsilon$-balls centered at the points of $\mathbb{S}(\mathbb{E}(l))$, $l = 1, \cdots, k+1$, covers $\hat{\mathcal{M}}$. Consequently (compare with (4.21)), the entropy $\mathbb{H}_\varepsilon$ of this covering satisfies

$$\mathbb{H}_\varepsilon(\hat{\mathcal{M}}, C([0,1], V_0)) \le \mathbb{H}_\varepsilon \le (k+1)\log_2 P + \log_2 N(k) + 2\log_2(k+1). \tag{4.32}$$

Computing $k = k(\varepsilon)$ from (4.30) and inserting this value into (4.32), we obtain an entropy estimate of the form (4.6) for the set $\hat{\mathcal{M}}$.

Now, we are in a position to estimate the entropy of $\mathcal{M}$ and to complete the proof of the lemma. To this end, we introduce the projector

$$\mathrm{Pr} : C([0,1], V_0) \to 2^{V_0}, \quad \mathrm{Pr}(v) := \{v(t) : t \in [0,1]\}.$$

Obviously, $\mathcal{M} = \mathrm{Pr}(\hat{\mathcal{M}})$ and, consequently

$$\mathrm{dist}_{V_0}(\mathcal{M}, \cup_{l \le k+1} \mathrm{Pr}(\mathbb{S}(\mathbb{E}(l)))) \le \varepsilon. \tag{4.33}$$

Thus, there remains to construct the appropriate covering of the sets $\mathcal{O}_\varepsilon(\mathrm{Pr}(\mathbb{S}(\mathbb{E}(l)))$, where $\mathcal{O}_\varepsilon$ denotes the $\varepsilon$-neighborhood in $V_0$. We recall that the sets $\mathbb{E}(l)$ contain a finite number of points. Therefore, it is sufficient to know how to cover the sets

$$\mathcal{O}_\varepsilon(\mathbb{S}(v,\xi)) = \{w \in V_0 : \exists t \in [0,1],\ \|v(t) - w\|_{V_0} \le \varepsilon,\ v(t) := U_\xi(t,0)v\}, \tag{4.34}$$

for every $(v,\xi) \in \mathbb{E}(k)$. In order to construct such a covering, we note that, according to our construction, $\mathbb{M} \subset \mathbb{S}_1\mathbb{B}$ (since we take a union in (4.16) not from $k = 0$, but from $k = 1$). Then, according to Corollary 1.2, $\mathcal{M}$ is bounded in $V_0^{\beta'}$ and, consequently (due to Theorem 1.4), every trajectory $u(t) := U_\xi(t,0)v$ starting from an arbitrary point $(v,\xi) \in \mathbb{M}$ is uniformly Holder continuous in $V_0$, i.e.

$$\|u(t+s) - u(t)\|_{V_0} \le Cs^\gamma, \quad t \ge 0,\ 0 < s < 1. \tag{4.35}$$

In particular, (4.35) holds uniformly with respect to $(v,\xi) \in \mathbb{E}(k),\ k \in \mathbb{N}$.

Let us fix $s_0 = s_0(\varepsilon) = (\varepsilon/C)^{1/\gamma}$ and let us consider the following discrete subset of (4.34):

$$L_{(v,\xi)} := \{v(ns_0) : n = 0, 1, \cdots, [1/s_0]\}. \tag{4.36}$$

Then, obviously, the $2\varepsilon$-balls centered at the points of $L_{(v,\xi)}$ cover (4.34). Moreover

$$\#L_{(v,\xi)} = 1 + [1/s_0] \le (C/\varepsilon)^{1/\gamma}, \tag{4.37}$$

and, consequently, the system of $2\varepsilon$-balls centered at the points of $L_{(v,\xi)}$, for every $(v,\xi) \in \mathbb{E}(l)$, $l \le k+1$, covers t$\mathcal{M}$. The entropy of this covering can be estimated by $\mathbb{H}_\varepsilon + 1/\gamma \log_2 C/\varepsilon$. Thus

$$\mathbb{H}_{2\varepsilon}(\mathcal{M}, V_0) \le (k+1)\log_2 P + \log_2 N(k) + 2\log_2(k+1) + \frac{1}{\gamma}\log_2 C/\varepsilon. \tag{4.38}$$

Computing $k(\varepsilon)$ from inequality (4.30) and inserting this expression into (4.38), we obtain estimate (4.6). Lemma 4.1, and thus Theorem 4.1, are proved.

**Remark 4.2.** We note that our construction of (infinite dimensional) exponential attractors is not based on the squeezing property (see also [13]) and is thus valid in a general Banach setting.

## §5 Examples

In this Section, we present several examples of external forces $g$ and discuss the corresponding global/uniform and exponential attractors. We start with the simplest case where $g$ is independent of $t$. Some related questions can be found in [4].

**Example 5.0 (autonomous external forces).** Let $g$ be independent of $t$. Then, equation (0.1) is autonomous and $\mathcal{H}(g) = \{g\}$. Consequently, the second term in estimate (4.6) vanishes and we have the *finite dimensional* global attractor $\mathcal{A}$ and exponential attractor $\mathcal{M}$ for the autonomous equation (0.1), where

$$\dim_F(\mathcal{A}, V_0) \le \dim_F(\mathcal{M}, V_0) \le C_{aut}. \tag{5.1}$$

As a next simple example, we consider the case of quasiperiodic external forces.

**Example 5.1 (quasiperiodic external forces).** We assume that the external force $g$ is of the form

$$g(t) = G(\mathcal{T}_t\alpha_0), \quad \alpha_0 \in \mathbb{T}^m, \tag{5.2}$$

where $G : \mathbb{T}^m \to L^2(\omega)$ is a smooth (at least $C^1$) function from the $m$-dimensional torus $\mathbb{T}^m$ into $L^2(\omega)$ and $\mathcal{T}_t$ is the standard linear flow on the $m$-dimensional torus $\mathbb{T}^m$ ($\mathcal{T}_t\alpha_0 := (\alpha_0 + t\beta)\, mod\, (2\pi)^m$, $\beta \in \mathbb{R}^m$ being a vector with rationally independent frequencies). In other words

$$g(t) = G(\beta^1 t + \alpha_0^1, \cdots, \beta^m t + \alpha_0^m),$$

and the function $G(z_1, \cdots, z_m)$ is $2\pi$-periodic with respect to each variable $z_l$. Then, the hull $\mathcal{H}(g)$ has the following description (see e.g [4]):

$$\mathcal{H}(g) = \{\xi_\alpha(t) := G(\mathcal{T}_t\alpha) : \alpha \in \mathbb{T}^m\}.$$

Furthermore, the hull $\mathcal{H}(g)$ is diffeomorphic to the torus $\mathbb{T}^m$ if $G$ is injective.

We note that the linear flow $\mathcal{T}_t$ preserves the distance between the points on $\mathbb{T}^m$. Consequently (since $G$ is $C^1$)

$$\mathbb{H}_\varepsilon(\mathcal{H}(g)\big|_{\Omega_{0,K\log_2 1/\varepsilon}}, L_b^2) \le \mathbb{H}_\varepsilon(\mathcal{H}(g), C_b(\mathbb{R}, L^2(\omega)) \le \\ \le H_{\varepsilon/C}(\mathbb{T}^m, \mathbb{R}^m) = (m + \overline{\overline{o}}(1)) \log_2 1/\varepsilon. \tag{5.3}$$

Inserting (5.3) into (3.11) and (4.6), we deduce that we again have the *finite dimensional* uniform attractor $\mathcal{A}$ and exponential attractor $\mathcal{M}$, with

$$\dim_F(\mathcal{A}, V_0) \le \dim_F(\mathcal{M}, V_0) \le C_{aut} + m, \tag{5.4}$$

where the constant $C_{aut} = C_{aut}(f, a, \|g\|_{L_b^2})$ is the same as that derived in the autonomous case (5.1).

Construction (5.3) can be generalized as follows.

**Example 5.2 (cascade systems).** Let $\mathcal{K}$ be a compact metric space with finite fractal dimension $\dim_F(\mathcal{K},\mathcal{K}) = m < \infty$ and let $\mathcal{T}_t$, $t \in \mathbb{R}$, be a flow on $\mathcal{K}$ that has a finite Lyapunov exponent $\mu \ge 0$:

$$d_{\mathcal{K}}(\mathcal{T}_t k_1, \mathcal{T}_t k_2) \le C 2^{\mu t} d_{\mathcal{K}}(k_1, k_2), \quad t \ge 0. \tag{5.5}$$

As in the previous example, we consider a uniformly Lipschitz continuous function $G : \mathcal{K} \to L^2(\omega)$ and an external force of the form

$$g(t) = G(\xi(t)), \quad \xi(t) = \mathcal{T}_t k_0, \quad k_0 \in \mathcal{K}. \tag{5.6}$$

For instance, the function $\xi(t)$ may be a complete bounded trajectory of the autonomous ODE

$$\xi'(t) = F(\xi(t)), \ \xi \in \mathbb{R}^m, \tag{5.7}$$

with $F$ of class $C^1$. (In other words, we consider an autonomous cascade system of ODE (5.7) and the RDE (0.1) with external force $g(t) = G(\xi(t))$.)

Then, $\mathcal{K} := [\xi(t), t \in \mathbb{R}]_{\mathbb{R}^m}$ is a compact set with finite ($\le m$) fractal dimension and condition (5.5) is also satisfied (since $F$ is $C^1$).

Let us now estimate the entropy of the hull $\mathcal{H}(g)$ of (5.6). We note that, due to the uniform Lipschitz continuity of $G$, we obtain

$$\mathbb{H}_{\varepsilon/L}(\mathcal{H}(g)\big|_{\Omega_{0,T(\varepsilon)}}, L^2_b) \le \mathbb{H}_{\varepsilon/C_1 L}(\mathcal{H}(\xi), L^\infty([0,T(\varepsilon)],\mathcal{K})), \tag{5.8}$$

where $\mathcal{H}(\xi)$ is the hull of $\xi(t)$ in $L^\infty_{loc}(\mathbb{R},\mathcal{K})$ and $T(\varepsilon) = K \log_2 1/\varepsilon$. Estimate (5.5), together with the fact that $\mathcal{K}$ is finite dimensional, imply the estimate

$$\begin{aligned} \mathbb{H}_{\varepsilon/C_1 L}(\mathcal{H}(\xi), L^\infty([0,T(\varepsilon)],\mathcal{K})) &\le \mathbb{H}_{2^{-\mu T(\varepsilon)}\varepsilon/C_2 L}(\mathcal{K},\mathcal{K}) \le \\ &\le (\dim_F \mathcal{K} + \bar{\bar{o}}(1)(\log_2 1/\varepsilon + \mu T(\varepsilon) + \log_2(C_2 L)) = (m(1+\mu K) + \bar{\bar{o}}(1)) \log_2 1/\varepsilon. \end{aligned} \tag{5.9}$$

Thus, we have proved that, in that case, equation (0.1) also possesses the *finite dimensional* uniform attractor $\mathcal{A}$ and exponential attractor $\mathcal{M}$ such that

$$\dim_F(\mathcal{A}, V_0) \le \dim_F(\mathcal{M}, V_0) \le C_{aut} + m(1 + K\mu), \tag{5.10}$$

where $K$ is the same as in (4.6).

**Example 5.3 (stabilizing external forces).** We assume here that the right-hand side $g$ belongs to $W^{1,2}_b(\mathbb{R}, L^2(\omega))$ and stabilizes as $t \to \pm\infty$ in the following sense: there exist $g^\pm \in L^2(\omega)$ and $\alpha > 0$ such that

$$\|g - g^\pm, \Omega_T\|_{0,2} \le C|T|^{-\alpha}. \tag{5.11}$$

Then, obviously

$$\mathcal{H}(g) = \{g^-\} \cup \{g^+\} \cup \{T_h g, h \in \mathbb{R}\}. \tag{5.12}$$

Let us construct the $\varepsilon$-net in $\mathcal{H}(g)\big|_{\Omega_{0,T(\varepsilon)}}$. To this end, we note that, on the one hand (due to the fact that $\partial_t g \in L^2_b(\Omega)$)

$$\|g - T_h g, \Omega_T\|_{0,2} \le C_1 h, \quad h \in \mathbb{R}, \tag{5.13}$$

for every $T \in \mathbb{R}$, and, on the other hand (due to estimate (5.11))

$$\|g^+ - T_h g\|_{L^2_b(\Omega_{0,T(\varepsilon)})} \le \varepsilon \text{ or } \|g^- - T_h g\|_{L^2_b(\Omega_{0,T(\varepsilon)})} \le \varepsilon, \tag{5.14}$$

if $h \notin [-t_c(\varepsilon), T(\varepsilon) + t_c(\varepsilon)]$, where $t_c(\varepsilon) = (C/\varepsilon)^{1/\alpha}$. Consequently, the set

$$\{g^-\} \cup \{g^+\} \cup \{T_{h_n} g, \quad h_n = -t_c + n\varepsilon/C_1, \quad n = 0, \cdots, [\frac{C_1(T(\varepsilon) + 2t_c)}{\varepsilon}] + 1\},$$

where $C_1$ is the same as in (5.13), is the $\varepsilon$-net in $\mathcal{H}(g)\big|_{\Omega_{0,T(\varepsilon)}}$. The entropy of this $\varepsilon$-net can be estimated as follows

$$\mathbb{H}_\varepsilon(\mathcal{H}(g), L^2_b(\Omega_{0,T(\varepsilon)})) \le \log_2 C/\varepsilon + \log_2(T(\varepsilon) + t_c(\varepsilon)) = (1 + \frac{1}{\alpha} + \overline{\overline{o}}(1)) \log_2 1/\varepsilon.$$

Thus, we have proved that equation (0.1) with a stabilizing external force $g$ possesses the *finite dimensional* uniform attractor $\mathcal{A}$ and exponential attractor $\mathcal{M}$ such that

$$\dim_F(\mathcal{A}, V_0) \le \dim_F(\mathcal{M}, V_0) \le C_{aut} + 1 + \frac{1}{\alpha}.$$

Let us now consider examples of infinite dimensional external forces. We start with the so-called frequency modulated external forces which are widespread in the information's industry.

**Example 5.4 (frequency modulated external forces).** Let $\mathbb{B}_L(\mathbb{R}) \subset C_b(\mathbb{R})$ be the space of functions $\xi \in C_b(\mathbb{R})$ whose Fourier transform (in the sense of distributions) $\widehat{\xi}$ have finite support

$$\operatorname{supp} \widehat{\xi} \subset [-L, L], \tag{5.15}$$

(see e.g. [23] for a detailed study of the space $\mathbb{B}_L(\mathbb{R})$). Let us fix an arbitrary $\xi \in \mathbb{B}_L(\mathbb{R})$. Then (due to the fact that $T_h\mathbb{B}_L(\mathbb{R}) = \mathbb{B}_L(\mathbb{R})$), the hull $\mathcal{H}(\xi)$ of this function in $C_{loc}(R)$ is a bounded subset of $\mathbb{B}_L(\mathbb{R})$ (it is known that $\mathbb{B}_L(\mathbb{R})$ consists of entire functions (see [23]). Consequently, $\xi(t)$ is obviously translation-compact).

It is proved in [38] that the restriction of the unit ball $B(1,0,\mathbb{B}_L(\mathbb{R}))$ in $\mathbb{B}_L(\mathbb{R})$ to the interval $[0,T]$ has the following entropy estimate:

$$\mathbb{H}_\varepsilon(B(0,1,\mathbb{B}_L(\mathbb{R})),C([0,T]))\le C(T+\log_2 1/\varepsilon)\log_2 1/\varepsilon, \tag{5.16}$$

(and this estimate is sharp, at least if $T\sim\log_2 1/\varepsilon$ or is larger). Consequently, we have the following estimate for the entropy of the hull $\mathcal{H}(\xi)$:

$$\mathbb{H}_\varepsilon(\mathcal{H}(\xi),C([0,T]))\le C_1(T+\log_2 1/\varepsilon)\log_2 1/\varepsilon. \tag{5.17}$$

We now assume that the right-hand side $g$ of (0.1) has the following form: $g(t)=G(\xi(t))$, where $G\in C^1(\mathbb{R},L^2(\omega))$ (for instance, $g(t)=\xi(t)g_0(x)$, with $g_0\in L^2(\omega)$). Then, (5.17) implies that

$$\mathbb{H}_{\varepsilon/L}(\mathcal{H}(g),L^2_b(\Omega_{0,T(\varepsilon)}))\le C_2\left(\log_2 1/\varepsilon\right)^2. \tag{5.18}$$

Thus, we have proved that equation (0.1) with a frequency modulated right-hand side $g$ possesses the (infinite dimensional) uniform attractor $\mathcal{A}$ and exponential attractor $\mathcal{M}$ having the following entropy estimates:

$$\mathbb{H}_\varepsilon(\mathcal{A},V_0)\le\mathbb{H}_\varepsilon(\mathcal{M},V_0)\le C\left(\log_2 1/\varepsilon\right)^2. \tag{5.19}$$

(compare with the typical asymptotics given in Section 3.)

We conclude our examples by considering a cascade system for which the right-hand side $g$ satisfies a RDE in an *unbounded domaim* (for simplicity, in $\mathbb{R}^n$).

**Example 5.5 (cascade systems with unbounded domains).** We assume that the external force $g(t)=\xi(t)\big|_{x\in\omega}$, where $\xi(t)=\xi(t,x)$, $x\in\mathbb{R}^n$, satisfies the RDE

$$\partial_t\xi=\Delta_x\xi-F(\xi),\quad x\in\mathbb{R}^n,\quad \xi(t)\in L^2_b(\mathbb{R}^n), \tag{5.20}$$

where, by definition, $L^2_b(\mathbb{R}^n)$ is a B-space generated by the following norm:

$$\|\xi(t)\|_{L^2_b}:=\sup_{x_0\in\mathbb{R}^n}\|\xi(t),B^1_{x_0}\|_{0,2}<\infty, \tag{5.21}$$

and $B^R_{x_0}$ denotes the $R$-ball centered at $x_0$. In other words, only bounded with respect to $|x|\to\infty$ solutions of (5.20) are considered.

It is known (see e.g [15], [28], [37] and [38]) that, under some natural dissipativity assumptions on $F$, this equation is well-posed and that the corresponding semigroup possesses the bounded global attractor $\mathcal{K}$ in $L^2_b(\Omega)$, which is compact in $L^2_{loc}(\mathbb{R}^n)$. We note that the attractor $\mathbb{A}$ is usually infinite dimensional but, as proved in [38] it possesses the following entropy estimate:

$$\mathbb{H}_\varepsilon(\mathbb{A}\big|_{B^R_{x_0}},C(B^R_{x_0}))\le C(R+\log_2 1/\varepsilon)^n\log_2 1/\varepsilon, \tag{5.22}$$

(moreover, this estimate is in a sense sharp).

We now assume that our external force $\xi(t)$, $t \in \mathbb{R}$, is a complete bounded trajectory of (5.20) which consequently belongs to the attractor $\mathbb{A}$. Then, it is not difficult to prove, using estimate (5.22) and estimate (5.23) below for the difference of two solutions on the attractor (which generalizes (5.5)):

$$\|u_1(t) - u_2(t), B^1_{x_0}\|^2_{0,2} \le Ce^{Lt} \int_{\mathbb{R}^n} |u_1(0,x) - u_2(0,x)|^2 e^{-\varepsilon|x-x_0|}\, dx, \tag{5.23}$$

(with $C$, $L$ and $\varepsilon$ independent of $x_0$ (see [14], [15] or [38] for the proof)), that the entropy of the hull $\mathcal{H}(g)$ satisfies:

$$H_{\varepsilon/L}(\mathcal{H}(g)\big|_{\Omega_{0,T(\varepsilon)}}, L^2_b(\Omega_{0,T(\varepsilon)})) \le C\,(\log_2 1/\varepsilon)^{n+1}. \tag{5.24}$$

Thus, equation (0.1), with right-hand side $g(t) = \xi(t,x)\big|_{x\in\omega}$ such that $\xi(t)$ is a solution of equation (5.20) in $\mathbb{R}^n$, possesses the (infinite dimensional) global attractor $\mathcal{A}$ and exponential attractor $\mathcal{M}$ having the following entropy estimates:

$$\mathbb{H}_\varepsilon(\mathcal{A}, V_0) \le \mathbb{H}_\varepsilon(\mathcal{M}, V_0) \le C\,(\log_2 1/\varepsilon)^{n+1}, \tag{5.25}$$

(compare with the typical asymptotics given in Section 3).

We also note that this result remains valid if $\xi$ is a solution of a damped hyperbolic equation in $\mathbb{R}^n$

$$\partial_t^2\xi + \gamma\partial_t\xi - \Delta_x\xi + F(\xi) = 0, \quad x \in \mathbb{R}^n, \quad \gamma > 0 \tag{5.26}$$

Indeed, the existence of a (locally compact) attractor $\mathbb{A}$ for the equation (5.26) is verified in [18] and the analogues of estimates (5.22) and (5.23) are obtained for this case in [39] and [40].

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