

EXPONENTIAL ATTRACTORS FOR SINGULARLY PERTURBED DAMPED WAVE EQUATIONS: A SIMPLE CONSTRUCTION

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ABSTRACT. This note is concerned with the damped wave equation

$$\varepsilon^2 \partial_{tt} u + \partial_t u - \Delta u + f(u) = g$$

depending on a small parameter ε and with the corresponding parabolic equation

$$\partial_t u - \Delta u + f(u) = g$$

obtained in the singular limit $\varepsilon \rightarrow 0$. The existence of a family \mathcal{M}_ε of exponential attractors which is Hölder continuous with respect to ε is proved.

1. INTRODUCTION

The longtime behavior of dynamical systems generated by dissipative partial differential equations can be better understood by using the concept of a *global attractor*. This is, by definition, the unique compact invariant set of the phase space which attracts the images of all bounded subsets as time tends to infinity (see [2, 3, 17] and references therein). However, the global attractor may present two essential drawbacks; namely, the rate of attraction may be arbitrarily slow and, as a rule, it cannot be estimated in terms of the physical parameters of the system under consideration. Consequently, the global attractor may be very sensitive to perturbations.

An alternative object to describe the longterm dynamics is an *inertial manifold*, which is free from the above-mentioned drawbacks (see [10]). Unfortunately, its existence can be proved only under very restrictive spectral gap assumptions, which can be verified in few particular dynamical systems, mainly arising from one-dimensional parabolic equations (see also [16] for nonexistence examples).

In order to overcome this difficulty, an intermediate object has been introduced in [6], an *exponential attractor* (or *inertial set*). Although, like the global attractor, it is not a manifold and usually has an irregular fractal structure, it is finite-dimensional and its rate of attraction is exponential and measurable in terms of the physical parameters, as in the case of an inertial manifold. Besides, an exponential attractor turns out to be almost as general as the global one, since its existence can be generally verified whenever a finite-dimensional global attractor is available.

It is also known that an exponential attractor is much more robust with respect to perturbations than the global one. Nonetheless, due to its essential nonuniqueness, only the relatively weak property of the *continuity up to time shifts* is available (see [6]), without the proper choice of a “one-valued branch” in the set of all admissible exponential attractors. Thus, the problem of a “clever” choice of an exponential attractor in dependence of perturbation parameters becomes crucial in order to obtain upper and lower semicontinuity or Hölder continuity results with respect to perturbations. This problem has been studied for different types of regular and singular perturbations in [7, 9, 11, 12, 13, 15]. However, in all these papers, the Hölder continuity of the family of exponential attractors \mathcal{M}_ε (ε is a perturbation parameter) has been proved only at

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one fixed point $\varepsilon = \varepsilon_0$, that is,

$$\text{dist}^{\text{sym}}(\mathcal{M}_\varepsilon, \mathcal{M}_{\varepsilon_0}) \leq C|\varepsilon - \varepsilon_0|^\vartheta, \quad \vartheta > 0,$$

where dist^{sym} is the symmetric Hausdorff distance in the proper phase space (see below). In addition, the unperturbed exponential attractor $\mathcal{M}_{\varepsilon_0}$ is needed in order to build \mathcal{M}_ε .

An alternative framework which allows to obtain a uniform Hölder continuity for all reasonable values of the perturbation parameter, that is,

$$\text{dist}^{\text{sym}}(\mathcal{M}_{\varepsilon_1}, \mathcal{M}_{\varepsilon_2}) \leq C|\varepsilon_1 - \varepsilon_2|^\vartheta, \quad \vartheta > 0,$$

has been recently suggested in [8]. The result of that paper deals with *regular* perturbations and is formally applicable to dynamical systems $S_\varepsilon(t)$ acting in the same phase space and depending smoothly on the perturbation parameter ε . Besides, the exponential attractor $\mathcal{M}_{\varepsilon_0}$ now only depends on the dynamical system $S_{\varepsilon_0}(t)$, and its construction requires no information on $S_\varepsilon(t)$ for $\varepsilon \neq \varepsilon_0$.

The main focus of the present work is to show that the above result can also be applied to the case of *singular* perturbations by using a simple scaling argument. For simplicity, we illustrate our method on the model example of a singularly perturbed damped wave equation, which is, in a sense, the standard model of a singularly perturbed dynamical system in the theory of attractors (see [2, 3, 17]). The applications of this method to more general classes of singularly perturbed systems will be considered in a forthcoming paper.

2. THE MAIN RESULT

For $\varepsilon \in (0, 1]$, we consider the Cauchy problem for the damped wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$

$$(2.1) \quad \begin{cases} \varepsilon^2 \partial_{tt} u + \partial_t u - \Delta u + f(u) = g, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \\ u|_{\partial\Omega} = 0, \end{cases}$$

along with the parabolic Cauchy problem

$$(2.2) \quad \begin{cases} \partial_t u - \Delta u + f(u) = g, \\ u(0) = u_0, \\ u|_{\partial\Omega} = 0, \end{cases}$$

obtained in the singular limit $\varepsilon \rightarrow 0$. Here, $g \in L^2(\Omega)$ is independent of time and $f \in C^2(\mathbb{R})$, with $f(0) = 0$, satisfies the standard growth and dissipation conditions

$$|f''(u)| \leq c_f(1 + |u|), \quad \liminf_{|u| \rightarrow \infty} \frac{f(u)}{u} > -\lambda_1,$$

where $c_f \geq 0$ and $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on $(L^2(\Omega), \|\cdot\|, \langle \cdot, \cdot \rangle)$ with Dirichlet boundary conditions. We introduce the product Banach spaces

$$X_\varepsilon^0 = H_0^1(\Omega) \times L^2(\Omega), \quad X_\varepsilon^1 = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega),$$

endowed with the norms

$$\|(u, v)\|_{X_\varepsilon^0}^2 = \|\nabla u\|^2 + \varepsilon^2 \|v\|^2, \quad \|(u, v)\|_{X_\varepsilon^1}^2 = \|\Delta u\|^2 + \varepsilon^2 \|v\|^2.$$

When $\varepsilon = 0$, the second components of the spaces X_0^0 and X_0^1 reduce to $\{0\}$. It is well known that (2.1)-(2.2) generate strongly continuous semigroups

$$S_\varepsilon(t) : X_\varepsilon^0 \rightarrow X_\varepsilon^0$$

satisfying the continuous dependence estimates

$$(2.3) \quad \|S_\varepsilon(t)\xi_1 - S_\varepsilon(t)\xi_2\|_{X_\varepsilon^0} \leq ce^{ct} \|\xi_1 - \xi_2\|_{X_\varepsilon^0},$$

for some $c \geq 0$ depending (increasingly) only on the X_ε^0 -norms of ξ_1 and ξ_2 (see [1, 2]).

Notation. We denote the Hausdorff semidistance and the symmetric Hausdorff distance of two subsets B_1, B_2 of a Banach space X by, respectively,

$$\text{dist}_X(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|x_1 - x_2\|_X$$

and

$$\text{dist}_X^{\text{sym}}(B_1, B_2) = \max \{ \text{dist}_X(B_1, B_2), \text{dist}_X(B_2, B_1) \}.$$

Throughout this paper, $C \geq 0$, $\omega > 0$ and $\vartheta \in (0, 1)$ will denote *generic* constants, while \mathcal{Q} will denote a *generic* increasing positive function. All these quantities, unless otherwise stated, are *independent* of ε . Further dependencies will be specified on occurrence. Besides, we set

$$\mathbb{B}_\varepsilon(r) = \{ \xi \in X_\varepsilon^1 : \|\xi\|_{X_\varepsilon^1} \leq r \}.$$

In [9, 14, 15, 18] the following further facts are established.

- There is $r_0 > 0$ such that the ball $\mathbb{B}_\varepsilon(r_0)$ is uniformly exponentially attracting for $S_\varepsilon(t)$, that is, for every bounded set $B \subset X_\varepsilon^0$,

$$(2.4) \quad \text{dist}_{X_\varepsilon^0}(S_\varepsilon(t)B, \mathbb{B}_\varepsilon(r_0)) \leq \mathcal{Q}(\|B\|_{X_\varepsilon^0})e^{-\omega t}.$$

- The semigroup $S_\varepsilon(t)$ fulfills a uniform asymptotic smoothing property: it can be split into the sum $S_\varepsilon(t) = L_\varepsilon(t) + K_\varepsilon(t)$ in such a way that, for every $\xi_1, \xi_2 \in \mathbb{B}_\varepsilon(r)$,

$$(2.5) \quad \begin{aligned} \|L_\varepsilon(t)\xi_1 - L_\varepsilon(t)\xi_2\|_{X_\varepsilon^0} &\leq Ce^{-\omega t}\|\xi_1 - \xi_2\|_{X_\varepsilon^0}, \\ \|K_\varepsilon(t)\xi_1 - K_\varepsilon(t)\xi_2\|_{X_\varepsilon^1} &\leq \mathcal{Q}(t)\|\xi_1 - \xi_2\|_{X_\varepsilon^0}. \end{aligned}$$

- For every $\xi = (u_0, u_1) \in \mathbb{B}_\varepsilon(r)$, we have the uniform estimate

$$(2.6) \quad \|\Delta u(t)\|^2 + \varepsilon^2 \|\nabla \partial_t u(t)\|^2 + \int_t^{t+1} \|\nabla \partial_t u(\tau)\|^2 d\tau \leq C.$$

- For every $t, \tau \in [0, T]$ and every $\xi \in \mathbb{B}_\varepsilon(r)$,

$$(2.7) \quad \|S_\varepsilon(t)\xi - S_\varepsilon(\tau)\xi\|_{X_\varepsilon^0} \leq \mathcal{Q}(T)|t - \tau|.$$

Here, \mathcal{Q} depends on ε .

The main result of this paper reads as follows.

Theorem 2.1. *For every $\varepsilon \in [0, 1]$, there exists an exponential attractor \mathcal{M}_ε for the semigroup $S_\varepsilon(t)$ acting on X_ε^0 which satisfies the following properties.*

(P1) \mathcal{M}_ε is bounded in X_ε^1 and in X_ε^0 , with bounds independent of ε .

(P2) The rate of attraction is uniformly exponential: for every bounded set $B \subset X_\varepsilon^0$,

$$\text{dist}_{X_\varepsilon^0}(S_\varepsilon(t)B, \mathcal{M}_\varepsilon) \leq \mathcal{Q}(\|B\|_{X_\varepsilon^0})e^{-\omega t}.$$

(P3) The fractal dimension of \mathcal{M}_ε in X_ε^0 is uniformly bounded, that is, $\dim_{X_\varepsilon^0}[\mathcal{M}_\varepsilon] \leq C$.

(P4) The map $\varepsilon \mapsto \mathcal{M}_\varepsilon$ is Hölder continuous in the following sense: If $\varepsilon_1 > \varepsilon_2$, then

$$\text{dist}_{X_{\varepsilon_1}^0}^{\text{sym}}(\mathcal{M}_{\varepsilon_1}, \mathcal{M}_{\varepsilon_2}) \leq C(\varepsilon_1 - \varepsilon_2)^\vartheta.$$

Recall that (see [5, 6, 7]), given a semigroup $S(t)$ on a Banach space X , an exponential attractor for $S(t)$ is a compact set $\mathcal{M} \subset X$ of finite fractal dimension in X , positively invariant for $S(t)$ and such that, for every bounded set $B \subset X$,

$$\text{dist}_X(S(t)B, \mathcal{M}) \leq \mathcal{Q}(\|B\|_X)e^{-\omega t}.$$

As recalled in the introduction, a weaker result than Theorem 2.1 has been proved in [9, 13], where the existence of a family of exponential attractors \mathcal{M}_ε has been shown, with a Hölder continuity property only at $\varepsilon = 0$. In this note, exploiting the ideas of [8], we establish the

Hölder continuity of \mathcal{M}_ε for all $\varepsilon \in [0, 1]$. We also mention that the growth restriction on f'' can be removed for ε small enough, by using the methods of [19].

3. THE ASSOCIATED DISCRETE SEMIGROUP

For $t_\star \geq 1$ to be determined, we consider the discrete semigroup

$$S_\varepsilon = S_\varepsilon(t_\star) : X_\varepsilon^0 \rightarrow X_\varepsilon^0.$$

Our first step is to show the existence of a family of discrete exponential attractors for S_ε , sharing analogous properties to those stated in Theorem 2.1. To this end, we will make use of the following abstract result from [8], concerned with the existence of a family of exponential attractors satisfying a Hölder continuity property with respect to (regular) perturbations.

Theorem 3.1. *Let X and X^1 be two Banach spaces with compact embedding $X^1 \Subset X$, and let P be a closed subset of X^0 bounded in X^1 . For every $\varepsilon \in [0, 1]$, assume that there exist a δ -neighborhood $\mathcal{O}_\delta(P)$ ($\delta > 0$) of the set P in X^1 and a family of maps $\widehat{S}_\varepsilon : \mathcal{O}_\delta(P) \rightarrow P$ satisfying the following conditions.*

(C1) *For every $x_1, x_2 \in \mathcal{O}_\delta(P)$,*

$$\widehat{S}_\varepsilon x_1 - \widehat{S}_\varepsilon x_2 = \mathcal{L}_\varepsilon(x_1, x_2) + \mathcal{K}_\varepsilon(x_1, x_2),$$

where

$$\|\mathcal{L}_\varepsilon(x_1, x_2)\|_X \leq \vartheta \|x_1 - x_2\|_X,$$

$$\|\mathcal{K}_\varepsilon(x_1, x_2)\|_{X^1} \leq C \|x_1 - x_2\|_X.$$

(C2) *The family \widehat{S}_ε is uniformly Hölder continuous with respect to ε , that is,*

$$\sup_{x \in \mathcal{O}_\delta(P)} \|\widehat{S}_{\varepsilon_1} x - \widehat{S}_{\varepsilon_2} x\|_X \leq C |\varepsilon_1 - \varepsilon_2|^\vartheta.$$

Then, there exists a family of closed sets $\widehat{\mathcal{M}}_\varepsilon^d \subset P$, positively invariant for \widehat{S}_ε , such that

$$(3.1) \quad \text{dist}_X(\widehat{S}_\varepsilon^n P, \widehat{\mathcal{M}}_\varepsilon^d) \leq C e^{-\omega n},$$

$$(3.2) \quad \dim_X[\widehat{\mathcal{M}}_\varepsilon^d] \leq C,$$

$$(3.3) \quad \text{dist}_X^{\text{sym}}(\widehat{\mathcal{M}}_{\varepsilon_1}^d, \widehat{\mathcal{M}}_{\varepsilon_2}^d) \leq C |\varepsilon_1 - \varepsilon_2|^\vartheta,$$

where $\widehat{S}_\varepsilon^n$ ($n \in \mathbb{N}$) is the family of discrete semigroups generated by the iterations of \widehat{S}_ε .

Remark 3.2. As $\widehat{\mathcal{M}}_\varepsilon^d$ is positively invariant, (C1) and (C2) imply that the thesis of Theorem 3.1 continues to hold if we replace $\widehat{\mathcal{M}}_\varepsilon^d$ with

$$\widehat{\mathcal{M}}_{\varepsilon,1}^d = \widehat{S}_\varepsilon \widehat{\mathcal{M}}_\varepsilon^d \subset \widehat{\mathcal{M}}_\varepsilon^d \subset P.$$

Indeed, $\widehat{\mathcal{M}}_{\varepsilon,1}^d$ is clearly positively invariant and compact, and the analogues of (3.1)-(3.2) are apparent. Concerning the analogue of (3.3), we observe that, if $x_j \in \widehat{\mathcal{M}}_{\varepsilon_j,1}^d$, then $x_j = \widehat{S}_\varepsilon w_j$, for some $w_j \in \widehat{\mathcal{M}}_{\varepsilon_j}^d$. Hence, from (C1) and (C2),

$$\|x_1 - x_2\|_X \leq \|\widehat{S}_{\varepsilon_1} w_1 - \widehat{S}_{\varepsilon_1} w_2\|_X + \|\widehat{S}_{\varepsilon_1} w_2 - \widehat{S}_{\varepsilon_2} w_2\|_X \leq C \|w_1 - w_2\|_X + C |\varepsilon_1 - \varepsilon_2|,$$

and the conclusion follows from (3.3).

However, we cannot directly employ this abstract result for our purposes, since the phase space under consideration depends explicitly on ε . In order to overcome this difficulty, we introduce, for any $\varepsilon \in [0, 1]$, the *scaling operators* $T_\varepsilon : X_\varepsilon^j \rightarrow X_1^j$, defined by

$$T_\varepsilon(\eta, \zeta) = (\eta, \varepsilon \zeta),$$

and we construct ε -scaled maps $\widehat{S}_\varepsilon : X_1^0 \rightarrow X_1^0$ associated with S_ε as

$$\widehat{S}_\varepsilon(y, z) = T_\varepsilon S_\varepsilon T_\varepsilon^{-1}(y, z).$$

When $\varepsilon = 0$, we set $T_0^{-1}(y, z) = (y, 0)$, so that

$$\widehat{S}_0(y, z) = S_0(y, 0).$$

With this position, if $\xi \in X_\varepsilon^j$, we have the equality $\|\xi\|_{X_\varepsilon^j} = \|T_\varepsilon \xi\|_{X_1^j}$.

Our goal is now to apply Theorem 3.1 to the ε -scaled maps \widehat{S}_ε , with $X = X_1^0$, $X^1 = X_1^1$ and $P = \mathbb{B}_1(r)$, for a fixed $r > r_0$. Note that, since X_1^1 is reflexive, $\mathbb{B}_1(r)$ is closed in X_0^1 . Fixing $\delta > 0$, it is apparent from (2.5) that, if we fix t_\star large enough,

$$\widehat{S}_\varepsilon : \mathcal{O}_\delta(\mathbb{B}_1(r)) \rightarrow \mathbb{B}_1(r),$$

with $\mathcal{O}_\delta(\mathbb{B}_1(r)) = \mathbb{B}_1(r + \delta)$, and condition (C1) of Theorem 3.1 is satisfied by setting

$$\begin{aligned} \mathcal{L}_\varepsilon(x_1, x_2) &= \widehat{L}_\varepsilon x_1 - \widehat{L}_\varepsilon x_2, \\ \mathcal{K}_\varepsilon(x_1, x_2) &= \widehat{K}_\varepsilon x_1 - \widehat{K}_\varepsilon x_2, \end{aligned}$$

where $\widehat{K}_\varepsilon = T_\varepsilon K_\varepsilon(t_\star) T_\varepsilon^{-1}$ and $\widehat{L}_\varepsilon = T_\varepsilon L_\varepsilon(t_\star) T_\varepsilon^{-1}$. If we can prove that (C2) holds as well, and we have the boundary layer condition

$$(3.4) \quad T_\varepsilon^{-1} \widehat{S}_\varepsilon \mathbb{B}_1(r) \text{ is bounded in } X_1^0 \text{ uniformly with respect to } \varepsilon > 0,$$

then we claim that there exists a family of compact sets

$$\mathcal{M}_\varepsilon^d = T_\varepsilon^{-1} \widehat{\mathcal{M}}_{\varepsilon,1}^d \subset T_\varepsilon^{-1} \mathbb{B}_1(r) = \mathbb{B}_\varepsilon(r),$$

positively invariant for S_ε and uniformly bounded in X_0^1 , such that

$$(3.5) \quad \text{dist}_{X_\varepsilon^0}(S_\varepsilon^n \mathbb{B}_\varepsilon, \mathcal{M}_\varepsilon^d) \leq C e^{-\omega n},$$

$$(3.6) \quad \dim_{X_\varepsilon^0}[\mathcal{M}_\varepsilon^d] \leq C,$$

$$(3.7) \quad \text{dist}_{X_{\varepsilon_1}^0}^{\text{sym}}(\mathcal{M}_{\varepsilon_1}^d, \mathcal{M}_{\varepsilon_2}^d) \leq C(\varepsilon_1 - \varepsilon_2)^\vartheta, \quad \varepsilon_1 > \varepsilon_2.$$

In order to prove the claim, we first note that, by construction and in view of (3.4), $\mathcal{M}_\varepsilon^d$ is compact, contained in $\mathbb{B}_\varepsilon(r)$ and uniformly bounded in X_1^0 . Moreover, using the positive invariance of $\widehat{\mathcal{M}}_{\varepsilon,1}^d$, we observe that

$$S_\varepsilon \mathcal{M}_\varepsilon^d = T_\varepsilon^{-1} \widehat{S}_\varepsilon \widehat{\mathcal{M}}_{\varepsilon,1}^d \subset T_\varepsilon^{-1} \widehat{\mathcal{M}}_{\varepsilon,1}^d = \mathcal{M}_\varepsilon^d.$$

Since T_ε is just a scaling, (3.1)-(3.2) easily imply (3.5)-(3.6). The Hölder continuity (3.7) is slightly more delicate. From (3.4),

$$\|\Pi_2 x\|_{Z^0} \leq C\varepsilon, \quad \forall x \in \widehat{\mathcal{M}}_{\varepsilon,1}^d,$$

where Π_2 is the projection of X_0^0 onto its second component Z^0 . Indeed, $x = \widehat{S}_\varepsilon w$, for some $w \in \widehat{\mathcal{M}}_\varepsilon^d \subset \mathbb{B}_1(r)$. Assume now $\varepsilon_1 > \varepsilon_2 > 0$, and let $\xi_j = (\eta_j, \zeta_j) \in \mathcal{M}_{\varepsilon_j}^d$. Setting $z_j = \varepsilon_j \zeta_j$, we have

$$T_{\varepsilon_j} \xi_j = (\eta_j, z_j) \in \widehat{\mathcal{M}}_{\varepsilon_j,1}^d.$$

Making use of the inequality

$$\frac{1}{\varepsilon_2^2} \|\varepsilon_2 z_1 - \varepsilon_1 z_2\|_{Z^0}^2 \leq 2\|z_1 - z_2\|_{Z^0}^2 + \frac{2(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon_2^2} \|z_2\|_{Z^0}^2 \leq 2\|z_1 - z_2\|_{Z^0}^2 + C(\varepsilon_1 - \varepsilon_2)^2,$$

we obtain

$$\|\xi_1 - \xi_2\|_{X_{\varepsilon_1}^0}^2 = \|\eta_1 - \eta_2\|_{Y^0}^2 + \frac{1}{\varepsilon_2^2} \|\varepsilon_2 z_1 - \varepsilon_1 z_2\|_{Z^0}^2 \leq 2\|T_{\varepsilon_1} \xi_1 - T_{\varepsilon_2} \xi_2\|_{X_1^0}^2 + C(\varepsilon_1 - \varepsilon_2)^2.$$

Hence, (3.7) follows from (3.3). The case $\varepsilon_1 > \varepsilon_2 = 0$ is immediate.

4. PROOF OF THEOREM 2.1

We preliminarily note that it is sufficient to construct uniform exponential attractors \mathcal{M}_ε having $\mathbb{B}_\varepsilon(r)$ as a basin of attraction rather than the whole space X_ε^0 . Namely, such that

$$(4.1) \quad \text{dist}_{X_\varepsilon^0}(S_\varepsilon(t)\mathbb{B}_\varepsilon(r), \mathcal{M}_\varepsilon) \leq Ce^{-\omega t}.$$

Indeed, due to the uniform exponential attraction property (2.4) and the Lipschitz continuity (2.3), we can appeal to the transitivity of the exponential attraction, devised in [9], which leads to the desired conclusion (P2).

The next two lemmas show that (3.4) and condition (C2) hold true for $S_\varepsilon = S_\varepsilon(t_*)$.

Lemma 4.1. *For $\varepsilon > 0$ and $t \geq 1$, the boundary layer estimate*

$$\|S_\varepsilon(t)\xi\|_{X_\varepsilon^0} \leq \mathcal{Q}(\rho)$$

holds for all $\xi \in X_\varepsilon^1$ such that $\|\xi\|_{X_\varepsilon^1} \leq \rho$.

Proof. Along this proof, $K \geq 0$ is a generic constant depending (increasingly) on ρ , but independent of ε . Let $u(t)$ be the solution to (2.1) with initial data $\xi = (u_0, u_1)$. From (2.6) and the continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$,

$$\|u(t)\|_{L^\infty} \leq C\|\Delta u(t)\| \leq C\|S(t)\xi\|_{X_\varepsilon^1} \leq \mathcal{Q}(\|\xi\|_{X_\varepsilon^1}) \leq K.$$

Hence, in light of the assumptions on f ,

$$\|\Delta u - f(u) + g\| \leq K.$$

Setting $w = \partial_t u$, we can rewrite (2.1) as

$$\varepsilon^2 \partial_t w + w = \Delta u - f(u) + g.$$

Since the right-hand side is uniformly bounded in $L^2(\Omega)$, we obtain, by solving explicitly the equation,

$$\|\partial_t u(t)\| \leq \|u_1\|e^{-t/\varepsilon^2} + K \leq \frac{\rho}{\varepsilon}e^{-t/\varepsilon^2} + K.$$

Thus, for $t \geq 1$, the required boundedness for $\|S_\varepsilon(t)\xi\|_{X_\varepsilon^0}$ follows. \square

Lemma 4.2. *For $t \geq 1$,*

$$\|\widehat{S}_{\varepsilon_1}(t)x - \widehat{S}_{\varepsilon_2}(t)x\|_{X_1^0} \leq \mathcal{Q}(\rho)e^{\mathcal{Q}(\rho)t}|\varepsilon_1 - \varepsilon_2|^{1/2},$$

for all $x \in X_1^1$ such that $\|x\|_{X_1^1} \leq \rho$. Here, $\widehat{S}_{\varepsilon_j}(t) = T_{\varepsilon_j}^{-1}S_{\varepsilon_j}(t)T_{\varepsilon_j}$.

Proof. Again, $K \geq 0$ is a generic constant depending on ρ , but independent of $\varepsilon_1, \varepsilon_2$. Let us first prove the result for the case $\varepsilon = \varepsilon_1 > \varepsilon_2 = 0$. For $x = (u_0, u_1)$, let $u^0(t)$ be the solution to (2.2), and $u^\varepsilon(t)$ the solution to (2.1) with initial data $\xi = (u_0, \varepsilon^{-1}u_1)$. Then, the difference $v = u^0 - u^\varepsilon$ solves the equation

$$\partial_t v - \Delta v = [f(u^\varepsilon) - f(u^0)] + \varepsilon^2 \partial_{tt} u^\varepsilon,$$

with $v(0) = 0$. Arguing as in the previous lemma,

$$\|u^0(t)\|_{L^\infty} + \|u^\varepsilon(t)\|_{L^\infty} \leq K,$$

and, from the assumptions on f , we learn that

$$\|f(u^0) - f(u^\varepsilon)\| \leq K\|v\|.$$

Analogously, by (2.6),

$$\int_t^{t+1} \|\nabla \partial_t u^\varepsilon\| \|\nabla \partial_t v\| d\tau \leq K.$$

Setting

$$\Phi = \frac{1}{2} \|\nabla v\|^2 - \varepsilon^2 \langle \nabla \partial_t u^\varepsilon, \nabla v \rangle,$$

and multiplying the equation above by $-\Delta v$, we obtain

$$\frac{d}{dt} \Phi + \|\Delta v\|^2 = \langle f(u^0) - f(u^\varepsilon), \Delta v \rangle - \varepsilon^2 \langle \nabla \partial_t u^\varepsilon, \nabla \partial_t v \rangle \leq K \|\nabla v\|^2 + \|\Delta v\|^2 + \varepsilon^2 \|\nabla \partial_t u^\varepsilon\| \|\nabla \partial_t v\|.$$

On account of (2.6),

$$\varepsilon^2 |\langle \nabla \partial_t u^\varepsilon, \nabla v \rangle| \leq \varepsilon^2 \|\nabla \partial_t u^\varepsilon\| \|\nabla v\| \leq K \varepsilon \|\nabla v\|.$$

Consequently,

$$\frac{1}{4} \|\nabla v\|^2 - K \varepsilon^2 \leq \Phi \leq \|\nabla v\|^2 + K \varepsilon^2.$$

Hence, we end up with the differential inequality

$$\frac{d}{dt} \Phi \leq K \Phi + K \varepsilon^2 + \varepsilon^2 \|\nabla \partial_t u^\varepsilon\| \|\nabla \partial_t v\|.$$

As $\Phi(0) = 0$, by (2.6) and the Gronwall lemma, we conclude that

$$\Phi(t) \leq K e^{Kt} \varepsilon^2,$$

which, in turn, implies that

$$\|\nabla u^0(t) - \nabla u^\varepsilon(t)\| \leq K e^{Kt} \varepsilon.$$

Concerning the norm of the second component, Lemma 4.1 gives, for $t \geq 1$,

$$\|\varepsilon \partial_t u^\varepsilon(t) - 0\| \leq K \varepsilon.$$

In conclusion, we have proved that, if $t \geq 1$,

$$\|\widehat{S}_\varepsilon(t)\xi - \widehat{S}_0(t)\xi\|_{X_1^0} \leq K e^{Kt} \varepsilon.$$

Let us now consider the remaining case $\varepsilon_1 > \varepsilon_2 > 0$. We first observe that, from the inequality above,

$$(4.2) \quad \|\widehat{S}_{\varepsilon_1}(t)\xi - \widehat{S}_{\varepsilon_2}(t)\xi\|_{X_1^0} \leq K e^{Kt} \varepsilon_1.$$

This is not enough to finish the proof of the lemma, and we need one more estimate. Let $u^1(t)$ and $u^2(t)$ be the solutions to (2.1) with $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$ respectively, with initial data $(u_0, \varepsilon_1^{-1} u_1)$ and $(u_0, \varepsilon_2^{-1} u_1)$. Then, the difference $v = u^2 - u^1$ solves the equation

$$\varepsilon_2^2 \partial_{tt} v + \partial_t v - \Delta v = [f(u^1) - f(u^2)] + (\varepsilon_1^2 - \varepsilon_2^2) \partial_{tt} u^1,$$

with initial data

$$v(0) = 0, \quad \partial_t v(0) = (\varepsilon_2^{-1} - \varepsilon_1^{-1}) u_1.$$

Reasoning as in the former case,

$$\|f(u^1) - f(u^2)\| \leq K \|v\|.$$

Besides, using (2.6) and equation (2.1) for u^1 , we find the control

$$\varepsilon_1^4 \int_t^{t+1} \|\partial_{tt} u^1(\tau)\|^2 d\tau \leq K.$$

Thus, multiplying the equation above by $\partial_t v$, and arguing in a standard way, we are led to

$$\frac{d}{dt} [\|\nabla v\|^2 + \varepsilon_2^2 \|\partial_t v\|^2] \leq K [\|\nabla v\|^2 + \varepsilon_2^2 \|\partial_t v\|^2] + \varepsilon_1^2 (\varepsilon_1 - \varepsilon_2)^2 \|\partial_{tt} u^1\|^2,$$

and the Gronwall lemma yields

$$\|\nabla v(t)\|^2 + \varepsilon_2^2 \|\partial_t v(t)\|^2 \leq K e^{Kt} \frac{(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon_1^2}.$$

Combining this estimate with the obvious equality

$$\varepsilon_2 \partial_t u^2 - \varepsilon_1 \partial_t u^1 = \varepsilon_2 \partial_t v + (\varepsilon_2 - \varepsilon_1) \partial_t u^1,$$

where $\|\partial_t u^1\|$ is bounded due to Lemma 4.1, we finally infer that, for $t \geq 1$,

$$(4.3) \quad \|\widehat{S}_{\varepsilon_1}(t)\xi - \widehat{S}_{\varepsilon_2}(t)\xi\|_{X_1^0} \leq K e^{Kt} \frac{(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1}.$$

The proof of the case $\varepsilon_1 > \varepsilon_2 > 0$ then follows from (4.2) and (4.3), just noting that

$$\min\{\varepsilon_1, (\varepsilon_1 - \varepsilon_2)/\varepsilon_1\} \leq (\varepsilon_1 - \varepsilon_2)^{1/2}. \quad \square$$

In summary, we have obtained the existence of a family $\mathcal{M}_\varepsilon^d \subset \mathbb{B}_\varepsilon(r)$ of discrete exponential attractors for S_ε , uniformly bounded in X_1^0 and satisfying (3.5)-(3.7).

The final step is to define the exponential attractors for the continuous semigroups $S_\varepsilon(t)$ via the standard expression

$$\mathcal{M}_\varepsilon = \bigcup_{t \in [t_\star, 2t_\star]} S_\varepsilon(t) \mathcal{M}_\varepsilon^d.$$

Indeed, the positive invariance of \mathcal{M}_ε is an immediate consequence of the positive invariance of $\mathcal{M}_\varepsilon^d$, while (P1) follows from (2.6) and Lemma 4.1. The uniform exponential attraction property (4.1) (which implies (P2), as shown before) and (P4) also follow from the analogous properties (3.5) and (3.7) of the discrete attractors together with the uniform Lipschitz continuity (2.3). Finally, collecting (2.3) and (2.7), we see that the map $(t, \xi) \mapsto S_\varepsilon(t)\xi$ is Lipschitz continuous from $[t_\star, 2t_\star] \times \mathcal{M}_\varepsilon^d$ into $\mathcal{M}_\varepsilon^d$. As a byproduct, \mathcal{M}_ε is compact in X_ε^0 and (P3) holds. This completes the proof of Theorem 2.1.

REFERENCES

- [1] J. Arrieta, A.N. Carvalho, J.K. Hale, *A damped hyperbolic equation with critical exponent*, Comm. Partial Differential Equations **17** (1992), 841–866.
- [2] A.V. Babin, M.I. Vishik, *Regular attractors of semigroups and evolution equations*, J. Math. Pures Appl. **62** (1983), 441–491.
- [3] A.V. Babin and M.I. Vishik, *Attractors of evolution equations*, North-Holland, Amsterdam, 1992.
- [4] M. Conti, V. Pata, M. Squassina, *Singular limit of differential systems with memory*, Indiana Univ. Math. J. **55** (2006), 169–215.
- [5] L. Dung, B. Nicolaenko, *Exponential attractors in Banach spaces*, J. Dynam. Differential Equations **13** (2001), 791–806.
- [6] A. Eden, C. Foias, B. Nicolaenko, R. Temam, *Exponential attractors for dissipative evolution equations*, Masson, Paris, 1994.
- [7] M. Efendiev, A. Miranville, S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in \mathbb{R}^3* , C.R. Acad. Sci. Paris Sér. I Math. **330** (2000), 713–718.
- [8] M. Efendiev, A. Miranville, S. Zelik, *Exponential attractors and finite-dimensional reduction for nonautonomous dynamical systems*, Proc. Roy. Soc. Edinburgh Sect. A **13** (2005), 703–730.
- [9] P. Fabrie, C. Galusinski, A. Miranville, S. Zelik, *Uniform exponential attractors for a singularly perturbed damped wave equation*, Discrete Contin. Dynam. Systems **10** (2004), 211–238.
- [10] C. Foias, G. Sell, R. Temam, *Inertial manifolds for nonlinear evolution equations*, J. Diff. Eqns. **73** (1988), 309–353.
- [11] S. Gatti, M. Grasselli, A. Miranville, V. Pata, *On the hyperbolic relaxation of the one-dimensional Cahn-Hilliard equation*, J. Math. Anal. Appl. **312** (2005), 230–247.
- [12] S. Gatti, M. Grasselli, A. Miranville, V. Pata, *Memory relaxation of first order evolution equations*, Nonlinearity **18** (2005), 1859–1883.
- [13] S. Gatti, M. Grasselli, A. Miranville, V. Pata, *A construction of a robust family of exponential attractors*, Proc. Amer. Math. Soc. **134** (2006), 117–127.
- [14] M. Grasselli, V. Pata, *On the damped semilinear wave equation with critical exponent*, Dynamical systems and differential equations (Wilmington, NC, 2002). Discrete Contin. Dynam. Systems (suppl.) (2003), 351–358.
- [15] M. Grasselli, V. Pata, *Asymptotic behavior of a parabolic-hyperbolic system*, Commun. Pure Appl. Anal. **3** (2004), 849–881.

- [16] X. Mora, J. Solà-Morales, *Existence and non-existence of finite-dimensional globally attracting invariant manifolds in semilinear damped wave equations*, in “Dynamics of infinite-dimensional systems” (Lisbon, 1986), NATO Adv. Sci. Inst. F. Comput. Systems Sci., Vol. 37, Springer-Verlag, Berlin, 1987.
- [17] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1997.
- [18] S. Zelik, *Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent*, Commun. Pure Appl. Anal. **3** (2004), 921–934.
- [19] S. Zelik, *Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities*, Discrete Contin. Dynam. Systems **11** (2004), 351–392.

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