Review of the 3-2-1 Euler Angles: a yaw–pitch–roll sequence

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1 Introduction

The $3-2-1$ Euler angles are one of the most widely used parameterisations of rotations. O'Reilly gives a history on page 184 of [4]. This review will give an overview of the important feautures of this set of Euler angles, and show that they are the ones used in [2] and [3].

Let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ be a basis for a fixed spatial frame, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for a body frame. The $3-2-1$ Euler angles provide an orthogonal matrix $\mathbf{Q}$ which maps $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$,

$$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\} \quad \xrightarrow{\mathbf{Q}} \quad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

by breaking the action of $\mathbf{Q}$ up into three steps

$$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\} \quad \longrightarrow \quad \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \quad \longrightarrow \quad \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \quad \longrightarrow \quad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\,,$$

where $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are intermediate frames of reference.

2 Map from $\mathbf{E}_j$ to $\mathbf{a}_j$: the yaw rotation

In this section the details of the map

$$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\} \quad \xrightarrow{\mathbf{L}(\psi,\mathbf{E}_3)} \quad \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$$

are constructed using the rotation tensor

$$\mathbf{L}(\psi, \mathbf{E}_3) = (\mathbf{I} - \mathbf{E}_3 \otimes \mathbf{E}_3)\cos\psi + \widehat{\mathbf{E}}_3 \sin\psi + \mathbf{E}_3 \otimes \mathbf{E}_3\,, \tag{2.1}$$

where

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c} := (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}\,, \quad \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3\,,$$

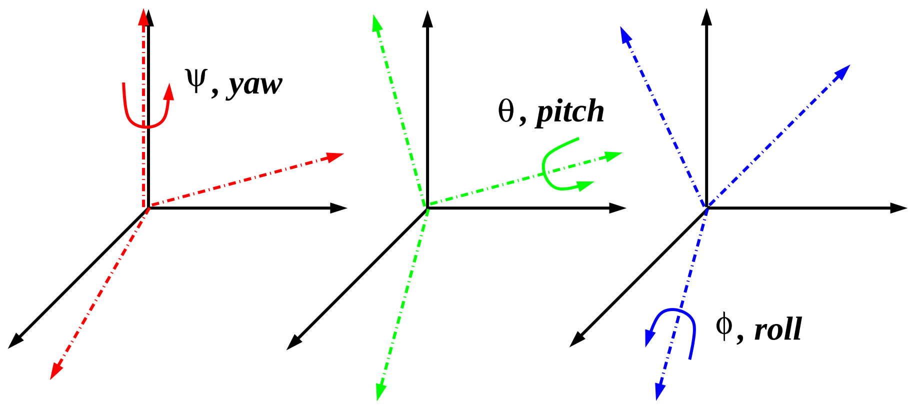

Figure 1: Schematic of the yaw-pitch-roll motion in terms of the Euler angles ψ, ϕ and θ.

and

$$\widehat{\mathbf{a}} := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} . \tag{2.2}$$

The hat-matrix has the property that

$$\widehat{\mathbf{a}}\,\mathbf{b} = \mathbf{a} \times \mathbf{b}\,, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\,.$$

The rotation (2.1) is a counterclockwise rotation about the $Z-$axis as shown schematically in Figure 1.

The new basis is

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{L}(\psi, \mathbf{E}_3)\mathbf{E}_1 = \mathbf{E}_1 \cos\psi + \mathbf{E}_3 \times \mathbf{E}_1 \sin\psi \\ \mathbf{a}_2 &= \mathbf{L}(\psi, \mathbf{E}_3)\mathbf{E}_2 = \mathbf{E}_2 \cos\psi + \mathbf{E}_3 \times \mathbf{E}_2 \sin\psi \\ \mathbf{a}_3 &= \mathbf{L}(\psi, \mathbf{E}_3)\mathbf{E}_3 = \mathbf{E}_3\,. \end{aligned}$$

Using the identities

$$\mathbf{E}_1 \times \mathbf{E}_2 = \mathbf{E}_3\,, \quad \mathbf{E}_3 \times \mathbf{E}_1 = \mathbf{E}_2\,, \quad \mathbf{E}_2 \times \mathbf{E}_3 = \mathbf{E}_1\,,$$

the matrix representation of the map is

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix} .$$

This is the first equation of (6.23) on page 184 of [4].

3 Map from $\mathbf{a}_j$ to $\mathbf{b}_j$: the pitch rotation

In this section the details of the map

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \quad \xrightarrow{\mathbf{L}(\theta, \mathbf{a}_2)} \quad \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$$

are constructed using the rotation tensor

$$\mathbf{L}(\theta, \mathbf{a}_2) = (\mathbf{I} - \mathbf{a}_2 \otimes \mathbf{a}_2) \cos\theta + \widehat{\mathbf{a}}_2 \sin\theta + \mathbf{a}_2 \otimes \mathbf{a}_2 \,, \tag{3.3}$$

where

$$\mathbf{a}_2 = -\mathbf{E}_1 \sin\psi + \mathbf{E}_2 \cos\psi \,.$$

The new basis is

$$\begin{aligned}
\mathbf{b}_1 &= \mathbf{L}(\theta, \mathbf{a}_2)\mathbf{a}_1 = \mathbf{a}_1 \cos\theta + \mathbf{a}_2 \times \mathbf{a}_1 \sin\theta \\
\mathbf{b}_2 &= \mathbf{L}(\theta, \mathbf{a}_2)\mathbf{a}_2 = \mathbf{a}_2 \\
\mathbf{b}_3 &= \mathbf{L}(\theta, \mathbf{a}_2)\mathbf{a}_3 = \mathbf{a}_3 \cos\theta + \mathbf{a}_2 \times \mathbf{a}_3 \sin\theta \,,
\end{aligned}$$

with matrix representation,

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} .$$

This is the second equation of (6.23) on page 184 of [4].

4 Map from $\mathbf{b}_j$ to $\mathbf{e}_j$: the roll rotation

In this section the details of the map

$$\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \quad \xrightarrow{\mathbf{L}(\phi, \mathbf{b}_1)} \quad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

are constructed using the rotation tensor

$$\mathbf{L}(\phi, \mathbf{b}_1) = (\mathbf{I} - \mathbf{b}_1 \otimes \mathbf{b}_1) \cos\phi + \widehat{\mathbf{b}}_1 \sin\phi + \mathbf{b}_1 \otimes \mathbf{b}_1 \,, \tag{4.4}$$

where

$$\mathbf{b}_1 = \mathbf{a}_1 \cos\theta - \mathbf{a}_3 \sin\theta \,.$$

The new basis is

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{L}(\phi, \mathbf{b}_1)\mathbf{b}_1 = \mathbf{b}_1 \\
\mathbf{e}_2 &= \mathbf{L}(\phi, \mathbf{b}_1)\mathbf{b}_2 = \mathbf{b}_2 \cos\phi + \mathbf{b}_1 \times \mathbf{b}_2 \sin\phi \\
\mathbf{e}_3 &= \mathbf{L}(\phi, \mathbf{b}_1)\mathbf{b}_3 = \mathbf{b}_3 \cos\phi + \mathbf{b}_1 \times \mathbf{b}_3 \sin\phi \,,
\end{aligned}$$

with matrix representation,

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix} .$$

This is the third equation of (6.23) on page 184 of [4].

5 The composite rotation from $\mathbf{E}_j$ to $\mathbf{e}_j$

In this section the details of the map

$$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\} \quad \xrightarrow{\mathbf{L}(\phi,\mathbf{b}_1)\mathbf{L}(\theta,\mathbf{a}_2)\mathbf{L}(\psi,\mathbf{a}_3)} \quad \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

are constructed. Using the above results

$$\begin{aligned}
\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} &= \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{pmatrix}\\
&= \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ \sin\phi\sin\theta & \cos\phi & \sin\phi\cos\theta\\ \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta\end{bmatrix}\begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{pmatrix}\\
&= \begin{bmatrix}\cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta\\ \sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi+\cos\phi\cos\psi & \sin\phi\cos\theta\\ \cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi & \cos\phi\cos\theta\end{bmatrix}\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{pmatrix}.
\end{aligned}$$

Write this as

$$\mathbf{e} = \mathbf{Q}^T\mathbf{E}\,.$$

Then

$$\mathbf{Q} = \begin{bmatrix}\cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi\\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi+\cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi\\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta\end{bmatrix}. \tag{5.5}$$

To see that this is the correct $\mathbf{Q}$, construct $\mathbf{Q}$ directly.

6 Direct construction of Q

By definition

$$\mathbf{Q} = \mathbf{L}(\phi,\mathbf{b}_1)\mathbf{L}(\theta,\mathbf{a}_2)\mathbf{L}(\psi,\mathbf{E}_3)\,.$$

In order to construct a matrix representation of the composite rotation, use the property of rotation tensors

$$\mathbf{L}(\theta,\mathbf{R}\mathbf{b}) = \mathbf{R}\,\mathbf{L}(\theta,\mathbf{b})\,\mathbf{R}^T\,.$$

when $\mathbf{R}$ is any proper rotation. Define

$$\mathbf{A}_3 = \mathbf{L}(\psi,\mathbf{E}_3)\,,\quad \mathbf{A}_2 = \mathbf{L}(\theta,\mathbf{E}_2)\,,\quad \mathbf{A}_1 = \mathbf{L}(\phi,\mathbf{E}_1)\,.$$

Then

$$\mathbf{L}(\theta,\mathbf{a}_2) = \mathbf{L}(\theta,\mathbf{A}_3\mathbf{E}_2) = \mathbf{A}_3\mathbf{A}_2\mathbf{A}_3^T\,,$$

and

$$
\begin{aligned}
\mathbf{L}(\phi, \mathbf{b}_1) &= \mathbf{L}(\phi, \mathbf{L}(\theta, \mathbf{a}_2)\mathbf{a}_1) \\
&= \mathbf{L}(\theta, \mathbf{a}_2)\mathbf{L}(\phi, \mathbf{a}_1)\mathbf{L}(\theta, \mathbf{a}_2)^T \\
&= \mathbf{A}_3\mathbf{A}_2\mathbf{A}_3^T\mathbf{L}(\phi, \mathbf{a}_1)\mathbf{A}_3\mathbf{A}_2^T\mathbf{A}_3^T \\
&= \mathbf{A}_3\mathbf{A}_2\mathbf{A}_3^T\mathbf{L}(\phi, \mathbf{A}_3\mathbf{E}_1)\mathbf{A}_3\mathbf{A}_2^T\mathbf{A}_3^T \\
&= \mathbf{A}_3\mathbf{A}_2\mathbf{A}_3^T\mathbf{A}_3\mathbf{A}_1\mathbf{A}_3^T\mathbf{A}_3\mathbf{A}_2^T\mathbf{A}_3^T \\
&= \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1\mathbf{A}_2^T\mathbf{A}_3^T\,.
\end{aligned}
$$

Hence

$$
\mathbf{Q} = \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1\mathbf{A}_2^T\mathbf{A}_3^T\mathbf{A}_3\mathbf{A}_2\mathbf{A}_3^T\mathbf{A}_3 = \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1 = \left(\mathbf{A}_1^T\mathbf{A}_2^T\mathbf{A}_3^T\right)^T\,.
$$

with matrix representation

$$
\mathbf{Q} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^T ,
$$

confirming the form of $\mathbf{Q}$.

7 Angular velocities

To compute angular velocities start with the representation

$$
\mathbf{Q} = \mathbf{A}_3\mathbf{A}_2\mathbf{A}_1 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} ,
$$

and use the properties

$$
\dot{\mathbf{A}}_1\mathbf{A}_1^T = \dot\phi\,\widehat{\mathbf{E}}_1\,, \quad \dot{\mathbf{A}}_2\mathbf{A}_2^T = \dot\theta\,\widehat{\mathbf{E}}_2\,, \quad \dot{\mathbf{A}}_3\mathbf{A}_3^T = \dot\psi\,\widehat{\mathbf{E}}_3\,.
$$

Now differentiate $\mathbf{Q}$,

$$
\dot{\mathbf{Q}} = \dot{\mathbf{A}}_3\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_3\dot{\mathbf{A}}_2\mathbf{A}_1 + \mathbf{A}_3\mathbf{A}_2\dot{\mathbf{A}}_1\,.
$$

The body angular velocity is

$$
\begin{aligned}
\widehat{\mathbf{\Omega}^b} &:= \mathbf{Q}^T\dot{\mathbf{Q}} \\
&= \mathbf{A}_1^T\mathbf{A}_2^T\mathbf{A}_3^T\left(\dot{\mathbf{Q}}\right) \\
&= \mathbf{A}_1^T\mathbf{A}_2^T\mathbf{A}_3^T\left(\dot{\mathbf{A}}_3\mathbf{A}_2\mathbf{A}_1 + \mathbf{A}_3\dot{\mathbf{A}}_2\mathbf{A}_1 + \mathbf{A}_3\mathbf{A}_2\dot{\mathbf{A}}_1\right) \\
&= \dot\psi\,\mathbf{A}_1^T\mathbf{A}_2^T\widehat{\mathbf{E}}_3\mathbf{A}_2\mathbf{A}_1 + \dot\theta\,\mathbf{A}_1^T\widehat{\mathbf{E}}_2\mathbf{A}_1 + \dot\phi\,\widehat{\mathbf{E}}_1\,.
\end{aligned}
$$

But

$$\mathbf{A}_1^T \widehat{\mathbf{E}}_2 \mathbf{A}_1 = (\widehat{\mathbf{A}_1^T \mathbf{E}_2})\,,$$

and

$$\begin{aligned}
\mathbf{A}_1^T \mathbf{E}_2 &= \Big[(\mathbf{I} - \mathbf{E}_1 \otimes \mathbf{E}_1)\cos\phi - \sin\phi \widehat{\mathbf{E}}_1 + \mathbf{E}_1 \otimes \mathbf{E}_1\Big]\,\mathbf{E}_2 \\
&= \mathbf{E}_2 \cos\phi - \sin\phi \mathbf{E}_1 \times \mathbf{E}_2 \\
&= \mathbf{E}_2 \cos\phi - \mathbf{E}_3 \sin\phi\,.
\end{aligned}$$

Similarly

$$\mathbf{A}_2^T \widehat{\mathbf{E}}_3 \mathbf{A}_2 = (\widehat{\mathbf{A}_2^T \mathbf{E}_3})\,,$$

and

$$\begin{aligned}
\mathbf{A}_2^T \mathbf{E}_3 &= \Big[(\mathbf{I} - \mathbf{E}_2 \otimes \mathbf{E}_2)\cos\theta - \sin\theta \widehat{\mathbf{E}}_2 + \mathbf{E}_2 \otimes \mathbf{E}_2\Big]\,\mathbf{E}_3 \\
&= \mathbf{E}_3 \cos\theta - \sin\theta \mathbf{E}_2 \times \mathbf{E}_3 \\
&= \mathbf{E}_3 \cos\theta - \mathbf{E}_1 \sin\theta\,.
\end{aligned}$$

Continuing

$$\mathbf{A}_1^T \mathbf{A}_2^T \widehat{\mathbf{E}}_3 \mathbf{A}_2 \mathbf{A}_1 = (\mathbf{A}_1^T (\mathbf{E}_3 \widehat{\cos\theta - \mathbf{E}_1} \sin\theta)\,,$$

and

$$\begin{aligned}
\mathbf{A}_1^T(\mathbf{E}_3 \cos\theta - \mathbf{E}_1 \sin\theta) &= -\mathbf{E}_1 \sin\theta + \cos\theta\, \mathbf{A}_1^T \mathbf{E}_3 \\
&= -\mathbf{E}_1 \sin\theta + \cos\theta(\mathbf{E}_3 \cos\phi - \mathbf{E}_1 \times \mathbf{E}_3 \sin\phi) \\
&= -\mathbf{E}_1 \sin\theta + \cos\theta(\mathbf{E}_3 \cos\phi + \mathbf{E}_2 \sin\phi) \\
&= -\mathbf{E}_1 \sin\theta + \mathbf{E}_2 \cos\theta \sin\phi + \mathbf{E}_3 \cos\theta \cos\phi\,.
\end{aligned}$$

Substitute into the expression for the body angular velocity,

$$\begin{aligned}
\boldsymbol{\Omega}^b &= \dot\psi(-\mathbf{E}_1 \sin\theta + \mathbf{E}_2 \cos\theta \sin\phi + \mathbf{E}_3 \cos\theta \cos\phi) \\
&\quad + \dot\theta(\mathbf{E}_2 \cos\phi - \mathbf{E}_3 \sin\phi) + \dot\phi \mathbf{E}_1 \\
&= \mathbf{E}_1(\dot\phi - \dot\psi \sin\theta) + \mathbf{E}_2(\dot\psi \cos\theta \sin\phi + \dot\theta \cos\phi) \\
&\quad + \mathbf{E}_3(\dot\psi \cos\theta \cos\phi - \dot\theta \sin\phi)\,.
\end{aligned}$$

In [4] the body angular velocity is denoted by $\boldsymbol{\omega}_{\mathbf{0R}}$ and the above expression for $\boldsymbol{\Omega}^b$ agrees with $\boldsymbol{\omega}_{\mathbf{0R}}$ on page 188 of [4].

In matrix form

$$\begin{pmatrix} \Omega_1^b \\ \Omega_2^b \\ \Omega_3^b \end{pmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta \sin\phi \\ 0 & -\sin\phi & \cos\theta \cos\phi \end{bmatrix} \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix}.$$

Write this is

$$\boldsymbol{\Omega}^b = \mathbf{B}^{-1} \dot\Theta\,.$$

Then

$$\mathbf{B} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} .$$

As shown below in §9, this expression agrees with $[B] := \mathbf{B}$ in [2] and in [3].

8 Small angle approximation

For small angle approximation rewrite the body representation of the angular velocity as

$$\begin{pmatrix} \Omega_1^b \\ \Omega_2^b \\ \Omega_3^b \end{pmatrix} = \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} + \begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & \cos\phi - 1 & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi - 1 \end{bmatrix} \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} .$$

Hence for small angles

$$\begin{pmatrix} \Omega_1^b \\ \Omega_2^b \\ \Omega_3^b \end{pmatrix} = \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} + \begin{bmatrix} 0 & 0 & -\theta \\ 0 & -\frac{1}{2}\phi^2 & \phi \\ 0 & -\phi & -\frac{1}{2}\theta^2 - \frac{1}{2}\phi^2 \end{bmatrix} \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} + \cdots .$$

Neglecting the higher order terms, the small angle approximation is

$$\begin{pmatrix} \Omega_1^b \\ \Omega_2^b \\ \Omega_3^b \end{pmatrix} \approx \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} .$$

The problem is how to approximate the rotation matrix. The rotation matrix lies on a manifold so standard linearization will result in a matrix which is no longer a rotation. Applying the small angle approximation to $\mathbf{Q}$ in (5.5)

$$\mathbf{Q}^{\text{approx}} = \begin{bmatrix} 1 & -\psi & \theta \\ \psi & 1 & -\phi \\ -\theta & \phi & 1 \end{bmatrix} = \mathbf{I} + \widehat{\mathbf{\Theta}} , \quad \mathbf{\Theta} = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} .$$

The problem is that $\mathbf{Q}^{\text{approx}}$ is no longer a rotation

$$(\mathbf{Q}^{\text{approx}})^T \neq (\mathbf{Q}^{\text{approx}})^{-1} .$$

Using this approximation will give some *motion* of a vehicle but not a rotation. There is however a standard way to approximate a rotation matrix using the Cayley transform. Redefine $\mathbf{Q}^{\text{approx}}$ as

$$\mathbf{Q}^{\text{cayley}} := (\mathbf{I} + \tfrac{1}{2}\widehat{\mathbf{\Theta}})(\mathbf{I} - \tfrac{1}{2}\widehat{\mathbf{\Theta}})^{-1} .$$

Then $\mathbf{Q}^{\text{cayley}}$ is orthogonal and so is a rotation, and it has the same order of accuracy as $\mathbf{Q}^{\text{approx}}$.

In general the angular velocity can be linearized, but extra care is necessary in linearizing rotations. On the other hand, there is no reason for these linearizations. In both analysis and numerics it is better to use the exact expressions, even for small angles..

9 Notation in [3]

On page 222 of [3] the ship translational displacements, denoted by (ξ_1, ξ_2, ξ_3), in the steady moving system and the Eulerian angles, denoted by (e_1, e_2, e_3), are related by

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{bmatrix} [R] & 0 \\ & \\ 0 & [B] \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} . \tag{9.1}$$

This is equation (5) in [3]. In equations (6)-(7) in [3], the matrices $[B]$ and $[R]$ are defined as follows

$$[B] = \begin{bmatrix} 1 & \sin e_1 \tan e_2 & \cos e_1 \tan e_2 \\ 0 & \cos e_1 & -\sin e_1 \\ 0 & \sin e_1 \sec e_2 & \cos e_1 \sec e_2 \end{bmatrix},$$

and

$$[R] = \begin{bmatrix} \cos e_2 \cos e_3 & \sin e_1 \sin e_2 \cos e_3 - \cos e_1 \sin e_3 & \cos e_1 \sin e_2 \cos e_3 + \sin e_1 \boxed{\sin e_2} \\ \cos e_2 \sin e_3 & \sin e_1 \sin e_2 \sin e_3 + \cos e_1 \cos e_3 & \boxed{\sin e_1} \sin e_2 \sin e_3 - \sin e_1 \cos e_3 \\ -\sin e_2 & \sin e_1 \cos e_2 & \cos e_1 \cos e_2 \end{bmatrix}.$$

Replacing $e_1 = \phi$, $e_2 = \theta$, $e_3 = \psi$ and $\xi_1 = q_1$, $\xi_2 = q_2$ and $\xi_3 = q_3$ it is clear (after correcting two boxed typos in $[R]$) that $[R] = \mathbf{Q}$ and $[B] = \mathbf{B}$. The first boxed term should be $\sin e_2 \rightarrow \sin e_3$ and the second boxed term should be $\sin e_1 \rightarrow \cos e_1$. In (9.1), the angular velocity (u_4, u_5, u_6) is related to the Euler angles by

$$\begin{pmatrix} u_4 \\ u_5 \\ u_6 \end{pmatrix} = [B]^{-1} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} .$$

Hence it is clear that $u_4 = \Omega_1^b$, $u_5 = \Omega_2^b$, $u_6 = \Omega_3^b$.

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