Dynamic coupling between shallow-water sloshing and a vehicle undergoing planar rigid-body motion

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1 Introduction

In this report the governing equations – for coupling between rigid-body motion of a vehicle containing fluid and the fluid motion – are derived. The motion of the fluid is determined by the Euler equations and these equations have been derived in [2]. Here, equations for the motion of the vehicle in the plane are derived. The strategy is the write down the Lagrangian functional for the vehicle containing fluid, and then the Euler-Lagrange equations give the vehicle motion equations.

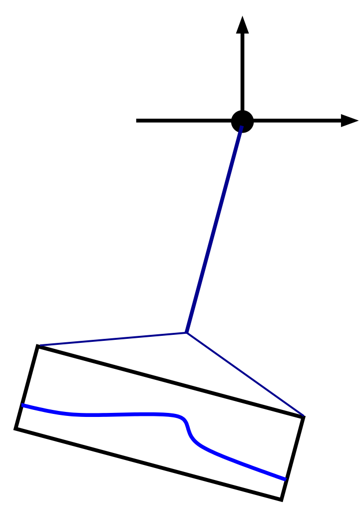
Let $\mathbf{X} = (X, Y)$ be planar coordinates for a fixed inertial frame and let $\mathbf{x} = (x, y)$ be coordinates for a moving frame attached to the vehicle. These coordinates are related by

$$\mathbf{X} = \mathbf{Q}\mathbf{x} + \mathbf{q}\,,$$

where $\mathbf{Q}$ is a proper orthogonal matrix and $\mathbf{q} \in \mathbb{R}^2$ is a vector representing the translation. Since every proper orthogonal matrix in the plane is of the form

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\,, \tag{1.1}$$

the rigid body motion can be represented by (θ, q_1, q_2).



A prototype of coupled motion is the "pendulum-slosh" problem shown to the right. Some experiments are (a) take quiescent initial conditions, but with some initial angular velocity, and determine the resulting fluid and angular motion; (b) take quiescent initial conditions, but with a forced harmonic translation

$$\mathbf{q}(t) = (q_1(t), q_2(t)) = (0, \varepsilon \sin\omega t)\,,$$

and the solve for the resulting angular displacement of the vessel and the fluid motion.

The purpose of this report is to explore the various reductions of the full equations to the shallow water equations, while maintaining the exact nonlinear equations for the vehicle motion. The report is a bit discursive and may contain typos. A distilled version is in progress.

2 Governing equations of the rigid tank motion

The governing equations of 3D rigid body motion coupled to sloshing have been derived in [5, 6] for the case of sloshing in spacecraft based on the balance of linear momentum and angular momentum. In this section it will be shown that these equations can derived directly from a variational principle. First the general form of the equations in 3D are derived and then restricted to 2D.

Two frames of reference are used. The spatial (inertial) frame has coordinates $\mathbf{X} = (X, Y, Z)$ and the body frame has coordinates $\mathbf{x} = (x, y, z)$. The distance between body frame origin to the point of rotation is denoted by $\mathbf{d} = (d_1, d_2, d_3)$. The fluid-tank system has a uniform translation $\mathbf{q}(t) = (q_1, q_2, q_3)$ relative to the spatial frame. The relations between spatial and body displacement, velocity and acceleration are derived in [2]. Introducing the following Lagrangian action and taking its first variation with respect to dependent variables gives the governing equations of rigid tank motion.

$$\mathscr{L} = \int_{t_1}^{t_2} \left(\mathrm{KE}^{fluid} - \mathrm{PE}^{fluid} + \mathrm{KE}^{vessel} - \mathrm{PE}^{vessel}\right) \mathrm{d}t\,,$$

where

$$\begin{aligned} \mathrm{KE}^{fluid} = & \int_{\mathsf{Vol}} \Big(\tfrac{1}{2}\|\dot{\mathbf{x}}\|^2 + \dot{\mathbf{x}}\cdot\big(\boldsymbol{\Omega}\times(\mathbf{x}+\mathbf{d}) + \mathbf{Q}^T\dot{\mathbf{q}}\big) + \mathbf{Q}^T\dot{\mathbf{q}}\cdot(\boldsymbol{\Omega}\times(\mathbf{x}+\mathbf{d})) \\ & + \tfrac{1}{2}\|\dot{\mathbf{q}}\|^2\Big)\rho \mathrm{d}\mathsf{Vol} + \tfrac{1}{2}\boldsymbol{\Omega}\cdot\mathbf{I}_f\boldsymbol{\Omega}\,, \end{aligned} \tag{2.2}$$

the kinetic energy of the dry vessel is

$$\mathrm{KE}^{vessel} = \tfrac{1}{2}m_v\|\dot{\mathbf{q}}\|^2 + (\boldsymbol{\Omega}\times m_v\overline{\mathbf{x}}_v)\cdot\mathbf{Q}^T\dot{\mathbf{q}} + \tfrac{1}{2}\boldsymbol{\Omega}\cdot\mathbf{I}_v\boldsymbol{\Omega}\,, \tag{2.3}$$

and $\mathbf{Q}(t)$ is a proper rotation in $\mathbb{R}^3$,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad \text{and} \quad \det[\mathbf{Q}] = 1\,,$$

$\mathbf{I}$ is the 3×3 identity matrix, $\boldsymbol{\Omega}$ is the axial vector of the skew-symmetric body angular velocity matrix $\widehat{\boldsymbol{\Omega}}$

$$\widehat{\boldsymbol{\Omega}} = \mathbf{Q}^T\dot{\mathbf{Q}} := \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}, \quad \boldsymbol{\Omega} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix},$$

and $\overline{\mathbf{x}}_v$ is the dry vessel center of mass coordinates relative to the body frame, $\mathbf{I}_f$ is the fluid mass moment of inertia tensor

$$\mathbf{I}_f = \int_{\mathsf{Vol}} \left(\|\mathbf{x}+\mathbf{d}\|^2\mathbf{I} - (\mathbf{x}+\mathbf{d})\otimes(\mathbf{x}+\mathbf{d})\right)\rho \mathrm{d}\mathsf{Vol},$$

and $\mathbf{I}_v$ is the dry vessel mass moment of inertia. Potential energy of the fluid is

$$\mathrm{PE}^{fluid} = \int_{\mathsf{Vol}} \rho g\left(\mathbf{Q}(\mathbf{x}+\mathbf{d}) + \mathbf{q}\right)\cdot\mathbf{E}_3 \mathrm{d}\mathsf{Vol}\,, \tag{2.4}$$

and potential energy of the dry vessel is

$$\mathrm{PE}^{vessel} \quad = \quad m_v g \left(\mathbf{Q}\overline{\mathbf{x}}_v + \mathbf{q} \right) \cdot \mathbf{E}_3 \,, \tag{2.5}$$

where $\mathbf{E}_3$ is the basis vector in the Z direction.

In the case of two dimensional motions, the equations simplify since rotations in 2D can be parameterized by a single angle as shown in (1.1. In two dimensions $\mathbf{X} = (X, Y)$, $\mathbf{x} = (x, y)$, $\mathbf{d} = (d_1, d_2)$, $\mathbf{q} = (q_1, q_2)$ and potential energies of the fluid and the vessel modify to

$$\mathrm{PE}^{fluid} \quad = \quad \int_{\mathsf{Vol}} \rho g \left(\mathbf{Q}\left(\mathbf{x} + \mathbf{d} \right) + \mathbf{q} \right) \cdot \mathbf{E}_2 \mathrm{d}\mathsf{Vol} \,,$$

and potential energy of the dry vessel is

$$\mathrm{PE}^{vessel} \quad = \quad m_v g \left(\mathbf{Q}\overline{\mathbf{x}}_v + \mathbf{q} \right) \cdot \mathbf{E}_2 \,,$$

where $\mathbf{E}_2$ is the basis vector in the Y direction. The kinetic and potential energies then read

$$\begin{aligned}
\mathrm{KE}^{fluid} \quad = \quad & \int_{\mathsf{Vol}} \Big[\tfrac{1}{2} \left(\dot{x}^2 + \dot{y}^2 \right) - \dot{\theta}\dot{x} \left(y + d_2 \right) + \Big(\dot{x} - \dot{\theta} \left(y + d_2 \right) \Big) \left(\dot{q}_1 \cos\theta + \dot{q}_2 \sin\theta \right) \\
& + \dot{\theta}\dot{y} \left(x + d_1 \right) + \Big(\dot{y} + \dot{\theta} \left(x + d_1 \right) \Big) \left(\dot{q}_2 \cos\theta - \dot{q}_1 \sin\theta \right) + \tfrac{1}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 \right) \\
& + \tfrac{1}{2} \dot{\theta}^2 \left(\left(x + d_1 \right)^2 + \left(y + d_2 \right)^2 \right) \Big] \rho \mathrm{d}\mathsf{Vol} \,, \\
\mathrm{KE}^{vessel} \quad = \quad & \tfrac{1}{2} m_v \left(\dot{q}_1^2 + \dot{q}_2^2 \right) - m_v \dot{\theta} \overline{\mathrm{y}}_v \left(\dot{q}_1 \cos\theta + \dot{q}_2 \sin\theta \right) + m_v \dot{\theta} \overline{\mathrm{x}}_v \left(\dot{q}_2 \cos\theta - \dot{q}_1 \sin\theta \right) \\
& + \tfrac{1}{2} I_v \dot{\theta}^2 \,, \\
\mathrm{PE}^{fluid} \quad = \quad & \int_{\mathsf{Vol}} \rho g \left(\sin\theta \left(x + d_1 \right) + \cos\theta \left(y + d_2 \right) + q_2 \right) \mathrm{d}\mathsf{Vol} \,, \\
\mathrm{PE}^{vessel} \quad = \quad & m_v g \left(\overline{\mathrm{x}}_v \sin\theta + \overline{\mathrm{y}}_v \cos\theta + q_2 \right) \,,
\end{aligned}$$

and so the Lagrangian action reads

$$\begin{aligned}
\mathscr{L} \quad = \quad & \int_{t_1}^{t_2} \int_0^L \int_0^h \Big[\tfrac{1}{2} \left(\dot{x}^2 + \dot{y}^2 \right) - \dot{\theta}\dot{x} \left(y + d_2 \right) + \Big(\dot{x} - \dot{\theta} \left(y + d_2 \right) \Big) \left(\dot{q}_1 \cos\theta + \dot{q}_2 \sin\theta \right) \\
& + \dot{\theta}\dot{y} \left(x + d_1 \right) + \Big(\dot{y} + \dot{\theta} \left(x + d_1 \right) \Big) \left(\dot{q}_2 \cos\theta - \dot{q}_1 \sin\theta \right) + \tfrac{1}{2} \left(\dot{q}_1^2 + \dot{q}_2^2 \right) \\
& + \tfrac{1}{2} \dot{\theta}^2 \left(\left(x + d_1 \right)^2 + \left(y + d_2 \right)^2 \right) - g \left(\sin\theta \left(x + d_1 \right) + \cos\theta \left(y + d_2 \right) + q_2 \right) \Big] \rho \mathrm{d}y \mathrm{d}x \mathrm{d}t \\
& + \int_{t_1}^{t_2} \Big[\tfrac{1}{2} m_v \left(\dot{q}_1^2 + \dot{q}_2^2 \right) - m_v \dot{\theta} \overline{\mathrm{y}}_v \left(\dot{q}_1 \cos\theta + \dot{q}_2 \sin\theta \right) + m_v \dot{\theta} \overline{\mathrm{x}}_v \left(\dot{q}_2 \cos\theta - \dot{q}_1 \sin\theta \right) \\
& + \tfrac{1}{2} I_v \dot{\theta}^2 - m_v g \left(\overline{\mathrm{x}}_v \sin\theta + \overline{\mathrm{y}}_v \cos\theta + q_2 \right) \Big] \mathrm{d}t \,.
\end{aligned} \tag{2.6}$$

Taking the variation of this Lagrangian action with respect to q_1, q_2 and θ then gives the governing equations of the tank motion in each direction

$$\frac{\delta\mathscr{L}}{\delta q_1}=0 \quad \Rightarrow$$
$$\int_0^L\int_0^h\Big[\ddot{y}\sin\theta-\ddot{x}\cos\theta+\dot{\theta}^2\left(\cos\theta\,(x+d_1)-\sin\theta\,(y+d_2)\right)$$
$$+\,\dot{\theta}\left(2\dot{x}\sin\theta+2\dot{y}\cos\theta\right)+\ddot{\theta}\left(\cos\theta\,(y+d_2)+\sin\theta\,(x+d_1)\right)-\ddot{q}_1\Big]\,\rho\mathrm{d}y\mathrm{d}x$$
$$+m_v\left(\ddot{\theta}\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)-\ddot{q}_1+\dot{\theta}^2\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)\right)=0\,,$$

$$\frac{\delta\mathscr{L}}{\delta q_2}=0 \quad \Rightarrow$$
$$\int_0^L\int_0^h\Big[\ddot{y}\cos\theta+\ddot{x}\sin\theta-\dot{\theta}^2\left(\cos\theta\,(y+d_2)+\sin\theta\,(x+d_1)\right)$$
$$+\,\dot{\theta}\left(2\dot{x}\cos\theta-2\dot{y}\sin\theta\right)+\ddot{\theta}\left(\cos\theta\,(x+d_1)-\sin\theta\,(y+d_2)\right)+\ddot{q}_2+g\Big]\,\rho\mathrm{d}y\mathrm{d}x$$
$$+m_v\left(\ddot{\theta}\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)+\ddot{q}_2-\dot{\theta}^2\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)+g\right)=0\,,$$

$$\frac{\delta\mathscr{L}}{\delta\theta}=0 \quad \Rightarrow$$
$$\int_0^L\int_0^h\big[\ddot{x}\,(y+d_2)+(y+d_2)\left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right)-\ddot{y}\,(x+d_1)$$
$$-\,(x+d_1)\left(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta\right)-\ddot{\theta}\left((x+d_1)^2+(y+d_2)^2\right)$$
$$-2\dot{\theta}\left(\dot{x}\,(x+d_1)+\dot{y}\,(y+d_2)\right)-g\left(\cos\theta\,(x+d_1)-\sin\theta\,(y+d_2)\right)\big]\,\rho\mathrm{d}y\mathrm{d}x$$
$$+m_v\overline{\mathrm{y}}_v\left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right)-m_v\overline{\mathrm{x}}_v\left(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta\right)$$
$$-I_v\ddot{\theta}-m_vg\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)=0\,.$$

Note that $\dot{x}$ and $\dot{y}$ represent the velocity and $\ddot{x}$ and $\ddot{y}$ represent the acceleration of a fluid particle relative to the body frame in the x and y directions. They are Lagrangian variables. Replacing them by their respective Eulerian quantities the governing equations read

$$\int_0^L\int_0^h\Big[\frac{Dv}{Dt}\sin\theta-\frac{Du}{Dt}\cos\theta+\dot{\theta}^2\left(\cos\theta\,(x+d_1)-\sin\theta\,(y+d_2)\right)$$
$$+\,\dot{\theta}\left(2u\sin\theta+2v\cos\theta\right)+\ddot{\theta}\left(\cos\theta\,(y+d_2)+\sin\theta\,(x+d_1)\right)-\ddot{q}_1\Big]\,\rho\mathrm{d}y\mathrm{d}x \tag{2.7}$$
$$+m_v\left(\ddot{\theta}\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)-\ddot{q}_1+\dot{\theta}^2\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)\right)=0\,,$$

$$\int_0^L\int_0^h\Big[\frac{Dv}{Dt}\cos\theta+\frac{Du}{Dt}\sin\theta-\dot{\theta}^2\left(\cos\theta\,(y+d_2)+\sin\theta\,(x+d_1)\right)$$
$$+\,\dot{\theta}\left(2u\cos\theta-2v\sin\theta\right)+\ddot{\theta}\left(\cos\theta\,(x+d_1)-\sin\theta\,(y+d_2)\right)+\ddot{q}_2+g\Big]\,\rho\mathrm{d}y\mathrm{d}x$$
$$+m_v\left(\ddot{\theta}\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)+\ddot{q}_2-\dot{\theta}^2\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)+g\right)=0\,, \tag{2.8}$$

$$
\begin{aligned}
&\int_0^L \int_0^h \left[\tfrac{Du}{Dt}\left(y+d_2\right) + \left(y+d_2\right)\left(\ddot{q}_1 \cos\theta + \ddot{q}_2 \sin\theta\right) - \tfrac{Dv}{Dt}\left(x+d_1\right) \right. \\
&- \left(x+d_1\right)\left(\ddot{q}_2 \cos\theta - \ddot{q}_1 \sin\theta\right) - \ddot{\theta}\left(\left(x+d_1\right)^2 + \left(y+d_2\right)^2\right) \\
&\left. -2\dot{\theta}\left(u\left(x+d_1\right) + v\left(y+d_2\right)\right) - g\left(\cos\theta\left(x+d_1\right) - \sin\theta\left(y+d_2\right)\right)\right] \rho \mathrm{d}y\mathrm{d}x \\
&+m_v \overline{\mathrm{y}}_v \left(\ddot{q}_1 \cos\theta + \ddot{q}_2 \sin\theta\right) - m_v \overline{\mathrm{x}}_v \left(\ddot{q}_2 \cos\theta - \ddot{q}_1 \sin\theta\right) \\
&-I_v \ddot{\theta} - m_v g\left(\overline{\mathrm{x}}_v \cos\theta - \overline{\mathrm{y}}_v \sin\theta\right) = 0\,.
\end{aligned} \tag{2.9}
$$

Now deriving the equations of $q_1(t)$ and $q_2(t)$ from equation (5) in [5] and equation of $\theta(t)$ from equation (6) in [5] for the case of roll-sway-heave motion, one gets

$$
\begin{aligned}
&\int_0^L \int_0^h \left(\ddot{q}_1 \cos\theta + \ddot{q}_2 \sin\theta - \ddot{\theta}\left(y+d_2\right) - \dot{\theta}^2\left(x+d_1\right) + \tfrac{Du}{Dt} - 2v\dot{\theta} + g\sin\theta\right) \rho \mathrm{d}y\mathrm{d}x \\
&+m_v \left(\ddot{q}_1 \cos\theta + \ddot{q}_2 \sin\theta - \overline{\mathrm{y}}_v \ddot{\theta} - \overline{\mathrm{x}}_v \dot{\theta}^2 + g\sin\theta\right) = 0\,,
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&\int_0^L \int_0^h \left(-\ddot{q}_1 \sin\theta + \ddot{q}_2 \cos\theta + \ddot{\theta}\left(x+d_1\right) - \dot{\theta}^2\left(y+d_2\right) + \tfrac{Dv}{Dt} + 2u\dot{\theta} + g\cos\theta\right) \rho \mathrm{d}y\mathrm{d}x \\
&+m_v \left(-\ddot{q}_1 \sin\theta + \ddot{q}_2 \cos\theta + \overline{\mathrm{x}}_v \ddot{\theta} - \overline{\mathrm{y}}_v \dot{\theta}^2 + g\cos\theta\right) = 0\,,
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
&\int_0^L \int_0^h \left[\tfrac{Du}{Dt}\left(y+d_2\right) - \tfrac{Dv}{Dt}\left(x+d_1\right) - 2\dot{\theta}\left(u\left(x+d_1\right) + v\left(y+d_2\right)\right) \right. \\
&- \ddot{\theta}\left(\left(x+d_1\right)^2 + \left(y+d_2\right)^2\right) + \left(x+d_1\right)\left(\ddot{q}_1 \sin\theta - \ddot{q}_2 \cos\theta - g\cos\theta\right) \\
&\left. + \left(y+d_2\right)\left(\ddot{q}_1 \cos\theta + \ddot{q}_2 \sin\theta + g\sin\theta\right)\right] \rho \mathrm{d}y\mathrm{d}x + m_v \sin\theta\left(g\overline{\mathrm{y}}_v + \ddot{q}_1 \overline{\mathrm{x}}_v + \ddot{q}_2 \overline{\mathrm{y}}_v\right) \\
&+m_v \cos\theta\left(-g\overline{\mathrm{x}}_v - \ddot{q}_2 \overline{\mathrm{x}}_v + \ddot{q}_1 \overline{\mathrm{y}}_v\right) - I_v \ddot{\theta} = 0\,.
\end{aligned} \tag{2.12}
$$

Equations (2.10) and (2.11) are relative to the body frame. By transforming these equations to the spatial frame it can be seen that they recover equations (2.7) and (2.8). Equation (2.12) is the same as equation (2.9).

3 Equations (2.7), (2.8), (2.9) in terms of wave height and depth averaged velocity field $(h, \overline{u}, \overline{v})$

Introducing the depth averaged horizontal and vertical velocities

$$
\begin{aligned}
\overline{u} &= \tfrac{1}{h}\int_0^h u\mathrm{d}y\,, \\
\overline{v} &= \tfrac{1}{h}\int_0^h v\mathrm{d}y\,,
\end{aligned}
$$

and using the relations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\int_0^h\left(v\sin\theta-u\cos\theta\right)\mathrm{d}y\mathrm{d}x &= \int_0^L\int_0^h\left(\frac{Dv}{Dt}\sin\theta-\frac{Du}{Dt}\cos\theta\right)\mathrm{d}y\mathrm{d}x\\
&\quad+\int_0^L\int_0^h\dot\theta\left(v\cos\theta+u\sin\theta\right)\mathrm{d}y\mathrm{d}x\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\int_0^h\left(v\cos\theta+u\sin\theta\right)\mathrm{d}y\mathrm{d}x &= \int_0^L\int_0^h\left(\frac{Dv}{Dt}\cos\theta+\frac{Du}{Dt}\sin\theta\right)\mathrm{d}y\mathrm{d}x\\
&\quad+\int_0^L\int_0^h\dot\theta\left(u\cos\theta-v\sin\theta\right)\mathrm{d}y\mathrm{d}x\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\int_0^h\left[u\left(y+d_2\right)-v\left(x+d_1\right)\right]\mathrm{d}y\mathrm{d}x &= \int_0^L\int_0^h\left[\frac{Du}{Dt}\left(y+d_2\right)-\frac{Dv}{Dt}\left(x+d_1\right)\right]\mathrm{d}y\mathrm{d}x\,,
\end{aligned}$$

then equations (2.7) and (2.8) in terms of depth averaged velocity field read

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\overline{v}\sin\theta-\overline{u}\cos\theta\right)\rho h\mathrm{d}x+\int_0^L\dot\theta\left(\overline{u}\sin\theta+\overline{v}\cos\theta\right)\rho h\mathrm{d}x\\
&+m_f\overline{x}_f\dot\theta^2\cos\theta-m_f\overline{y}_f\dot\theta^2\sin\theta+m_f\overline{y}_f\ddot\theta\cos\theta+m_f\overline{x}_f\ddot\theta\sin\theta-\left(m_f+m_v\right)\ddot q_1\\
&+m_v\ddot\theta\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)+m_v\dot\theta^2\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)=0\,,
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\overline{v}\cos\theta+\overline{u}\sin\theta\right)\rho h\mathrm{d}x+\int_0^L\dot\theta\left(\overline{u}\cos\theta-\overline{v}\sin\theta\right)\rho h\mathrm{d}x\\
&-m_f\overline{y}_f\dot\theta^2\cos\theta-m_f\overline{x}_f\dot\theta^2\sin\theta+m_f\overline{x}_f\ddot\theta\cos\theta-m_f\overline{y}_f\ddot\theta\sin\theta+\left(m_f+m_v\right)\ddot q_2\\
&+\left(m_f+m_v\right)g+m_v\ddot\theta\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)-m_v\dot\theta^2\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)=0\,,
\end{aligned}\tag{3.14}$$

where

$$\begin{aligned}
\overline{x}_f &= \frac{1}{m_f}\int_0^L\left(x+d_1\right)\rho h\mathrm{d}x\,,\\
\overline{y}_f &= \frac{1}{m_f}\int_0^L\left(\frac{h^2}{2}+d_2h\right)\rho\mathrm{d}x\\
m_f &= \int_0^L\int_0^h\rho\mathrm{d}y\mathrm{d}x=\rho h_0L\\
h_0 &= \frac{1}{L}\int_0^L h\left(x,t\right)\rho\mathrm{d}x\,.
\end{aligned}$$

Equation (2.9) in terms of depth averaged velocity field reads

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\left(\frac{h^2}{2}+d_2h\right)\overline{u}-h\left(x+d_1\right)\overline{v}\right)\rho\mathrm{d}x+\left(\ddot q_1\cos\theta+\ddot q_2\sin\theta\right)m_f\overline{y}_f\\
&-\left(\ddot q_2\cos\theta-\ddot q_1\sin\theta\right)m_f\overline{x}_f-\int_0^L\ddot\theta\left(\left(x+d_1\right)^2h+\frac{h^3}{3}+d_2^2h+d_2h^2\right)\rho\mathrm{d}x\\
&-\int_0^L2\dot\theta\left(x+d_1\right)h\overline{u}\rho\mathrm{d}x-\int_0^L2\dot\theta\left(\frac{h^2}{2}+d_2h\right)\overline{v}\rho\mathrm{d}x-g\cos\theta m_f\overline{x}_f+g\sin\theta m_f\overline{y}_f\\
&+m_v\overline{\mathrm{y}}_v\left(\ddot q_1\cos\theta+\ddot q_2\sin\theta\right)-m_v\overline{\mathrm{x}}_v\left(\ddot q_2\cos\theta-\ddot q_1\sin\theta\right)-I_v\ddot\theta\\
&-m_vg\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)=0\,.
\end{aligned}\tag{3.15}$$

Note that the following assumptions are considered in deriving equation (3.15)

$$\int_0^L\left[\left(\frac{1}{2}h+d_2\right)\int_0^h yu_y\mathrm{d}y\right]\rho\mathrm{d}x-\int_0^L\int_0^h\left(\frac{1}{2}y^2+d_2y\right)u_y\rho\mathrm{d}y\mathrm{d}x \approx 0\,,\tag{3.16}$$

which results in

$$\int_0^L \left(\tfrac{1}{2}h^2 + d_2 h\right)\rho U \mathrm{d}x - \int_0^L \left(\tfrac{1}{2}h^2 + d_2 h\right)\rho \overline{u}\mathrm{d}x - \int_0^L \int_0^h \left(\tfrac{1}{2}y^2 + d_2 y\right) u_y \rho \mathrm{d}y\mathrm{d}x \quad \approx \quad 0\,,$$

and the second assumption is

$$\int_0^L \left[\left(\tfrac{1}{2}h + d_2\right)\int_0^h y v_y \mathrm{d}y\right]\rho \mathrm{d}x - \int_0^L \int_0^h \left(\tfrac{1}{2}y^2 + d_2 y\right) v_y \rho \mathrm{d}y\mathrm{d}x \quad \approx \quad 0\,, \tag{3.17}$$

which results in

$$\int_0^L \left(\tfrac{1}{2}h^2 + d_2 h\right)\rho V \mathrm{d}x - \int_0^L \left(\tfrac{1}{2}h^2 + d_2 h\right)\rho \overline{v}\mathrm{d}x - \int_0^L \int_0^h \left(\tfrac{1}{2}y^2 + d_2 y\right) v_y \rho \mathrm{d}y\mathrm{d}x \quad \approx \quad 0\,,$$

where the *surface velocity field* is defined by

$$\begin{aligned} U(x,t) \quad &:= \quad u(x, h(x,t), t)\,, \\ V(x,t) \quad &:= \quad v(x, h(x,t), t)\,. \end{aligned}$$

4 Equations (2.7), (2.8), (2.9) in terms of wave height and depth averaged horizontal velocity $(h, \overline{u})$

Equations (2.7), (2.8) and (2.9) can be derived in terms of $(h, \overline{u})$. For this purpose write these equations in terms of (h, U, V), substitute for V from the kinematic free surface boundary condition $V = h_t + U h_x$, and use the following relation

$$U \quad = \quad \overline{u} + \frac{1}{h}\int_0^h y u_y \mathrm{d}y\,,$$

to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left[(h_t + \overline{u}h_x)\sin\theta - \overline{u}\cos\theta\right]\rho h \mathrm{d}x + \int_0^L \dot{\theta}\left(\overline{u}\sin\theta + (h_t + \overline{u}h_x)\cos\theta\right)\rho h \mathrm{d}x \\ &+ m_f \overline{x}_f \dot{\theta}^2 \cos\theta - m_f \overline{y}_f \dot{\theta}^2 \sin\theta + m_f \overline{y}_f \ddot{\theta}\cos\theta + m_f \overline{x}_f \ddot{\theta}\sin\theta - (m_f + m_v)\,\ddot{q}_1 \\ &+ m_v \ddot{\theta}\left(\overline{\mathrm{x}}_v \sin\theta + \overline{\mathrm{y}}_v \cos\theta\right) + m_v \dot{\theta}^2 \left(\overline{\mathrm{x}}_v \cos\theta - \overline{\mathrm{y}}_v \sin\theta\right) = 0\,, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left[(h_t + \overline{u}h_x)\cos\theta + \overline{u}\sin\theta\right]\rho h \mathrm{d}x + \int_0^L \dot{\theta}\left(\overline{u}\cos\theta - (h_t + \overline{u}h_x)\sin\theta\right)\rho h \mathrm{d}x \\ &- m_f \overline{y}_f \dot{\theta}^2 \cos\theta - m_f \overline{x}_f \dot{\theta}^2 \sin\theta + m_f \overline{x}_f \ddot{\theta}\cos\theta - m_f \overline{y}_f \ddot{\theta}\sin\theta + (m_f + m_v)\,\ddot{q}_2 \\ &+ (m_f + m_v)\,g + m_v \ddot{\theta}\left(\overline{\mathrm{x}}_v \cos\theta - \overline{\mathrm{y}}_v \sin\theta\right) - m_v \dot{\theta}^2 \left(\overline{\mathrm{x}}_v \sin\theta + \overline{\mathrm{y}}_v \cos\theta\right) = 0\,, \end{aligned} \tag{4.19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\left(\frac{h^2}{2}+d_2h\right)\overline{u}-(x+d_1)\,h\,(h_t+\overline{u}h_x)\right)\rho\mathrm{d}x+(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta)\,m_f\overline{y}_f$$
$$-(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta)\,m_f\overline{x}_f-\int_0^L\ddot{\theta}\left((x+d_1)^2\,h+\frac{h^3}{3}+d_2^2h+d_2h^2\right)\rho\mathrm{d}x$$
$$-\int_0^L 2\dot{\theta}\,(x+d_1)\,h\overline{u}\rho\mathrm{d}x-\int_0^L 2\dot{\theta}\left(\frac{h^2}{2}+d_2h\right)(h_t+\overline{u}h_x)\,\rho\mathrm{d}x-g\cos\theta m_f\overline{x}_f$$
$$+g\sin\theta m_f\overline{y}_f+m_v\overline{\mathrm{y}}_v\,(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta)-m_v\overline{\mathrm{x}}_v\,(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta)-I_v\ddot{\theta}$$
$$-m_vg\,(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta)=0\,. \tag{4.20}$$

Note that the following assumptions are considered in deriving equations (4.18) and (4.19)

$$\begin{aligned}\int_0^L h_x\int_0^h yu_y\mathrm{d}y\mathrm{d}x &\approx 0\,,\\ \int_0^L\int_0^h yv_y\mathrm{d}y\mathrm{d}x &\approx 0\,.\end{aligned} \tag{4.21}$$

And the simplifying assumptions in deriving equation (4.20) are (3.16) with

$$\int_0^L (x+d_1)\,h_x\int_0^h yu_y\rho\mathrm{d}y\mathrm{d}x-\int_0^L\int_0^h (x+d_1)\,yv_y\rho\mathrm{d}y\mathrm{d}x \approx 0\,, \tag{4.22}$$

and

$$\int_0^L \left(\tfrac{1}{2}h+d_2\right)h_x\int_0^h yu_y\rho\mathrm{d}y\mathrm{d}x-\int_0^L\int_0^h\left(\tfrac{1}{2}y^2+d_2y\right)v_y\rho\mathrm{d}y\mathrm{d}x \approx 0\,. \tag{4.23}$$

5 Equations (2.7), (2.8), (2.9) in terms of wave height and surface velocity field (h,U,V)

Equations (2.7), (2.8) and (2.9) in terms of (h,U,V) read

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (V\sin\theta-U\cos\theta)\,\rho h\mathrm{d}x+\int_0^L\dot{\theta}\,(U\sin\theta+V\cos\theta)\,\rho h\mathrm{d}x$$
$$+m_f\overline{x}_f\dot{\theta}^2\cos\theta-m_f\overline{y}_f\dot{\theta}^2\sin\theta+m_f\overline{y}_f\ddot{\theta}\cos\theta+m_f\overline{x}_f\ddot{\theta}\sin\theta-(m_f+m_v)\,\ddot{q}_1$$
$$+m_v\ddot{\theta}\,(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta)+m_v\dot{\theta}^2\,(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta)=0\,, \tag{5.24}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (V\cos\theta+U\sin\theta)\,\rho h\mathrm{d}x+\int_0^L\dot{\theta}\,(U\cos\theta-V\sin\theta)\,\rho h\mathrm{d}x$$
$$-m_f\overline{y}_f\dot{\theta}^2\cos\theta-m_f\overline{x}_f\dot{\theta}^2\sin\theta+m_f\overline{x}_f\ddot{\theta}\cos\theta-m_f\overline{y}_f\ddot{\theta}\sin\theta+(m_f+m_v)\,\ddot{q}_2$$
$$+(m_f+m_v)\,g+m_v\ddot{\theta}\,(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta)-m_v\dot{\theta}^2\,(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta)=0\,, \tag{5.25}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\left(\frac{h^2}{2}+d_2h\right)U-h\left(x+d_1\right)V\right)\rho\mathrm{d}x+\left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right)m_f\overline{y}_f\\
&-\left(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta\right)m_f\overline{x}_f-\int_0^L\ddot{\theta}\left(\left(x+d_1\right)^2h+\frac{h^3}{3}+d_2^2h+d_2h^2\right)\rho\mathrm{d}x\\
&-\int_0^L2\dot{\theta}\left(x+d_1\right)hU\rho\mathrm{d}x-\int_0^L2\dot{\theta}\left(\frac{h^2}{2}+d_2h\right)V\rho\mathrm{d}x-g\cos\theta m_f\overline{x}_f+g\sin\theta m_f\overline{y}_f\\
&+m_v\overline{\mathrm{y}}_v\left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right)-m_v\overline{\mathrm{x}}_v\left(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta\right)-I_v\ddot{\theta}\\
&-m_vg\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)=0\,.
\end{aligned}\tag{5.26}$$

The simplifying assumptions in deriving equations (5.24) and (5.25) are

$$\begin{aligned}
\int_0^L\int_0^h yu_y\mathrm{d}y\mathrm{d}x &\approx 0\,,\\
\int_0^L\int_0^h yv_y\mathrm{d}y\mathrm{d}x &\approx 0\,,
\end{aligned}\tag{5.27}$$

and for equation (5.26) are

$$\begin{aligned}
\int_0^L\int_0^h\rho\left(x+d_1\right)yu_y\mathrm{d}y\mathrm{d}x &\approx 0\,,\\
\int_0^L\int_0^h\rho\left(x+d_1\right)yv_y\mathrm{d}y\mathrm{d}x &\approx 0\,,\\
\int_0^L\int_0^h\rho\left(\tfrac{1}{2}y^2+d_2y\right)u_y\mathrm{d}y\mathrm{d}x &\approx 0\,,\\
\int_0^L\int_0^h\rho\left(\tfrac{1}{2}y^2+d_2y\right)v_y\mathrm{d}y\mathrm{d}x &\approx 0\,.
\end{aligned}\tag{5.28}$$

6 Equations (2.7), (2.8), (2.9) in terms of wave height and horizontal surface velocity (h, U)

To obtain equations (2.7), (2.8) and (2.9) in terms of (h, U) substitute for V in equations (5.24), (5.25) and (5.26) from the kinematic condition $V = h_t + Uh_x$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\left(h_t+Uh_x\right)\sin\theta-U\cos\theta\right)\rho h\mathrm{d}x+\int_0^L\dot{\theta}\left(U\sin\theta+\left(h_t+Uh_x\right)\cos\theta\right)\rho h\mathrm{d}x\\
&+m_f\overline{x}_f\dot{\theta}^2\cos\theta-m_f\overline{y}_f\dot{\theta}^2\sin\theta+m_f\overline{y}_f\ddot{\theta}\cos\theta+m_f\overline{x}_f\ddot{\theta}\sin\theta-\left(m_f+m_v\right)\ddot{q}_1\\
&+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)+m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)=0\,,
\end{aligned}\tag{6.29}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\left(\left(h_t+Uh_x\right)\cos\theta+U\sin\theta\right)\rho h\mathrm{d}x+\int_0^L\dot{\theta}\left(U\cos\theta-\left(h_t+Uh_x\right)\sin\theta\right)\rho h\mathrm{d}x\\
&-m_f\overline{y}_f\dot{\theta}^2\cos\theta-m_f\overline{x}_f\dot{\theta}^2\sin\theta+m_f\overline{x}_f\ddot{\theta}\cos\theta-m_f\overline{y}_f\ddot{\theta}\sin\theta+\left(m_f+m_v\right)\ddot{q}_2\\
&+\left(m_f+m_v\right)g+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)-m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta\right)=0\,,
\end{aligned}\tag{6.30}$$

and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left(\left(\frac{h^2}{2}+d_2h\right)U - h\left(x+d_1\right)\left(h_t+Uh_x\right)\right)\rho\mathrm{d}x + \left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right)m_f\overline{y}_f \\
&-\left(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta\right)m_f\overline{x}_f - \int_0^L \ddot{\theta}\left(\left(x+d_1\right)^2h+\frac{h^3}{3}+d_2^2h+d_2h^2\right)\rho\mathrm{d}x \\
&-\int_0^L 2\dot{\theta}\left(x+d_1\right)hU\rho\mathrm{d}x - \int_0^L 2\dot{\theta}\left(\frac{h^2}{2}+d_2h\right)\left(h_t+Uh_x\right)\rho\mathrm{d}x - g\cos\theta m_f\overline{x}_f \\
&+g\sin\theta m_f\overline{y}_f + m_v\overline{\mathrm{y}}_v\left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right) - m_v\overline{\mathrm{x}}_v\left(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta\right) \\
&-I_v\ddot{\theta} - m_vg\left(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta\right)=0\,.
\end{aligned}
\tag{6.31}
$$

7 Summary of the roll-sway-heave equations coupled to shallow-water sloshing equations

Armenio and Larroca shallow water equations (ALR SWEs) [4] including the translation accelerations $\ddot{\mathbf{q}}$ and assuming $(y=h)$ are

$$
\begin{aligned}
&h_t+\overline{u}h_x+h\overline{u}_x=0\,,\\
&\overline{u}_t+\left(\overline{u}+2\dot{\theta}h\right)\overline{u}_x+\Big(\ddot{\theta}\left(x+d_1\right)-\dot{\theta}^2\left(h+d_2\right)+2\dot{\theta}\overline{u}\\
&\qquad\qquad +g\cos\theta-\ddot{q}_1\sin\theta+\ddot{q}_2\cos\theta\Big)h_x\\
&=-g\sin\theta+\ddot{\theta}d_2+\dot{\theta}^2\left(x+d_1\right)-\ddot{q}_1\cos\theta-\ddot{q}_2\sin\theta\,,
\end{aligned}
\tag{7.32}
$$

A review of ALR SWEs derivation identifying the key assumptions is presented in [3]. A closed set of equations for $(h,\overline{u},q_1,q_2,\theta)$ are the ALR SWEs (7.32) coupled to equations (4.18), (4.19) and (4.20) with the key assumptions (3.16), (4.21), (4.22) and (4.23). And the key assumptions in deriving ALR SWEs are [4]

$$
\begin{aligned}
&\frac{Dv}{Dt}\approx 0\,,\\
&2\dot{\theta}v\approx 0\,,\\
&\frac{1}{\rho}\frac{\partial p}{\partial x}\quad \text{in the } x\text{−momentum equation is evaluated at } y=0\,,\\
&\frac{Du}{Dt}\approx\overline{u}_t+\overline{u}\overline{u}_x\,.
\end{aligned}
\tag{7.33}
$$

Note that in the derivation of equations (4.18), (4.19) and (4.20) the ALR shallow-water assumptions are not considered. See §9 for the simplified version of roll, sway and heave equations considering the assumptions (7.33).

Shallow-water sloshing equations in terms of surface velocity are [2]

$$
\begin{aligned}
&h_t+Uh_x+hU_x=0\,,\\
&U_t+\left(U+2\dot{\theta}h\right)U_x+\Big(\ddot{\theta}\left(x+d_1\right)-\dot{\theta}^2\left(h+d_2\right)+2\dot{\theta}U\\
&\qquad\qquad +g\cos\theta-\ddot{q}_1\sin\theta+\ddot{q}_2\cos\theta\Big)h_x\\
&=-g\sin\theta+\ddot{\theta}\left(h+d_2\right)+\dot{\theta}^2\left(x+d_1\right)-\ddot{q}_1\cos\theta-\ddot{q}_2\sin\theta\,.
\end{aligned}
\tag{7.34}
$$

The key assumptions in deriving surface SWEs (7.34) are

$$\begin{array}{l} \left|\frac{Dv}{Dt}\right|^h\Big| \quad \text{is small compared with} \quad |a\,(x,t)|\;, \\ |V + hU_x| \ll 1\,, \end{array} \tag{7.35}$$

where $a\,(x,t)$ is defined in [2]. A closed set of equations for (h, U, q_1, q_2, θ) are the surface SEWs (7.34) coupled to equations (6.29), (6.30) and (6.31) with the key assumptions (5.27) and (5.28).

8 Variational principle: ALR SWEs

ALR SWEs have a variational formulation. Introduce the Lagrangian functional

$$\begin{aligned} \mathscr{L}\,(h,u,\phi) \;=\; & \int_{t_1}^{t_2}\int_0^L \Big[\tfrac{1}{2}h\overline{u}^2 - \dot{\theta}h\overline{u}\,(h+d_2) + \tfrac{1}{2}\dot{\theta}^2\left(\tfrac{1}{3}h^3 + d_2h^2 + d_2^2h\right) \\ & + \tfrac{1}{2}h\dot{\theta}^2\,(x+d_1)^2 - \tfrac{1}{2}\ddot{\theta}\,(x+d_1)\,h^2 - g\cos\theta\left(\tfrac{1}{2}h^2 + d_2h\right) \\ & - gh\sin\theta\,(x+d_1) - \tfrac{1}{2}\left(-\ddot{q}_1\sin\theta + \ddot{q}_2\cos\theta\right)h^2 \\ & - \left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right)h\,(x+d_1) + \phi\left(h_t + h\overline{u}_x + \overline{u}h_x\right)\Big]\,\mathrm{d}x\mathrm{d}t\,. \end{aligned} \tag{8.36}$$

The first variation of this Lagrangian functional with respect to h, $\overline{u}$ and ϕ recovers the ALR SWEs (7.32) [3].

8.1 Lagrangian particle path formulation of the ALR SWEs

To derive the LPP form of the equations (7.32) consider the mapping

$$(t,a) \mapsto (t, x(a,t))\,, \quad \text{with} \quad 0 \le a \le L\,, \quad t \ge 0\,.$$

Assuming non-degeneracy ($x_a \neq 0$) the derivatives in (7.32) are mapped to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - x_t x_a^{-1}\frac{\partial}{\partial a} \quad \text{and} \quad \frac{\partial}{\partial x} = x_a^{-1}\frac{\partial}{\partial a}\,.$$

Substitute into (7.32) and set $\overline{u} = x_t$ then the mass equation simplifies to

$$x_a h_t + h x_{at} \;=\; 0 \quad \Rightarrow \quad \tfrac{d}{dt}(hx_a) = 0\,. \tag{8.37}$$

Integrating this equation

$$hx_a = \chi(a)\,,$$

where $\chi(a) := hx_a\big|_{t=0}$ is determined by the initial data. The second equation of (7.32) then reduces to

$$\begin{aligned} & \ddot{x} + 2\dot{\theta}\tfrac{\chi}{x_a^2}\dot{x}_a + \Big(\ddot{\theta}\,(x+d_1) - \dot{\theta}^2\left(\tfrac{\chi}{x_a} + d_2\right) + 2\dot{\theta}\dot{x} + g\cos\theta \\ & -\ddot{q}_1\sin\theta + \ddot{q}_2\cos\theta\Big)\left(\tfrac{\chi_a}{x_a^2} - \tfrac{\chi x_{aa}}{x_a^3}\right) + g\sin\theta - \ddot{\theta}d_2 - \dot{\theta}^2\,(x+d_1) \\ & +\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta = 0\,. \end{aligned} \tag{8.38}$$

Introduce the Lagrangian action

$$\begin{aligned}\mathscr{L} \quad = \quad & \int_{t_1}^{t_2}\int_0^L \Big[\tfrac{1}{2}\dot{x}^2 - \dot{\theta}\dot{x}\left(\tfrac{\chi}{x_a}+d_2\right) + \tfrac{1}{2}\dot{\theta}^2\left(\tfrac{1}{3}\tfrac{\chi^2}{x_a^2}+\tfrac{d_2\chi}{x_a}+d_2^2\right)\\ & +\tfrac{1}{2}\dot{\theta}^2\left(x+d_1\right)^2 - \tfrac{1}{2}\ddot{\theta}\left(x+d_1\right)\tfrac{\chi}{x_a} - g\cos\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\\ & -g\sin\theta\left(x+d_1\right) - \tfrac{1}{2}\left(-\ddot{q}_1\sin\theta+\ddot{q}_2\cos\theta\right)\tfrac{\chi}{x_a}\\ & -\left(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta\right)\left(x+d_1\right)\Big]\chi \mathrm{d}a\mathrm{d}t\,.\end{aligned} \tag{8.39}$$

Taking the first variation of the Lagrangian functional (8.39) with respect to x recovers equation (8.38).

9 Roll-sway-heave equations in Lagrangian coordinates

In this section a Lagrangian functional in Lagrangian coordinates is defined which recovers the Lagrangian form of the roll, sway and heave equations (4.18), (4.19) and (4.20) with a few diffrences.

Introduce the Lagrangian action in Lagrangian coordinates

$$\begin{aligned}\mathscr{L} \quad = \quad & \int_{t_1}^{t_2}\int_0^L \Bigg[\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\left(\tfrac{\chi}{x_a}\right)^2\left(\tfrac{\dot{x}_a}{x_a}\right)^2 - \dot{\theta}\dot{x}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \dot{x}\left(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta\right)\\ & -\dot{\theta}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\left(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta\right) - \dot{\theta}\tfrac{\chi\dot{x}_a}{x_a^2}\left(x+d_1\right) - \tfrac{\chi\dot{x}_a}{x_a^2}\left(\dot{q}_2\cos\theta-\dot{q}_1\sin\theta\right)\\ & +\dot{\theta}\left(x+d_1\right)\left(\dot{q}_2\cos\theta-\dot{q}_1\sin\theta\right) + \tfrac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right) + \tfrac{1}{2}\dot{\theta}^2\left(x+d_1\right)^2\\ & +\tfrac{1}{2}\dot{\theta}^2\left(\tfrac{1}{3}\left(\tfrac{\chi}{x_a}\right)^2 + d_2^2 + \tfrac{d_2\chi}{x_a}\right) - g\sin\theta\left(x+d_1\right) - g\cos\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\\ & -gq_2\Big]\rho\chi\mathrm{d}a\mathrm{d}t + \int_{t_1}^{t_2}\Big[\tfrac{1}{2}m_v\left(\dot{q}_1^2+\dot{q}_2^2\right) - m_v\overline{\mathrm{y}}_v\dot{\theta}\left(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta\right)\\ & +m_v\overline{\mathrm{x}}_v\dot{\theta}\left(\dot{q}_2\cos\theta-\dot{q}_1\sin\theta\right) + \tfrac{1}{2}I_v\dot{\theta}^2 - m_v g\left(\overline{\mathrm{x}}_v\sin\theta+\overline{\mathrm{y}}_v\cos\theta+q_2\right)\Big]\,\mathrm{d}t\,.\end{aligned} \tag{9.40}$$

Taking the first variation of this Lagrangian functional with respect to q_1 gives

$$\begin{aligned}& \int_0^L\Big[-\ddot{x}\cos\theta + 2\dot{x}\dot{\theta}\sin\theta + \ddot{\theta}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\cos\theta \boxed{-\tfrac{3}{2}}\tfrac{\chi\dot{x}_a}{x_a^2}\dot{\theta}\cos\theta - \left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\dot{\theta}^2\sin\theta\\ & -\tfrac{\chi\ddot{x}_a}{x_a^2}\sin\theta + \tfrac{2\chi\dot{x}_a^2}{x_a^3}\sin\theta + \ddot{\theta}\left(x+d_1\right)\sin\theta + \dot{\theta}^2\left(x+d_1\right)\cos\theta - \ddot{q}_1\Big]\rho\chi\mathrm{d}a\\ & +m_v\overline{\mathrm{y}}_v\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right) - m_v\ddot{q}_1 + m_v\overline{\mathrm{x}}_v\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right) = 0\,.\end{aligned} \tag{9.41}$$

Taking the first variation of the Lagrangian functional (9.40) with respect to q_2 gives

$$\int_0^L \Big[-\ddot{x}\sin\theta - 2\dot{x}\dot{\theta}\cos\theta + \ddot{\theta}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\sin\theta \boxed{-\tfrac{3}{2}} \tfrac{\chi\dot{x}_a}{x_a^2}\dot{\theta}\sin\theta + \left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\dot{\theta}^2\cos\theta \\ +\tfrac{\chi\ddot{x}_a}{x_a^2}\cos\theta - \tfrac{2\chi\dot{x}_a^2}{x_a^3}\cos\theta - \ddot{\theta}\left(x+d_1\right)\cos\theta + \dot{\theta}^2\left(x+d_1\right)\sin\theta - \ddot{q}_2 - g\Big]\rho\chi\mathrm{d}a \\ +m_v\overline{\mathrm{y}}_v\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) - m_v\ddot{q}_2 - m_v\overline{\mathrm{x}}_v\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = 0\,. \tag{9.42}$$

Taking the first variation of the Lagrangian functional (9.40) with respect to θ gives

$$\int_0^L \Big[\ddot{x}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \tfrac{1}{2}\tfrac{\chi\dot{x}\dot{x}_a}{x_a^2} \boxed{+\tfrac{1}{2}\tfrac{\chi\dot{x}_a}{x_a^2}\left(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta\right)} - 2\left(x+d_1\right)\tfrac{\chi\dot{x}_a^2}{x_a^3} \\ +\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) + \left(x+d_1\right)\tfrac{\chi\ddot{x}_a}{x_a^2} - \left(x+d_1\right)\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) \\ -\ddot{\theta}\left(x+d_1\right)^2 - 2\dot{\theta}\dot{x}\left(x+d_1\right) - \ddot{\theta}\left(\tfrac{1}{3}\tfrac{\chi^2}{x_a^2} + d_2^2 + \tfrac{d_2\chi}{x_a}\right)\boxed{+\tfrac{2}{3}}\tfrac{\dot{\theta}\chi^2\dot{x}_a}{x_a^3}\boxed{+1}\tfrac{\dot{\theta}d_2\chi\dot{x}_a}{x_a^2} \\ -g\cos\theta\left(x+d_1\right) + g\sin\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\Big]\rho\chi\mathrm{d}a + m_v\overline{\mathrm{y}}_v\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) \\ -m_v\overline{\mathrm{x}}_v\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) - I_v\ddot{\theta} - m_v g\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) = 0\,. \tag{9.43}$$

Transforming equation (4.18) to Lagrangian coordinates gives

$$\int_0^L \Big[-\ddot{x}\cos\theta + 2\dot{x}\dot{\theta}\sin\theta + \ddot{\theta}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\cos\theta\boxed{-2}\tfrac{\chi\dot{x}_a}{x_a^2}\dot{\theta}\cos\theta - \left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\dot{\theta}^2\sin\theta \\ -\tfrac{\chi\ddot{x}_a}{x_a^2}\sin\theta + \tfrac{2\chi\dot{x}_a^2}{x_a^3}\sin\theta + \ddot{\theta}\left(x+d_1\right)\sin\theta + \dot{\theta}^2\left(x+d_1\right)\cos\theta - \ddot{q}_1\Big]\rho\chi\mathrm{d}a \\ +m_v\overline{\mathrm{y}}_v\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) - m_v\ddot{q}_1 + m_v\overline{\mathrm{x}}_v\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) = 0\,. \tag{9.44}$$

Transforming equation (4.19) to Lagrangian coordinates gives

$$\int_0^L \Big[-\ddot{x}\sin\theta - 2\dot{x}\dot{\theta}\cos\theta + \ddot{\theta}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\sin\theta\boxed{-2}\tfrac{\chi\dot{x}_a}{x_a^2}\dot{\theta}\sin\theta + \left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\dot{\theta}^2\cos\theta \\ +\tfrac{\chi\ddot{x}_a}{x_a^2}\cos\theta - \tfrac{2\chi\dot{x}_a^2}{x_a^3}\cos\theta - \ddot{\theta}\left(x+d_1\right)\cos\theta + \dot{\theta}^2\left(x+d_1\right)\sin\theta - \ddot{q}_2 - g\Big]\rho\chi\mathrm{d}a \\ +m_v\overline{\mathrm{y}}_v\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) - m_v\ddot{q}_2 - m_v\overline{\mathrm{x}}_v\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = 0\,. \tag{9.45}$$

Transforming equation (4.20) to Lagrangian coordinates gives

$$\int_0^L \Big[\ddot{x}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \tfrac{1}{2}\tfrac{\chi\dot{x}\dot{x}_a}{x_a^2} + \left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) - 2\left(x+d_1\right)\tfrac{\chi\dot{x}_a^2}{x_a^3} \\ +\left(x+d_1\right)\tfrac{\chi\ddot{x}_a}{x_a^2} - \left(x+d_1\right)\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) - \ddot{\theta}\left(x+d_1\right)^2 - 2\dot{\theta}\dot{x}\left(x+d_1\right) \\ -\ddot{\theta}\left(\tfrac{1}{3}\tfrac{\chi^2}{x_a^2} + d_2^2 + \tfrac{d_2\chi}{x_a}\right)\boxed{+1}\tfrac{\dot{\theta}\chi^2\dot{x}_a}{x_a^3}\boxed{+2}\tfrac{\dot{\theta}d_2\chi\dot{x}_a}{x_a^2} - g\cos\theta\left(x+d_1\right) \\ +g\sin\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\Big]\rho\chi\mathrm{d}a + m_v\overline{\mathrm{y}}_v\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) \\ -m_v\overline{\mathrm{x}}_v\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) - I_v\ddot{\theta} - m_v g\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) = 0\,. \tag{9.46}$$

Comparison of equations (9.41) and (9.44), (9.42) and (9.45), and (9.43) and (9.46) shows that only the boxed terms do not match. So the Lagrangian functionals (9.40) and (8.39) can be a starting point for the construction of a numerical scheme which includes the integration of these functionals together at each time step.

10 Roll-sway-heave equations satisfying the ALR shallow-water assumptions

Equations (4.18), (4.19) and (4.20) are the approximated versions of roll, sway and heave equations in terms of $(h, \overline{u})$ which can be coupled to ALR SWEs (7.32) to get a system of differential equations for variables $(h, \overline{u}, q_1, q_2, \theta)$. The point is that in derivation of equations (4.18), (4.19) and (4.20) the ALR shallow-water assumptions are not considered. So in this section the simplified version of the roll, sway and heave equations satisfying the shallow-water assumptions are derived.

Equations (2.7), (2.8) and (2.9) can be simplified considering the ALR shallow-water assumptions (7.33). Equation (2.7) with assumptions (7.33) reads

$$\begin{aligned} &\int_0^L \Big[-\left(\overline{u}_t + \overline{u}\overline{u}_x\right)\cos\theta + \dot{\theta}^2\cos\theta\left(x+d_1\right) - \dot{\theta}^2\sin\theta\left(\tfrac{1}{2}h + d_2\right) \\ &+2\dot{\theta}\sin\theta\overline{u} + \ddot{\theta}\cos\theta\left(\tfrac{1}{2}h+d_2\right) + \ddot{\theta}\sin\theta\left(x+d_1\right) - \ddot{q}_1\Big]\,\rho h\mathrm{d}x \\ &+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\sin\theta + \overline{\mathrm{y}}_v\cos\theta\right) - m_v\ddot{q}_1 + m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) = 0\,. \end{aligned} \tag{10.47}$$

Equation (2.8) with assumptions (7.33) reads

$$\begin{aligned} &\int_0^L \Big[\left(\overline{u}_t + \overline{u}\overline{u}_x\right)\sin\theta - \dot{\theta}^2\cos\theta\left(\tfrac{1}{2}h+d_2\right) - \dot{\theta}^2\sin\theta\left(x+d_1\right) \\ &+2\dot{\theta}\cos\theta\overline{u} + \ddot{\theta}\cos\theta\left(x+d_1\right) - \ddot{\theta}\sin\theta\left(\tfrac{1}{2}h+d_2\right) + \ddot{q}_2 + g\Big]\,\rho h\mathrm{d}x \\ &+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) + m_v\ddot{q}_2 - m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\sin\theta + \overline{\mathrm{y}}_v\cos\theta\right) = 0\,. \end{aligned} \tag{10.48}$$

Equation (2.9) with assumptions (7.33) reads

$$\begin{aligned} &\int_0^L \Big[\left(\overline{u}_t + \overline{u}\overline{u}_x\right)\left(\tfrac{1}{2}h+d_2\right) + \left(\tfrac{1}{2}h+d_2\right)\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) \\ &-\left(x+d_1\right)\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) - \ddot{\theta}\left(x+d_1\right)^2 - \ddot{\theta}\left(\tfrac{1}{3}h^2 + d_2^2 + d_2 h\right) \\ &-2\dot{\theta}\overline{u}\left(x+d_1\right) - g\cos\theta\left(x+d_1\right) + g\sin\theta\left(\tfrac{1}{2}h+d_2\right)\Big]\,\rho h\mathrm{d}x \\ &+m_v\overline{\mathrm{y}}_v\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) - m_v\overline{\mathrm{x}}_v\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) \\ &-I_v\ddot{\theta} - m_v g\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) = 0\,. \end{aligned} \tag{10.49}$$

Equation (10.47) in Lagrangian coordinates reads

$$\begin{aligned} &\int_0^L \Big[-\ddot{x}\cos\theta + \dot{\theta}^2\cos\theta\left(x+d_1\right) - \dot{\theta}^2\sin\theta\left(\frac{1}{2}\frac{\chi}{x_a}+d_2\right) + 2\dot{\theta}\dot{x}\sin\theta \\ &+\ddot{\theta}\cos\theta\left(\frac{1}{2}\frac{\chi}{x_a}+d_2\right) + \ddot{\theta}\sin\theta\left(x+d_1\right) - \ddot{q}_1\Big]\,\rho\chi\mathrm{d}a \\ &+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\sin\theta + \overline{\mathrm{y}}_v\cos\theta\right) - m_v\ddot{q}_1 + m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) = 0\,. \end{aligned} \tag{10.50}$$

Equation (10.48) in Lagrangian coordinates reads

$$\begin{aligned}&\int_0^L \Big[\ddot{x}\sin\theta - \dot{\theta}^2\cos\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) - \dot{\theta}^2\sin\theta\,(x+d_1) + 2\dot{\theta}\dot{x}\cos\theta \\ &+\ddot{\theta}\cos\theta\,(x+d_1) - \ddot{\theta}\sin\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \ddot{q}_2 + g\Big]\,\rho\chi\mathrm{d}a \\ &+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) + m_v\ddot{q}_2 - m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\sin\theta + \overline{\mathrm{y}}_v\cos\theta\right) + m_v g = 0\,.\end{aligned} \tag{10.51}$$

Equation (10.49) in Lagrangian coordinates reads

$$\begin{aligned}&\int_0^L \Big[\ddot{x}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) \\ &-(x+d_1)\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) - \ddot{\theta}\,(x+d_1)^2 - \ddot{\theta}\left(\tfrac{1}{3}\tfrac{\chi^2}{x_a^2}+d_2^2+\tfrac{d_2\chi}{x_a}\right) \\ &-2\dot{\theta}\dot{x}\,(x+d_1) - g\cos\theta\,(x+d_1) + g\sin\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\Big]\,\rho\chi\mathrm{d}a \\ &+m_v\overline{\mathrm{y}}_v\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) - m_v\overline{\mathrm{x}}_v\left(\ddot{q}_2\cos\theta - \ddot{q}_1\sin\theta\right) \\ &-I_v\ddot{\theta} - m_v g\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) = 0\,.\end{aligned} \tag{10.52}$$

Introduce the Lagrangian action

$$\begin{aligned}\mathscr{L} = &\int_{t_1}^{t_2}\int_0^L \Big[\tfrac{1}{2}\dot{x}^2 - \dot{\theta}\dot{x}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \dot{x}\left(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta\right) + \tfrac{1}{2}\theta\tfrac{\chi\dot{x}\dot{x}_a}{x_a^2} \\ &-\dot{\theta}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\left(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta\right) + \dot{\theta}\,(x+d_1)\left(\dot{q}_2\cos\theta - \dot{q}_1\sin\theta\right) \\ &+\tfrac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right) + \tfrac{1}{2}\dot{\theta}^2\,(x+d_1)^2 + \tfrac{1}{2}\dot{\theta}^2\left(\tfrac{1}{3}\tfrac{\chi^2}{x_a^2}+d_2^2+\tfrac{d_2\chi}{x_a}\right) \\ &-g\sin\theta\,(x+d_1) - g\cos\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) - gq_2\Big]\,\rho\chi\mathrm{d}a\mathrm{d}t \\ &+\int_{t_1}^{t_2}\Big[\tfrac{1}{2}m_v\left(\dot{q}_1^2+\dot{q}_2^2\right) - m_v\overline{\mathrm{y}}_v\dot{\theta}\left(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta\right) + \tfrac{1}{2}I_v\dot{\theta}^2 \\ &+m_v\overline{\mathrm{x}}_v\dot{\theta}\left(\dot{q}_2\cos\theta - \dot{q}_1\sin\theta\right) - m_v g\left(\overline{\mathrm{x}}_v\sin\theta + \overline{\mathrm{y}}_v\cos\theta + q_2\right)\Big]\,\mathrm{d}t\,.\end{aligned} \tag{10.53}$$

Taking the first variation of the Lagrangian functional (10.53) with respect to q_1 gives

$$\begin{aligned}&\int_0^L \Big[-\ddot{x}\cos\theta + \dot{\theta}^2\cos\theta\,(x+d_1) - \dot{\theta}^2\sin\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + 2\dot{\theta}\dot{x}\sin\theta \\ &+\ddot{\theta}\cos\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \ddot{\theta}\sin\theta\,(x+d_1) - \ddot{q}_1\boxed{-\tfrac{1}{2}\tfrac{\chi\dot{x}_a}{x_a^2}\dot{\theta}\cos\theta}\Big]\,\rho\chi\mathrm{d}a \\ &+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\sin\theta + \overline{\mathrm{y}}_v\cos\theta\right) - m_v\ddot{q}_1 + m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) = 0\,.\end{aligned} \tag{10.54}$$

Taking the first variation of the Lagrangian functional (10.53) with respect to q_2 gives

$$\begin{aligned}&\int_0^L \Big[\ddot{x}\sin\theta - \dot{\theta}^2\cos\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) - \dot{\theta}^2\sin\theta\,(x+d_1) + 2\dot{\theta}\dot{x}\cos\theta \\ &+\ddot{\theta}\cos\theta\,(x+d_1) - \ddot{\theta}\sin\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right) + \ddot{q}_2 + g\boxed{+\tfrac{1}{2}\tfrac{\chi\dot{x}_a}{x_a^2}\dot{\theta}\sin\theta}\Big]\,\rho\chi\mathrm{d}a \\ &+m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v\cos\theta - \overline{\mathrm{y}}_v\sin\theta\right) + m_v\ddot{q}_2 - m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v\sin\theta + \overline{\mathrm{y}}_v\cos\theta\right) + m_v g = 0\,.\end{aligned} \tag{10.55}$$

Taking the first variation of the Lagrangian functional (10.53) with respect to θ gives

$$
\begin{aligned}
&\int_0^L \Big[\ddot{x}\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)+\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta)\\
&-(x+d_1)(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta)-\ddot{\theta}(x+d_1)^2-\ddot{\theta}\left(\tfrac{1}{3}\tfrac{\chi^2}{x_a^2}+d_2^2+\tfrac{d_2\chi}{x_a}\right)\\
&\boxed{-\tfrac{1}{2}\tfrac{\chi\dot{x}_a}{x_a^2}(\dot{q}_1\cos\theta+\dot{q}_2\sin\theta)+\dot{\theta}\left(\tfrac{2}{3}\tfrac{\chi^2\dot{x}_a}{x_a^3}+\tfrac{d_2\chi\dot{x}_a}{x_a^2}\right)}\\
&-2\dot{\theta}\dot{x}(x+d_1)-g\cos\theta(x+d_1)+g\sin\theta\left(\tfrac{1}{2}\tfrac{\chi}{x_a}+d_2\right)\Big]\rho\chi\mathrm{d}a\\
&+m_v\overline{\mathrm{y}}_v(\ddot{q}_1\cos\theta+\ddot{q}_2\sin\theta)-m_v\overline{\mathrm{x}}_v(\ddot{q}_2\cos\theta-\ddot{q}_1\sin\theta)\\
&-I_v\ddot{\theta}-m_v g(\overline{\mathrm{x}}_v\cos\theta-\overline{\mathrm{y}}_v\sin\theta)=0\,.
\end{aligned}
\tag{10.56}
$$

Boxed terms of equations (10.54), (10.55) and (10.56) are the extra terms in comparison with equations (10.50), (10.51) and (10.52).

11 Lagrangian functional when q_1 and q_2 are relative to the body frame

The fluid kinetic energy when q_1 and q_2 are relative to the body frame reads

$$
\begin{aligned}
\mathrm{KE}^{fluid} &= \int_{\mathsf{Vol}}\Big(\tfrac{1}{2}\|\dot{\mathbf{x}}+\dot{\mathbf{q}}\|^2+(\dot{\mathbf{x}}+\dot{\mathbf{q}})\cdot(\mathbf{\Omega}\times(\mathbf{x}+\mathbf{d}+\mathbf{q}))+(\mathbf{\Omega}\times(\mathbf{x}+\mathbf{d}))\cdot(\mathbf{\Omega}\times\mathbf{q})\\
&\quad+\tfrac{1}{2}\|\mathbf{\Omega}\times\mathbf{q}\|^2\Big)\rho\mathrm{d}\mathsf{Vol}+\tfrac{1}{2}\mathbf{\Omega}\cdot\mathbf{I}_f\mathbf{\Omega}\,,
\end{aligned}
\tag{11.57}
$$

the kinetic energy of the dry vessel with q_1 and q_2 relative to the body frame reads

$$
\begin{aligned}
\mathrm{KE}^{vessel} &= \tfrac{1}{2}m_v\|\dot{\mathbf{q}}\|^2+\dot{\mathbf{q}}\cdot(\mathbf{\Omega}\times m_v(\overline{\mathbf{x}}_v+\mathbf{q}))+(\mathbf{\Omega}\times m_v\overline{\mathbf{x}}_v)\cdot(\mathbf{\Omega}\times\mathbf{q})\\
&\quad+\tfrac{1}{2}m_v\|\mathbf{\Omega}\times\mathbf{q}\|^2+\tfrac{1}{2}\mathbf{\Omega}\cdot\mathbf{I}_v\mathbf{\Omega}\,,
\end{aligned}
\tag{11.58}
$$

the fluid potential energy reads

$$
\mathrm{PE}^{fluid} = \int_{\mathsf{Vol}}\rho g\mathbf{Q}(\mathbf{x}+\mathbf{d}+\mathbf{q})\cdot\mathbf{E}_3\mathrm{d}\mathsf{Vol}\,,
\tag{11.59}
$$

and the vessel potential energy reads

$$
\mathrm{PE}^{vessel} = m_v g\mathbf{Q}(\overline{\mathbf{x}}_v+\mathbf{q})\cdot\mathbf{E}_3\,.
\tag{11.60}
$$

When the vessel motion is restricted to roll-sway-heave motion the kinetic and potential energies become

$$
\begin{aligned}
\mathrm{KE}^{fluid} &= \int_{\mathsf{Vol}} \Big[\tfrac{1}{2}\left(\dot{x}+\dot{q}_1\right)^2 + \tfrac{1}{2}\left(\dot{y}+\dot{q}_2\right)^2 - \dot{\theta}\left(\dot{x}+\dot{q}_1\right)\left(y+d_2+q_2\right) \\
&\quad +\dot{\theta}\left(\dot{y}+\dot{q}_2\right)\left(x+d_1+q_1\right) + \dot{\theta}^2\left(q_2\left(y+d_2\right)+q_1\left(x+d_1\right)\right) \\
&\quad +\tfrac{1}{2}\dot{\theta}^2\left(q_1^2+q_2^2\right) + \tfrac{1}{2}\dot{\theta}^2\left(\left(x+d_1\right)^2+\left(y+d_2\right)^2\right)\Big]\,\rho \mathrm{d}\mathsf{Vol}\,, \\
\mathrm{KE}^{vessel} &= \tfrac{1}{2}m_v\left(\dot{q}_1^2+\dot{q}_2^2\right) - m_v\dot{\theta}\dot{q}_1\left(\overline{\mathrm{y}}_v+q_2\right) + m_v\dot{\theta}\dot{q}_2\left(\overline{\mathrm{x}}_v+q_1\right) \\
&\quad +m_v\dot{\theta}^2\left(q_2\overline{\mathrm{y}}_v+q_1\overline{\mathrm{x}}_v\right) + \tfrac{1}{2}m_v\dot{\theta}^2\left(q_1^2+q_2^2\right) + \tfrac{1}{2}I_v\dot{\theta}^2\,, \\
\mathrm{PE}^{fluid} &= \int_{\mathsf{Vol}} \rho g\left[\sin\theta\left(x+d_1+q_1\right)+\cos\theta\left(y+d_2+q_2\right)\right]\mathrm{d}\mathsf{Vol}\,, \\
\mathrm{PE}^{vessel} &= m_v g\left[\sin\theta\left(\overline{\mathrm{x}}_v+q_1\right)+\cos\theta\left(\overline{\mathrm{y}}_v+q_2\right)\right]\,,
\end{aligned}
$$

and so the Lagrangian action reads

$$
\begin{aligned}
\mathscr{L} &= \int_{t_1}^{t_2}\int_0^L\int_0^h \Big[\tfrac{1}{2}\left(\dot{x}+\dot{q}_1\right)^2 + \tfrac{1}{2}\left(\dot{y}+\dot{q}_2\right)^2 - \dot{\theta}\left(\dot{x}+\dot{q}_1\right)\left(y+d_2+q_2\right) \\
&\quad +\dot{\theta}\left(\dot{y}+\dot{q}_2\right)\left(x+d_1+q_1\right) + \dot{\theta}^2\left(q_2\left(y+d_2\right)+q_1\left(x+d_1\right)\right) \\
&\quad +\tfrac{1}{2}\dot{\theta}^2\left(q_1^2+q_2^2\right) + \tfrac{1}{2}\dot{\theta}^2\left(\left(x+d_1\right)^2+\left(y+d_2\right)^2\right) \\
&\quad -g\sin\theta\left(x+d_1+q_1\right) - g\cos\theta\left(y+d_2+q_2\right)\Big]\,\rho \mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&\quad +\int_{t_1}^{t_2}\Big[\tfrac{1}{2}m_v\left(\dot{q}_1^2+\dot{q}_2^2\right) - m_v\dot{\theta}\dot{q}_1\left(\overline{\mathrm{y}}_v+q_2\right) + m_v\dot{\theta}\dot{q}_2\left(\overline{\mathrm{x}}_v+q_1\right) \\
&\quad +m_v\dot{\theta}^2\left(q_2\overline{\mathrm{y}}_v+q_1\overline{\mathrm{x}}_v\right) + \tfrac{1}{2}m_v\dot{\theta}^2\left(q_1^2+q_2^2\right) + \tfrac{1}{2}I_v\dot{\theta}^2 \\
&\quad -m_v g\sin\theta\left(\overline{\mathrm{x}}_v+q_1\right) - m_v g\cos\theta\left(\overline{\mathrm{y}}_v+q_2\right)\Big]\,\mathrm{d}t\,.
\end{aligned}
\tag{11.61}
$$

Now taking the first variation of this Lagrangian functional with respect to q_1, q_2 and θ gives respectively the sway, heave and roll equations

$$
\begin{aligned}
&\tfrac{\delta\mathscr{L}}{\delta q_1} = 0 \quad \Rightarrow \\
&\int_0^h\int_0^L \Big[-\tfrac{Du}{Dt} - \ddot{q}_1 + \ddot{\theta}\left(y+d_2+q_2\right) + 2\dot{\theta}\left(v+\dot{q}_2\right) \\
&+\dot{\theta}^2\left(x+d_1+q_1\right) - g\sin\theta\Big]\,\rho \mathrm{d}y\mathrm{d}x - m_v\ddot{q}_1 + m_v\ddot{\theta}\left(\overline{\mathrm{y}}_v+q_2\right) \\
&+2m_v\dot{\theta}\dot{q}_2 + m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v+q_1\right) - m_v g\sin\theta = 0\,,
\end{aligned}
\tag{11.62}
$$

$$
\begin{aligned}
&\tfrac{\delta\mathscr{L}}{\delta q_2} = 0 \quad \Rightarrow \\
&\int_0^h\int_0^L \Big[-\tfrac{Dv}{Dt} - \ddot{q}_2 - 2\dot{\theta}\left(u+\dot{q}_1\right) - \ddot{\theta}\left(x+d_1+q_1\right) \\
&+\dot{\theta}^2\left(y+d_2+q_2\right) - g\cos\theta\Big]\,\rho \mathrm{d}y\mathrm{d}x - m_v\ddot{q}_2 - 2m_v\dot{\theta}\dot{q}_1 \\
&-m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v+q_1\right) + m_v\dot{\theta}^2\left(\overline{\mathrm{y}}_v+q_2\right) - m_v g\cos\theta = 0\,,
\end{aligned}
\tag{11.63}
$$

$$\frac{\delta \mathscr{L}}{\delta \theta} = 0 \quad \Rightarrow$$
$$\int_0^h \int_0^L \Big[\left(\tfrac{Du}{Dt} + \ddot{q}_1\right)(y + d_2 + q_2) - \left(\tfrac{Dv}{Dt} + \ddot{q}_2\right)(x + d_1 + q_1) - 2\ddot{\theta} q_2 (y + d_2)$$
$$-2\ddot{\theta} q_1 (x + d_1) - 2\dot{\theta}\dot{q}_2 (y + d_2 + q_2) - 2\dot{\theta}\dot{q}_1 (x + d_1 + q_1)$$
$$-2\dot{\theta} u (x + d_1 + q_1) - 2\dot{\theta} v (y + d_2 + q_2) - \ddot{\theta}\left(q_1^2 + q_2^2\right)$$
$$-\ddot{\theta}\left((x + d_1)^2 + (y + d_2)^2\right) - g\cos\theta (x + d_1 + q_1) + g\sin\theta (y + d_2 + q_2)\Big] \rho \mathrm{d}y\mathrm{d}x$$
$$+m_v \ddot{q}_1 (\overline{\mathrm{y}}_v + q_2) - m_v \ddot{q}_2 (\overline{\mathrm{x}}_v + q_1) - 2 m_v \ddot{\theta} (q_2 \overline{\mathrm{y}}_v + q_1 \overline{\mathrm{x}}_v)$$
$$-2 m_v \dot{\theta} (\dot{q}_2 \overline{\mathrm{y}}_v + \dot{q}_1 \overline{\mathrm{x}}_v) - m_v \ddot{\theta} (q_1^2 + q_2^2) - 2 m_v \dot{\theta} (q_1 \dot{q}_1 + q_2 \dot{q}_2)$$
$$-I_v \ddot{\theta} - m_v g \cos\theta (\overline{\mathrm{x}}_v + q_1) + m_v g \sin\theta (\overline{\mathrm{y}}_v + q_2) = 0 \,. \tag{11.64}$$

12 Equations (11.62), (11.63) and (11.64) with ALR shallow-water assumptions

Equations (11.62), (11.63) and (11.64) with the ALR shallow-water assumptions (7.33) read respectively

$$\int_0^L \Big[-(\overline{u}_t + \overline{u}\overline{u}_x) - \ddot{q}_1 + \ddot{\theta}\left(\tfrac{1}{2}h + d_2 + q_2\right) + 2\dot{\theta}\dot{q}_2 + \dot{\theta}^2 (x + d_1 + q_1)$$
$$-g\sin\theta\Big] \rho h \mathrm{d}x - m_v \ddot{q}_1 + m_v \ddot{\theta} (\overline{\mathrm{y}}_v + q_2) + 2 m_v \dot{\theta}\dot{q}_2 \tag{12.65}$$
$$+m_v \dot{\theta}^2 (\overline{\mathrm{x}}_v + q_1) - m_v g \sin\theta = 0 \,,$$

$$\int_0^L \Big[-\ddot{q}_2 - 2\dot{\theta}(\overline{u} + \dot{q}_1) - \ddot{\theta}(x + d_1 + q_1) + \dot{\theta}^2\left(\tfrac{1}{2}h + d_2 + q_2\right) - g\cos\theta \Big] \rho h \mathrm{d}x$$
$$-m_v \ddot{q}_2 - 2 m_v \dot{\theta}\dot{q}_1 - m_v \ddot{\theta} (\overline{\mathrm{x}}_v + q_1) + m_v \dot{\theta}^2 (\overline{\mathrm{y}}_v + q_2) - m_v g \cos\theta = 0 \,, \tag{12.66}$$

$$\int_0^L \Big[(\overline{u}_t + \overline{u}\overline{u}_x + \ddot{q}_1)\left(\tfrac{1}{2}h + d_2 + q_2\right) - \ddot{q}_2 (x + d_1 + q_1) - 2\ddot{\theta}\Big(q_2\left(\tfrac{1}{2}h + d_2\right) +$$
$$q_1 (x + d_1)\Big) - 2\dot{\theta}\dot{q}_2\left(\tfrac{1}{2}h + d_2 + q_2\right) - 2\dot{\theta}\dot{q}_1 (x + d_1 + q_1) - 2\dot{\theta}\overline{u} (x + d_1 + q_1)$$
$$-\ddot{\theta}\left(q_1^2 + q_2^2\right) - \ddot{\theta}(x + d_1)^2 - \ddot{\theta}\left(\tfrac{1}{3}h^2 + d_2^2 + d_2 h\right) - g\cos\theta (x + d_1 + q_1)$$
$$+g\sin\theta\left(\tfrac{1}{2}h + d_2 + q_2\right)\Big] \rho h \mathrm{d}x$$
$$+m_v \ddot{q}_1 (\overline{\mathrm{y}}_v + q_2) - m_v \ddot{q}_2 (\overline{\mathrm{x}}_v + q_1) - 2 m_v \ddot{\theta} (q_2 \overline{\mathrm{y}}_v + q_1 \overline{\mathrm{x}}_v)$$
$$-2 m_v \dot{\theta} (\dot{q}_2 \overline{\mathrm{y}}_v + \dot{q}_1 \overline{\mathrm{x}}_v) - m_v \ddot{\theta} (q_1^2 + q_2^2) - 2 m_v \dot{\theta} (q_1 \dot{q}_1 + q_2 \dot{q}_2)$$
$$-I_v \ddot{\theta} - m_v g \cos\theta (\overline{\mathrm{x}}_v + q_1) + m_v g \sin\theta (\overline{\mathrm{y}}_v + q_2) = 0 \,. \tag{12.67}$$

Equations (12.65), (12.66) and (12.67) in Lagrangian coordinates become respectively

$$\begin{aligned}
&\int_0^L \left[-\ddot{x} - \ddot{q}_1 + \ddot{\theta}\left(\frac{1}{2}\frac{\chi}{x_a} + d_2 + q_2\right) + 2\dot{\theta}\dot{q}_2 + \dot{\theta}^2\left(x + d_1 + q_1\right) - g\sin\theta\right]\rho\chi \mathrm{d}a \\
&-m_v\ddot{q}_1 + m_v\ddot{\theta}\left(\overline{\mathrm{y}}_v + q_2\right) + 2m_v\dot{\theta}\dot{q}_2 + m_v\dot{\theta}^2\left(\overline{\mathrm{x}}_v + q_1\right) - m_v g\sin\theta = 0\,,
\end{aligned} \tag{12.68}$$

$$\begin{aligned}
&\int_0^L \left[-\ddot{q}_2 - 2\dot{\theta}\left(\dot{x} + \dot{q}_1\right) - \ddot{\theta}\left(x + d_1 + q_1\right) + \dot{\theta}^2\left(\frac{1}{2}\frac{\chi}{x_a} + d_2 + q_2\right) - g\cos\theta\right]\rho\chi \mathrm{d}a \\
&-m_v\ddot{q}_2 - 2m_v\dot{\theta}\dot{q}_1 - m_v\ddot{\theta}\left(\overline{\mathrm{x}}_v + q_1\right) + m_v\dot{\theta}^2\left(\overline{\mathrm{y}}_v + q_2\right) - m_v g\cos\theta = 0\,,
\end{aligned} \tag{12.69}$$

$$\begin{aligned}
&\int_0^L \Bigg[\left(\ddot{x} + \ddot{q}_1\right)\left(\frac{1}{2}\frac{\chi}{x_a} + d_2 + q_2\right) - \ddot{q}_2\left(x + d_1 + q_1\right) - 2\ddot{\theta}\left(q_2\left(\frac{1}{2}\frac{\chi}{x_a} + d_2\right) + \right. \\
&\left. q_1\left(x + d_1\right)\right) - 2\dot{\theta}\dot{q}_2\left(\frac{1}{2}\frac{\chi}{x_a} + d_2 + q_2\right) - 2\dot{\theta}\dot{q}_1\left(x + d_1 + q_1\right) - 2\dot{\theta}\dot{x}\left(x + d_1 + q_1\right) \\
&-\ddot{\theta}\left(q_1^2 + q_2^2\right) - \ddot{\theta}\left(x + d_1\right)^2 - \ddot{\theta}\left(\frac{1}{3}\frac{\chi^2}{x_a^2} + d_2^2 + \frac{d_2\chi}{x_a}\right) - g\cos\theta\left(x + d_1 + q_1\right) \\
&+g\sin\theta\left(\frac{1}{2}\frac{\chi}{x_a} + d_2 + q_2\right)\Bigg]\rho\chi \mathrm{d}a \\
&+m_v\ddot{q}_1\left(\overline{\mathrm{y}}_v + q_2\right) - m_v\ddot{q}_2\left(\overline{\mathrm{x}}_v + q_1\right) - 2m_v\ddot{\theta}\left(q_2\overline{\mathrm{y}}_v + q_1\overline{\mathrm{x}}_v\right) \\
&-2m_v\dot{\theta}\left(\dot{q}_2\overline{\mathrm{y}}_v + \dot{q}_1\overline{\mathrm{x}}_v\right) - m_v\ddot{\theta}\left(q_1^2 + q_2^2\right) - 2m_v\dot{\theta}\left(q_1\dot{q}_1 + q_2\dot{q}_2\right) \\
&-I_v\ddot{\theta} - m_v g\cos\theta\left(\overline{\mathrm{x}}_v + q_1\right) + m_v g\sin\theta\left(\overline{\mathrm{y}}_v + q_2\right) = 0\,.
\end{aligned} \tag{12.70}$$

Introduce the Lagrangian functional

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_0^L \Bigg[\frac{1}{2}\left(\dot{x} + \dot{q}_1\right)^2 + \frac{1}{2}\dot{q}_2^2 - \dot{\theta}\left(\dot{x} + \dot{q}_1\right)\left(\frac{1}{2}\frac{\chi}{x_a} + d_2 + q_2\right) + \dot{\theta}\dot{q}_2\left(x + d_1 + q_1\right) \\
&+\dot{\theta}^2 q_2\left(\frac{1}{2}\frac{\chi}{x_a} + d_2\right) + \dot{\theta}^2 q_1\left(x + d_1\right) + \frac{1}{2}\dot{\theta}^2\left(q_1^2 + q_2^2\right) + \frac{1}{2}\dot{\theta}^2\left(x + d_1\right)^2 \\
&+\frac{1}{2}\dot{\theta}^2\left(\frac{1}{3}\frac{\chi^2}{x_a^2} + d_2^2 + \frac{d_2\chi}{x_a}\right) - g\sin\theta\left(x + d_1 + q_1\right) \\
&-g\cos\theta\left(\frac{1}{2}\frac{\chi}{x_a} + d_2 + q_2\right)\Bigg]\rho\chi \mathrm{d}a\mathrm{d}t \\
&+\int_{t_1}^{t_2}\Big[\frac{1}{2}m_v\left(\dot{q}_1^2 + \dot{q}_2^2\right) - m_v\dot{\theta}\dot{q}_1\left(\overline{\mathrm{y}}_v + q_2\right) + m_v\dot{\theta}\dot{q}_2\left(\overline{\mathrm{x}}_v + q_1\right) \\
&+m_v\dot{\theta}^2\left(q_2\overline{\mathrm{y}}_v + q_1\overline{\mathrm{x}}_v\right) + \frac{1}{2}m_v\dot{\theta}^2\left(q_1^2 + q_2^2\right) + \frac{1}{2}I_v\dot{\theta}^2 \\
&-m_v g\sin\theta\left(\overline{\mathrm{x}}_v + q_1\right) - m_v g\cos\theta\left(\overline{\mathrm{y}}_v + q_2\right)\Big]\mathrm{d}t\,.
\end{aligned} \tag{12.71}$$

Taking the first variation of the Lagrangian functional (12.71) with respect to q_1 recovers equation (12.68) with an extra term on the left hand side which is $-\frac{1}{2}\frac{\dot{\theta}\chi\dot{x}_a}{x_a^2}$. Taking the first variation of this Lagrangian functional with respect to q_2 recovers equation

(12.69). And taking the first variation of this action integral with respect to θ gives

$$\int_0^L \Big[(\ddot{x}+\ddot{q}_1)\left(\frac{1}{2}\frac{\chi}{x_a}+d_2+q_2\right) - \ddot{q}_2\,(x+d_1+q_1) - 2\ddot{\theta}\left(q_2\left(\frac{1}{2}\frac{\chi}{x_a}+d_2\right)+ q_1\,(x+d_1)\right) - 2\dot{\theta}\dot{q}_2\left(\frac{1}{2}\frac{\chi}{x_a}+d_2+q_2\right) - 2\dot{\theta}\dot{q}_1\,(x+d_1+q_1) - 2\dot{\theta}\dot{x}\,(x+d_1+q_1)$$
$$-\ddot{\theta}\left(q_1^2+q_2^2\right) - \ddot{\theta}\,(x+d_1)^2 - \ddot{\theta}\left(\frac{1}{3}\frac{\chi^2}{x_a^2}+d_2^2+\frac{d_2\chi}{x_a}\right) - g\cos\theta\,(x+d_1+q_1)$$
$$\boxed{-\frac{1}{2}\frac{\chi\dot{x}_a}{x_a^2}(\dot{x}+\dot{q}_1) + \frac{\dot{\theta}q_2\chi\dot{x}_a}{x_a^2} + \dot{\theta}\left(\frac{2}{3}\frac{\chi^2\dot{x}_a}{x_a^3}+\frac{d_2\chi\dot{x}_a}{x_a^2}\right)} + g\sin\theta\left(\frac{1}{2}\frac{\chi}{x_a}+d_2+q_2\right)\Big]\rho\chi \mathrm{d}a$$
$$+m_v\ddot{q}_1\left(\overline{\mathrm{y}}_v+q_2\right) - m_v\ddot{q}_2\left(\overline{\mathrm{x}}_v+q_1\right) - 2m_v\ddot{\theta}\left(q_2\overline{\mathrm{y}}_v+q_1\overline{\mathrm{x}}_v\right)$$
$$-2m_v\dot{\theta}\left(\dot{q}_2\overline{\mathrm{y}}_v+\dot{q}_1\overline{\mathrm{x}}_v\right) - m_v\ddot{\theta}\left(q_1^2+q_2^2\right) - 2m_v\dot{\theta}\left(q_1\dot{q}_1+q_2\dot{q}_2\right)$$
$$-I_v\ddot{\theta} - m_v g\cos\theta\left(\overline{\mathrm{x}}_v+q_1\right) + m_v g\sin\theta\left(\overline{\mathrm{y}}_v+q_2\right) = 0\,, \tag{12.72}$$

where the boxed terms are the extra terms in comparison with equation (12.70).

13 ALR SWEs with q_1 and q_2 respect to the body frame

Armenio and Larroca shallow water equations (ALR SWEs) [4] including the translation accelerations $\ddot{\mathbf{q}}$ with respect to the body frame and assuming $(y = h)$ are

$$h_t + \overline{u}h_x + h\overline{u}_x = 0\,,$$
$$\overline{u}_t + \left(\overline{u}+2\dot{\theta}h\right)\overline{u}_x + \Big(\ddot{\theta}\,(x+d_1+q_1) - \dot{\theta}^2\,(h+d_2+q_2) + 2\dot{\theta}\,(\overline{u}+\dot{q}_1) + g\cos\theta + \ddot{q}_2\Big)h_x \tag{13.73}$$
$$= -g\sin\theta + \ddot{\theta}\,(d_2+q_2) + \dot{\theta}^2\,(x+d_1+q_1) - \ddot{q}_1 + 2\dot{\theta}\dot{q}_2\,.$$

Introduce the Lagrangian functional

$$\begin{aligned}\mathscr{L}(h,u,\phi) = \int_{t_1}^{t_2}\int_0^L \Big[& \frac{1}{2}h\overline{u}^2 - \dot{\theta}h\overline{u}\,(h+d_2+q_2) + \frac{1}{2}\dot{\theta}^2\left(\frac{1}{3}h^3+d_2h^2+d_2^2h+q_2h^2\right) \\ & +\frac{1}{2}h\dot{\theta}^2\,(x+d_1)^2 - \frac{1}{2}\ddot{\theta}\,(x+d_1+q_1)\,h^2 - g\cos\theta\left(\frac{1}{2}h^2+d_2h+q_2h\right) \\ & -gh\sin\theta\,(x+d_1+q_1) - \left(\dot{\theta}\dot{q}_1+\frac{1}{2}\ddot{q}_2\right)h^2 + \left(\dot{\theta}^2q_1-\dot{\theta}\dot{q}_2\right)(x+d_1)\,h \\ & -\left(\ddot{q}_1-2\dot{\theta}\dot{q}_2\right)h\,(x+d_1) + \phi\left(h_t+(h\overline{u})_x\right)\Big]\mathrm{d}x\mathrm{d}t\,.\end{aligned} \tag{13.74}$$

The first variation of this Lagrangian functional with respect to h, $\overline{u}$ and ϕ recovers (13.73).

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