SINGULARITY FORMATION AND GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS FOR ONE DIMENSIONAL ROTATING SHALLOW WATER SYSTEM

BIN CHENG, PENG QU, AND CHUNJING XIE

Abstract. We study classical solutions of one dimensional rotating shallow water system which plays an important role in geophysical fluid dynamics. The main results contain two contrasting aspects. First, when the solution crosses certain thresholds, we prove finite-time singularity formation for the classical solutions by studying the weighted gradients of Riemann invariants and utilizing conservation of physical energy. In fact, the singularity formation will take place for a large class of C^1 initial data whose gradients and physical energy can be arbitrarily small and higher order derivatives should be large. Second, when the initial data have constant potential vorticity, global existence of small classical solutions is established via studying an equivalent form of a quasilinear Klein-Gordon equation satisfying certain null conditions. In this global existence result, the smallness condition is in terms of the higher order Sobolev norms of the initial data.

1. Introduction and Main Results

The one dimensional rotating shallow water system plays an important role in the study of geostrophic adjustment and zonal jets (e.g. [34, 11]) in large scale geophyisical fluid dynamics. It is the simplest model for studying both phenomena since it captures the two most essential driving forces: pressure gradient and Coriolis forces. The one dimensional rotating shallow water system was first analyzed in the pioneering work by Rossby in [30] (and hence named by "Rossby adjustment") and was also used in classical textbooks such as [13, Section 7.2] to find analytical solution to a one dimensional adjustment problem. Interests in nearly one dimensional geophysical flows have been renewed recently thanks to high resolution data from both observations and simulations, showing that robust and persistent zonal flow pattern appears ubiquitously in planet scale circulations – c.f. [12, 4] and references therein.

Due to the crucial presence of the Coriolis force resulting from the use of a rotating frame, the one dimensional version of the rotating shallow water system must retain both components

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of the horizontal velocity field. Upon suitable rescaling, the system in Eulerian form reads

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x h^\gamma/\gamma = hv, \\ \partial_t(hv) + \partial_x(huv) = -hu, \end{cases} \tag{1}$$

where h denotes the height of the fluid surface, u denotes the velocity component in the x-direction, and v is the other horizontal velocity component that is in the direction orthogonal to the x-direction. The Coriolis force, caused by a rotating frame, is represented by the $(hv, -hu)^T$ terms on the right-hand side of the last two equations of (1). For the rotating shallow water model, one has $\gamma = 2$ in the pressure law ([7]), but in this paper we will prove results that are valid for the general case $\gamma \geq 1$.

The system (1) is a typical one dimensional system of balance laws, which has attracted plenty of studies pioneered by Riemann [8]. One of the most important features of nonlinear hyperbolic system of conservation laws is that the wave speed depends on the solution itself so that the classical solutions in general are expected to form singularity in finite time (cf. [21, 18, 16, 22]). In fact, the system (1) can also be regarded as one dimensional compressible Euler system with source terms. The formation of singularity and critical threshold phenomena for the compressible Euler system without or with certain special source terms were studied in [24, 33, 3] and references therein. The additional source terms very often demand substantial novel techniques in addition to the classical singularity formation theory developed in [21, 18] and subsequent literature.

The novelty of our results and techniques is two-fold. First, with the extra Coriolis force added to the conservation laws, allowing *arbitrarily small threshold* on the initial data for singularity formation is very tricky. The challenge is essentially that, in the Riccati type ODE, one needs to tightly control any extra source term resulting from the Coriolis force so that it does not override the $O\Big(\dfrac{1}{C+t}\Big)$ growth rate that a source-less Riccati ODE would otherwise enjoy (with the constant C having the same sign and size as the *most negative* gradient of the Riemann invariants at $t = 0$). In fact, a crude upper bound for the source term (c.f. $-\dfrac{1}{h}$ in (38)) we establish (c.f. (27)) approaches zero at an exponential rate so that we simply treat it as zero in the proof of Theorem 1 which then subsequently shows an $O(1)$ threshold on the initial data. We also mention that a different Riccati type ODE is established in [34, eq. (64)] which presents the same (if not more) challenge. Their very brief, informal discussion following that seems to suggest an $O(1)$ threshold on the initial data for singularity formation.

On the other hand, with (h, u, v) depending only on (t, x)-variables, the system (1) can be regarded as a special case of the two dimensional rotating shallow water system ([28]). The latter is a widely used approximation of the three dimensional incompressible Euler system and the Boussinesq system in the regime of large scale geophysical fluid motion. In the two dimensional setting, it is shown in [6] that there exist global classical solutions for a large class of small initial data subject to the constant potential vorticity constraint, which is analogue of the irrotational constraint for the compressible Euler system. Since classical solutions of two dimensional compressible Euler system in general form singularity (c.f. [29]), the result of [6] evidences that the Coriolis forcing term plays a decisive role in the global well-posednesss theory for classical solutions of compressible flows. A key ingredient in the proof of [6] is that the rotating shallow water system and in fact also its one dimensional reduction, when subject to the (invariant) constant potential vorticity constraint, can be reformulated into a quasilinear Klein-Gordon system. Note that the solutions of Klein-Gordon equations are of faster dispersive decay than those of the corresponding nonlinear wave system, the latter of which is derived from non-rotating fluid models. The rate of this dispersive decay, however, is tied to the spatial dimension. In fact, in the one dimensional setting, the decay rate of Klein-Gordon system is not fast enough for the global existence theory of [6] to be directly applied to the system (1).

In short, for studying the lifespan and global existence of classical solutions for the one dimensional rotating shallow water system, we have to introduce novel techniques and investigate the nonlinear structure of the system (1) more carefully.

The main objectives of this paper are two-fold. First, we study the formation of singularities for a general class of C^1 initial data by capturing the nonlinear interactions in the system (1). These initial data can have arbitrarily small gradients and physical energy but higher order derivatives should be large. Second, we take a careful look at the structure of the system (1), exploit the dispersion provided by the Coriolis forcing terms, and show the global existence of classical solutions for a class of initial data that are of small size in terms of higher order Sobolev norms.

Before stating the main theorems, we first rewrite the system (1) in the Lagrangian coordinates, which takes a simpler form in one dimensional setting.

Assume the initial height field $h(0,x) = h_0(x) \in C^1(\mathbb{R})$ is strictly away from vacuum, i.e., $0 < c_h \leq h_0(x) = h(0,x) \leq C_h$. It then induces a "coordinate stretching" at $t = 0$ specified by a C^2 bijection $\phi : \mathbb{R} \to \mathbb{R}$ defined as

$$\xi = \phi(x) := \int_0^x h(0,s)\,ds, \tag{2}$$

whose inverse function can be written as $x = \phi^{-1}(\xi)$. Assume $u \in C^1([0,T) \times \mathbb{R})$. Let $\sigma(t,\xi)$ be the unique particle path determined by

$$\begin{cases} \partial_t \sigma(t,\xi) = u(t,\sigma(t,\xi)), \\ \sigma(0,\xi) = \phi^{-1}(\xi), \end{cases} \tag{3}$$

so that for each fixed $t \in [0,T)$, we have a C^1 bijection $x = \sigma(t,\xi) : \mathbb{R} \to \mathbb{R}$. Define

$$\widetilde{h}(t,\xi) := h(t,\sigma(t,\xi)), \quad \widetilde{u}(t,\xi) := u(t,\sigma(t,\xi)), \quad \text{and} \quad \widetilde{v}(t,\xi) := v(t,\sigma(t,\xi)). \tag{4}$$

It is easy to see that (h,u,v) solves the system (1) if and only if $(\widetilde{h},\widetilde{u},\widetilde{v})$ defined in (4) is a solution of the following rotating shallow water system in the Lagrangian form (cf. [34]),

$$\begin{cases} \partial_t \widetilde{h} + \widetilde{h}^2 \partial_\xi \widetilde{u} = 0, \\ \partial_t \widetilde{u} + \partial_\xi \widetilde{h}^\gamma/\gamma - \widetilde{v} = 0, \\ \partial_t \widetilde{v} + \widetilde{u} = 0. \end{cases} \tag{5}$$

For the detailed proof of this equivalence, please refer to [7].

For the rest of the paper, we deal with the system (5). For convenience, we drop the tilde signs in $\widetilde{h}$, $\widetilde{u}$, $\widetilde{v}$ when there is no ambiguity for the presentation. Thus, in the Lagrangian coordinates, the rotating shallow water system (5) can be written as

$$\begin{cases} \partial_t h + h^2 \partial_\xi u = 0, \\ \partial_t u + \partial_\xi \big(h^\gamma/\gamma\big) - v = 0, \\ \partial_t v + u = 0\,. \end{cases} \tag{6}$$

The objective in this paper is then to study the Cauchy problem for the system (6) with initial data

$$(h,u,v)(0,\xi) = (h_0(\xi), u_0(\xi), v_0(\xi)) \qquad \text{for} \quad \xi \in \mathbb{R}\,. \tag{7}$$

For any C^1 solution of (6), it follows from (6) that one has

$$\partial_t \Big(\frac{1}{h} + \partial_\xi v\Big) = 0. \tag{8}$$

This is a key geophysical property of rotating fluids known as conservation of potential vorticity in the Lagrangian form, which allows us to define the invariance

$$\omega_0(\xi) := \frac{1}{h(0,\xi)} + \partial_\xi v(0,\xi) = \frac{1}{h(t,\xi)} + \partial_\xi v(t,\xi)\,. \tag{9}$$

We now introduce the "weighted gradients of Riemann invariants", inspired by the techniques of Tadmor and Wei in [33],

$$Z_j := \sqrt{h}\,\big[\partial_\xi u + (-1)^j h^{\frac{\gamma-3}{2}} \partial_\xi h\big] \quad \text{for } j = 1,2. \tag{10}$$

Also, define

$$Z^\sharp(t) := \sup_{\xi\in\mathbb{R}} \max_{j=i,2} Z_j(t,\xi),\ \ Z^\flat(t) := \inf_{\xi\in\mathbb{R}} \min_{j=1,2} Z_j(t,\xi),\ \ Z_0^\sharp = Z^\sharp(0),\ \text{and}\ \omega_0^\sharp := \sup_{\xi\in\mathbb{R}} \omega_0(\xi)\,. \tag{11}$$

The first main result is on the formation of singularities for classical solutions and consists of two theorems.

Theorem 1. *Fix $T' \geq 0$. Consider a classical solution $(h,u,v) \in C^1([0,T'] \times \mathbb{R})$ to the rotating shallow water system (6) with initial data satisfying $\inf_\xi h_0 > 0$ and $(h_0 - 1, u_0, v_0) \in C_0^1(\mathbb{R})$. If*

$$Z^\flat(T') \leq -\sqrt{2\omega_0^\sharp}\,, \tag{12}$$

then the solution must develop a singularity in finite time $t = T^\sharp > T'$ in the following sense

$$\inf_{\substack{0\le t<T^\sharp,\\ \xi\in\mathbb{R}}} h(t,\xi) > 0, \qquad \sup_{\substack{0\le t<T^\sharp,\\ \xi\in\mathbb{R}}} \max\{h(t,\xi), |u(t,\xi)|, |v(t,\xi)|\} < \infty, \tag{13}$$

and

$$\sup_{0\le t<T^\sharp} Z^\sharp(t) < \infty, \quad \lim_{t\nearrow T^\sharp} Z^\flat(t) = -\infty\,. \tag{14}$$

The proof of Theorem 1 is given in Section 3.2.

Remark 1. It follows from the conservation of potential vorticity and the bounds for h in (13) that $\partial_\xi v$ is bounded even when the singularity is formed. Furthermore, it follows from the definition of Z_j ($j = 1, 2$) in (10) and the estimates in (13)-(14) that we have

$$\lim_{t\nearrow T^\sharp} \inf_{\xi\in\mathbb{R}} \partial_\xi u(t,\xi) = -\infty\,.$$

Obviously, if the initial data satisfy (12), then the solution of the problem (6) forms singularities in finite time. In fact, we can also characterize a class of initial data which do not satisfy (12) at the initial time, but rather evolve to satisfy (12), and eventually form a singularity according to the above theorem.

We define physical energy of the rotating shallow water system as

$$E(t) := \int_{-\infty}^{\infty} \frac{1}{2}(u^2+v^2)(t,\xi) + \mathcal{Q}(h(t,\xi))\,d\xi, \tag{15}$$

where

$$\mathcal{Q}(h) := \frac{1}{\gamma}\int_1^h (s^{\gamma-2} - s^{-2})\,ds \ge 0\,, \tag{16}$$

and define

$$E_0 := E(0) \quad \text{and} \quad G_0 := \sqrt{2\omega_0^\sharp} + \max\left\{Z_0^\sharp,\ \sqrt{2\omega_0^\sharp}\right\}, \tag{17}$$

where $\omega_0^\sharp$ and $Z_0^\sharp$ are defined in (11).

Theorem 2. *Consider the Cauchy problem for the system* (6) *subject to initial data* (7) *satisfying $(h_0 - 1, u_0, v_0) \in C_0^1(\mathbb{R})$ and $\inf_\xi h_0 > 0$. Suppose*

$$Z^\flat(0) < -\sqrt{2}\sqrt{\omega_0^\sharp - \left[\mathcal{F}_\gamma^{-1}\big(G_0\,E_0\big) + 1\right]^{-\frac{2}{\gamma}}} \tag{18}$$

where $\mathcal{F}_\gamma^{-1}$ is the inverse function of the function $\mathcal{F}_\gamma(\cdot)$ defined by

$$\mathcal{F}_\gamma(\alpha) := \frac{16}{3\gamma^3}\,\frac{\alpha^3}{(\alpha+1)^3}\left\{(\alpha+1)^{3-\frac{2}{\gamma}} + (\alpha+1)^{3-\frac{2}{\gamma}-1}\right\}. \tag{19}$$

Then the solution must develop the singularity in finite time $t = T^\sharp$ in the sense of (13) *and* (14).

The proof of Theorem 2 is a straightforward combination of Theorems 1 and 9. We have the following remarks on these two theorems.

Remark 2. Straightforward computations show that for $\gamma \geq 1$, $\mathcal{F}_\gamma$ defined in (19) is a strictly increasing function mapping $(0, \infty)$ to $(0, \infty)$. Hence the inverse function $\mathcal{F}_\gamma^{-1}$ is always well-defined on $(0, \infty)$.

Remark 3. If $(h_0 - 1, v_0)$ are compactly supported, it follows from (9) that one always has $\omega_0^\sharp \geq 1$, so all the square roots in (12), (17), and (18) are always real.

Remark 4. We only consider such initial data that $(h_0 - 1, u_0, v_0)$ are compactly supported. As the propagation of information for general data is at a finite speed, the results in Theorems 1 and 2 can also be extended to general initial data without compact support in the same spirit of [16, 22]. Also thanks to the finite speed of propagation, when the initial data are indeed compactly supported and a singularity does develop in finite time as in (13) and (14), we actually have the singularity occur at a finite location as well.

Remark 5. The singularity formation criterion (18) allows arbitrarily small initial gradients at the order of $O(E_0^{1/3})$. Indeed, by definition of $\mathcal{F}_\gamma$ in (19), we have

$$\lim_{\alpha \searrow 0} \frac{\mathcal{F}_\gamma(\alpha)}{\alpha^3} = \frac{32}{3\gamma^3}\,.$$

Therefore, with $G_0 > 0$ bounded above by a constant, we can find positive constants C and $\overline{E}_0$ so that

$$C^{-1} {E_0}^{1/3} \leq \mathcal{F}_\gamma^{-1}\big(G_0\, E_0\big) \leq C {E_0}^{1/3} \quad \text{for all} \quad E_0 < \overline{E}_0. \tag{20}$$

Then, by choosing arbitrarily small $E_0 < \overline{E}_0$ and choosing $\omega_0^\sharp, h_0$ to be arbitrarily close to 1 (with the most convenient choice being $h_0 \equiv 1, v_0 \equiv 0$), we make the right hand side of (18) at order $O(E_0^{1/3})$. This in turn allows us to choose initial data having small gradients, namely, it is possible to choose initial data satisfying

$$\big|(Z_1, Z_2)\big|(0,\xi) \sim O(E_0^{1/3}) \quad \text{and} \quad \omega_0^\sharp,\ h_0 \approx 1$$

so that

$$\big|\partial_\xi(h_0, u_0, v_0)\big|(0,\xi) \sim O(E_0^{1/3}),$$

which also satisfy the condition (18) for singularity formation.

Remark 6. The singularity formation criterion (18) also reflects the fact that we utilize total physical energy and its conservation to prove pointwise singularity formation.

On the other hand, if the initial data have not only small gradients, but also small higher derivatives in Sobolev spaces, then classical solutions can exist globally for one dimensional rotating shallow water system. This is our next main result.

Theorem 3. *Consider the Cauchy problem* (6) *and* (7) *subject to compactly supported initial data* $(h_0 - 1, u_0, v_0)$ *with* $\inf_\xi h_0 > 0$. *Suppose that* $\omega_0 \equiv 1$. *Then, there exists a small positive* *number δ so that if the Sobolev norm $\|u_0\|_{H^k(\mathbb{R})} + \|v_0\|_{H^{k+1}(\mathbb{R})} < \delta$ for some sufficiently large integer k, then there is a global classical solution for the problem* (6) *and* (7) *for all time $t \geq 0$.*

Theorem 3 is proved in Section 4. Here are a few remarks on Theorem 3.

Remark 7. In Theorem 3, we consider only the data close to the constant state $(1, 0, 0)$. In fact, the results also hold for any data close to $(\bar{H}_0, 0, 0)$ with a constant $\bar{H}_0 > 0$.

Remark 8. Although the singularity formation result in Theorem 2 allows arbitrarily small initial gradients at the order of $O(E_0^{1/3})$ for any sufficiently small E_0, Theorem 2 and Theorem 3 are compatible, or more precisely, they characterize different sets of initial data. To see this, we recall Gagliardo-Nirenberg interpolation inequality to have

$$\|\partial_\xi u_0\|_{L^\infty(\mathbb{R})} \leq C\|u_0\|_{H^2(\mathbb{R})} \leq C\|u_0\|_{H^k(\mathbb{R})}^{\frac{2}{k}} \|u_0\|_{L^2(\mathbb{R})}^{1-\frac{2}{k}} \leq C\,\delta^{\frac{2}{k}}\, E_0^{\frac{1}{2}-\frac{1}{k}}\,.$$

For any initial data satisfying the assumptions in Theorem 3 with $k \geq 7$ so that $\omega_0 \equiv 1 = \omega_0^\sharp$ and small $E_0 < \delta$, one has

$$\|\partial_\xi u_0\|_{L^\infty(\mathbb{R})} \leq C\delta^{\frac{2}{k}} E_0^{5/14}.$$

Applying similar argument to $1 - \frac{1}{h_0} = \omega_0 - \frac{1}{h_0} = \partial_\xi v_0$ shows that

$$\|h_0 - 1\|_{L^\infty(\mathbb{R})} + \|\partial_\xi h_0\|_{L^\infty(\mathbb{R})} \leq C\delta^{\frac{2}{k}} E_0^{5/14},$$

which is much smaller than $O(E_0^{1/3})$. By the lower bound in (20), it is impossible for such initial data to also satisfy the assumption (18) of Theorem 2 as long as we choose δ in Theorem 3 to be sufficiently small.

Remark 9. Global existence of classical solutions for the rotating shallow water system is fundamentally different from the non-rotating, compressible Euler system [18]. Rotating force plays an important role in the well-posedness theory of classical solutions to PDEs modeling compressible flows.

Remark 10. The methods on both singularity formation and global existence in this paper also work in the similar fashion for the one dimensional Euler-Poisson system with a nonzero background charge for hydrodynamical model in semiconductor devices and plasmas. We would also like to mention the recent work [14] where the global existence of classical solutions for the Euler-Poisson system with small initial data was proved via a different method.

The rest of the paper is organized as follows. In Section 2, we introduce the Riemann invariants and weighted gradients of Riemann invariants, and give some basic estimates for these quantities. In Section 3, we prove the finite time formation of singularity via investigating the weighted gradients of Riemann invariants and utilizing conservation of physical energy. In Section 4, we reformulate the Lagrangian rotating shallow water system subject to constant potential vorticity into a one dimensional Klein-Gordon equation which is then shown to satisfy the null conditions in [9]. The results in [9] help establish the global existence of small classical solutions. We also provide an appendix where we prove two elementary lemmas used in Section 3.

2. Riemann Invariants and Their Basic Estimates

2.1. **Riemann invariants and rough upper bounds.** The system (6) is a 3×3 system of balance laws. One way to diagonalize the system of balance laws is to write the system in terms of the Riemann invariants. However, a generic 3×3 system usually does not have 3 full Riemann invariant coordinates [8]. Fortunately, the particular system (6) has 3 full Riemann invariant coordinates R_i $(i = 1, 2, 3)$, so that it can be recast into a "diagonalized" form,

$$\begin{cases} \partial_t R_1 - h^{\frac{\gamma+1}{2}} \partial_\xi R_1 - R_3 = 0\,, \\ \partial_t R_2 + h^{\frac{\gamma+1}{2}} \partial_\xi R_2 - R_3 = 0\,, \\ \partial_t R_3 + \dfrac{R_1 + R_2}{2} = 0\,, \end{cases} \tag{21}$$

where, by borrowing notations from the so-called p-system, we let $p(\frac{1}{h}) = \frac{h^\gamma}{\gamma}$, i.e., $p(s) := \frac{s^{-\gamma}}{\gamma}$ and define Riemann invariants as

$$\begin{cases} R_1 := u + \displaystyle\int_1^{\frac{1}{h}} \sqrt{-p'(s)}\, ds = u - \mathcal{K}(h), \\ R_2 := u - \displaystyle\int_1^{\frac{1}{h}} \sqrt{-p'(s)}\, ds = u + \mathcal{K}(h), \\ R_3 := v, \end{cases} \tag{22}$$

with, apparently,

$$\mathcal{K}(h) := \int_1^h s^{\frac{\gamma-3}{2}}\, ds\,. \tag{23}$$

Note that h can be expressed in terms of the Riemann invariants as

$$h = \vartheta\Big(\frac{R_2 - R_1}{2}\Big) \qquad \text{with} \qquad \vartheta(z) = \mathcal{K}^{-1}(z) = \begin{cases} \Big(\frac{\gamma-1}{2}\, z + 1\Big)^{\frac{2}{\gamma-1}}, & \gamma > 1, \\ e^z, & \gamma = 1. \end{cases} \tag{24}$$

Based on the Riemann invariants formulation above, we have the following estimates related to the L^∞ bounds of the solutions which then lead to an important upper bound for h and consequently the finite speed of propagation.

Lemma 4. *Fix $T > 0$. Let $(h, u, v) \in C^1([0, T] \times \mathbb{R})$ with $h > 0$ solve the system* (6) *and equivalently* (21)*. Suppose*

$$Z_0^\sharp,\ \omega_0^\sharp,\ \inf_\xi h_0 \text{ and } \sup_\xi \{h_0, |u_0|, |v_0|\} \quad \text{are all finite and positive} \tag{25}$$

and

$$M_0 := \sup_{\xi \in \mathbb{R}} \{|R_1(0,\xi)|, |R_2(0,\xi)|, |R_3(0,\xi)|\} < \infty.$$

Then, at any $t \in [0, T]$, we have

$$\sup_{\xi\in\mathbb{R}}\{|R_1(t,\xi)|, |R_2(t,\xi)|, |R_3(t,\xi)|\} \le M_0 e^t \tag{26}$$

and

$$\sup_{\xi\in\mathbb{R}} h(t,\xi) \le \theta^\sharp(t) := \vartheta(M_0 e^t) = \begin{cases} \left(\frac{\gamma-1}{2} M_0 e^t + 1\right)^{\frac{2}{\gamma-1}}, & \gamma > 1, \\ e^{(M_0 e^t)}, & \gamma = 1. \end{cases} \tag{27}$$

Proof. Obviously, the estimate (27) is a consequence of the representation (24) and the estimate (26). So we need only to prove (26). Then, it suffices to show that, for any $\varepsilon > 0$, $N > 0$, we have

$$\max_{(t,\xi)\in A_{N,T,\varepsilon}} \max_{1\le i\le 3} |e^{-t} R_i|(t,\xi) < M_0 + \varepsilon, \tag{28}$$

where $A_{N,T,\varepsilon}$ (see Fig. 1) is the trapezoid

$$A_{N,T,\varepsilon} := \left\{(t,\xi) \in [0,T]\times\mathbb{R} \,\middle|\, |\xi| \le N + (T-t)\big[\vartheta(e^T(M_0+2\varepsilon))\big]^{\frac{\gamma+1}{2}}\right\}. \tag{29}$$

Suppose that the estimate (28) is not true. By the compactness of $A_{N,T,\varepsilon}$, there must exist an earliest time t' so that

$$\max_{\substack{(t,\xi)\in A_{N,T,\varepsilon}\\ t\in[0,t']}} \max_{1\le i\le 3} |e^{-t} R_i|(t,\xi) = M_0 + \varepsilon. \tag{30}$$

Note the definition of M_0 implies that $t' > 0$. The speeds of the characteristics of (21) are $-h^{\frac{\gamma+1}{2}}$, 0, and $h^{\frac{\gamma+1}{2}}$, respectively. It follows from (24) and (30) that one has

$$h^{\frac{\gamma+1}{2}} < \big[\vartheta(e^T(M_0+2\varepsilon))\big]^{\frac{\gamma+1}{2}} \quad \text{for } (t,\xi)\in A_{N,T,\varepsilon} \text{and } t\in[0,t'].$$

Thus, the above estimate together with the definition of $A_{N,T,\varepsilon}$ in (29) guarantees that all the characteristics of (21) emitting from $(t',\xi') \in A_{N,T,\varepsilon}$ and going backward in time always stay within $A_{N,T,\varepsilon}$. Now, introduce

$$m^\sharp(t) := \max_{(t,\xi)\in A_{N,T,\varepsilon}} \max_{1\le i\le 3} R_i(t,\xi) \quad \text{and} \quad m^\flat(t) := \min_{(t,\xi)\in A_{N,T,\varepsilon}} \min_{1\le i\le 3} R_i(t,\xi)\,.$$

Then, for any $t \in (0, t']$, upon integrating each equation of (21) along the associated characteristic from 0 to t, we have

$$m^\sharp(t) \le m^\sharp(0) + \int_0^t \max\{m^\sharp(s),\, -m^\flat(s)\}\, ds$$

and

$$m^\flat(t) \ge m^\flat(0) + \int_0^t \min\{-m^\sharp(s),\, m^\flat(s)\}\, ds\,.$$

Combine these two inequalities to have

$$\max\{m^\sharp(t),\, -m^\flat(t)\} \le \max\{m^\sharp(0),\, -m^\flat(0)\} + \int_0^t \max\{m^\sharp(s),\, -m^\flat(s)\}\, ds\,.$$

Therefore, $\max\{m^\sharp(t),\,-m^\flat(t)\}$ satisfies a Gronwall's inequality which leads to

$$\max\{m^\sharp(t),\,-m^\flat(t)\}\le e^t\max\{m^\sharp(0),\,-m^\flat(0)\}\le e^t M_0\quad\text{for all}\quad t\in(0,t']\,.$$

This contradicts (30). Hence the lemma is proved. $\square$

The upper bound of h in (27) allows us to define the following trapezoidal regions, similar to (29), in the spirit of domain of dependence and domain of influence.

$$\Omega^{\mathrm{bw}}_{N,T}:=\Big\{(t,\xi)\in[0,T]\times\mathbb{R}\,\Big|\,|\xi|\le N+(T-t)\big[\vartheta(e^T(M_0+1))\big]^{\frac{\gamma+1}{2}}\Big\},\tag{31}$$

$$\Omega^{\mathrm{fw}}_{N,T}:=\Big\{(t,\xi)\in[0,T]\times\mathbb{R}\,\Big|\,|\xi|\le N+t\big[\vartheta(e^T(M_0+1))\big]^{\frac{\gamma+1}{2}}\Big\}.\tag{32}$$

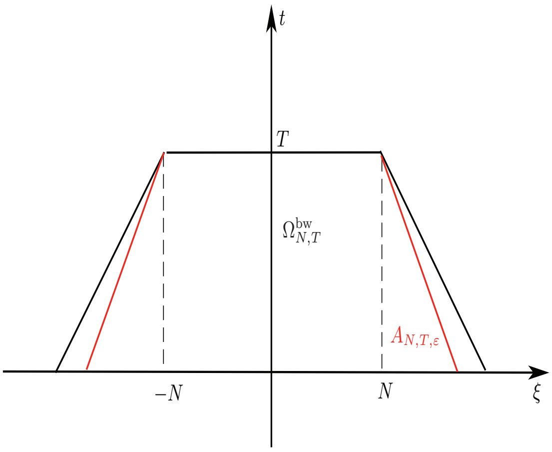

Fig. 1 $\quad\Omega^{\mathrm{bw}}_{N,T}$ and $A_{N,T,\varepsilon}$

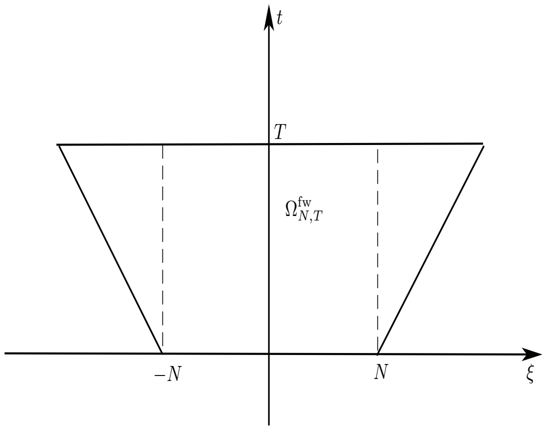

Fig. 2 $\quad\Omega^{\mathrm{fw}}_{N,T}$

Under the assumptions of Lemma 4, we have that characteristics with speed $\pm h^{\frac{\gamma+1}{2}}$ or 0 emitting from a point within $\Omega^{\mathrm{bw}}_{N,T}$ (resp. $\Omega^{\mathrm{fw}}_{N,T}$) and going backward (resp. forward) in time always stay within $\Omega^{\mathrm{bw}}_{N,T}$ (resp. $\Omega^{\mathrm{fw}}_{N,T}$) till $t=0$ (resp. $t=T$).

Note that it is crucial that we shall also prove the lower bound of h to be strictly above 0, which will be dealt with later.

2.2. **Dynamics of gradients of Riemann invariants.** Here, we follow the original idea of Lax ([21]) to study the dynamics of gradients of Riemann invariants and will further reformulate the system inspired by the method in [33] by Tadmor and Wei. Note that despite the similarity of our equations with those of [33], their ODEs [33, (3.12)] for weighted gradients of the Riemann invariants do not have a term that corrsponds to $1/h$ term in our ODEs (38). This is in fact one of the main technical difficulties we have to tackle here.

First, differentiating the first two equations of (21) with respect to ξ gives

$$\begin{cases} D_1(\partial_\xi R_1)-\partial_\xi(h^{\frac{\gamma+1}{2}})(\partial_\xi R_1)=\partial_\xi R_3,\\ D_2(\partial_\xi R_2)+\partial_\xi(h^{\frac{\gamma+1}{2}})(\partial_x R_2)=\partial_\xi R_3,\end{cases}\tag{33}$$

where

$$D_1 := \partial_t - h^{\frac{\gamma+1}{2}}\partial_\xi \quad \text{and} \quad D_2 := \partial_t + h^{\frac{\gamma+1}{2}}\partial_\xi.$$

It follows from (23) and (24) that one has

$$\partial_\xi h = \frac{1}{\mathcal{K}'(h)}\,\frac{\partial_\xi R_2 - \partial_\xi R_1}{2} = h^{-\frac{\gamma-3}{2}}\,\frac{\partial_\xi R_2 - \partial_\xi R_1}{2},$$

so

$$\partial_\xi(h^{\frac{\gamma+1}{2}}) = \frac{\gamma+1}{2}h^{\frac{\gamma-1}{2}}\,\partial_\xi h = \frac{\gamma+1}{2}\,h\,\frac{\partial_\xi R_2 - \partial_\xi R_1}{2}.$$

Combine this with potential vorticity conservation (9) and transform (33) into

$$\begin{cases} D_1(\partial_\xi R_1) = \dfrac{\gamma+1}{4}h\,(\partial_\xi R_2 - \partial_\xi R_1)(\partial_\xi R_1) + \omega_0 - \dfrac{1}{h}, \\ D_2(\partial_\xi R_2) = \dfrac{\gamma+1}{4}h\,(\partial_\xi R_1 - \partial_\xi R_2)(\partial_\xi R_2) + \omega_0 - \dfrac{1}{h}. \end{cases} \tag{34}$$

We also use $\partial_\xi R_1, \partial_\xi R_2$ and the first equation in (6) to rewrite the dynamics of h as

$$\partial_t h + \frac{h^2(\partial_\xi R_2 + \partial_\xi R_1)}{2} = 0. \tag{35}$$

It follows from the definition of R_1 and R_2 that one has

$$\frac{h^2(\partial_\xi R_2 - \partial_\xi R_1)}{2} = h^2\partial_\xi\mathcal{K}(h) = h^{\frac{\gamma+1}{2}}\partial_\xi h. \tag{36}$$

Substituting (36) into (35) gives

$$D_1 h = -h^2\partial_\xi R_2 \qquad \text{and equivalently} \qquad D_2 h = -h^2\partial_\xi R_1\,. \tag{37}$$

Recall the definitions of Riemann invariants in (22) and of weighted gradients of Riemann invariants Z_j $(j = 1, 2)$ in (10) to rewrite

$$Z_1 = \sqrt{h}\,\partial_\xi R_1 \quad \text{and} \quad Z_2 = \sqrt{h}\,\partial_\xi R_2\,.$$

Then, combine (34) and (37) to derive dynamics of Z_j along the characteristics,

$$D_j Z_j = \sqrt{h}\left[-(\overline{\gamma} + \tfrac{1}{2})Z_j^2 + \overline{\gamma}Z_1 Z_2 + \omega_0(\xi) - \frac{1}{h}\right], \qquad j = 1, 2, \tag{38}$$

where $\overline{\gamma} = \frac{\gamma-1}{4} \geq 0$. Furthermore, it follows from (35) that we have

$$\partial_t h = -\frac{1}{2}\,h^{3/2}\,(Z_1 + Z_2), \;\text{ i.e. }\; \partial_t\frac{1}{\sqrt{h}} = \frac{1}{4}(Z_1 + Z_2). \tag{39}$$

2.3. **Upper bound for weighted gradients of Riemann invariants.** The following lemma uses the above formulation in terms of the weighted gradients of Riemann invariants to show an upper bound of Z_j and consequently a positive lower bound of h.

Lemma 5. *Fix $T > 0$. Under the same assumptions and notations as in Lemma 4, we have*

$$Z^\sharp(t) \le W_0^\sharp := \max\left\{Z_0^\sharp,\ \sqrt{2\omega_0^\sharp}\right\} \quad \text{for any } t \in [0, T] \tag{40}$$

and

$$h(t,\xi) \ge \left(\frac{1}{\sqrt{\inf_\xi h_0}} + \frac{t}{2}W_0^\sharp\right)^{-2} \quad \text{for any } t \in [0,T] \text{ and } \xi \in \mathbb{R}. \tag{41}$$

Proof. Consider any large but compact region $\Omega^{\mathrm{bw}}_{N,T}$ as defined in (31). It is a domain of dependence for its every time slice. Then it follows from (39) and $\inf_\xi h_0 > 0$ that h is always positive in $\Omega^{\mathrm{bw}}_{N,T}$. Now, it suffices to show

$$\max_{\Omega^{\mathrm{bw}}_{N,T}}\{Z_1, Z_2\} \le W_0^\sharp\,, \tag{42}$$

and

$$\max_{\Omega^{\mathrm{bw}}_{N,T}} \frac{1}{\sqrt{h}} \le \frac{1}{\sqrt{\inf_\xi h_0}} + \frac{t}{2}W_0^\sharp\,. \tag{43}$$

Noting that $Z_0^\sharp > 0$, we assume without loss of generality that at some $(t', \xi') \in \Omega^{\mathrm{bw}}_{N,T}$,

$$Z_1(t',\xi') = \max_{\Omega^{\mathrm{bw}}_{N,T}}\{Z_1, Z_2\} > 0. \tag{44}$$

If the maximum in (44) is attained at $t' = 0$, then the estimate (42) is apparently true. Otherwise, the maximum in (44) is achieved for $t' > 0$. Therefore, one has $D_1 Z_1(t',\xi') \ge 0$. This, together with (38), yields

$$\sqrt{h}\left(-(\overline{\gamma} + \tfrac{1}{2})Z_1^2 + \overline{\gamma} Z_1 Z_2 + \omega_0(\xi) - \frac{1}{h}\right) \ge 0 \quad \text{at } (t', \xi').$$

Since $Z_1(t',\xi') \ge Z_2(t',\xi')$ and $Z_1(t',\xi') > 0$, we have

$$-\frac{1}{2}Z_1^2 + \omega_0^\sharp \ge \frac{1}{h} > 0 \quad \text{at } (t',\xi').$$

This proves the estimate (42). The estimate (43) is a direct consequence of equation (39) and the estimate (42). $\square$

Although the results proved so far are regarding closed time interval $[0, T]$, this is not so essential. In fact, for any small $\varepsilon > 0$, we can replace every T by $T - \varepsilon$ in Lemmas 4 and 5 and still obtain the corresponding estimates. Since these estimates are regardless of ε, we can let ε approach zero and establish that all estimates in Lemmas 4 and 5 are still valid if we replace every occurrence of $[0, T]$ by $[0, T)$ in their conditions and conclusions. Also, these two lemmas together with (9) imply $|\partial_\xi v|$ is bounded from above. Therefore, we obtain the following corollary that characterizes the type of possible singularities that a solution may develop.

Corollary 6. *Given the same type of initial conditions as in Lemma 4, suppose a C^1 solution exists over time interval $[0,T]$ (resp. $[0,T)$). Then, for all $t \in [0,T]$ (resp. $t \in [0,T)$) and all $\xi \in \mathbb{R}$, we have $(h, |u|, |v|, |\partial_\xi v|)$ as well as Z_1 and Z_2 to be uniformly bounded from above, and have h to be uniformly bounded from below. These bounds are positive and only depend on γ, M_0, $\inf_\xi h_0$, $W_0^\sharp$, and T.*

Furthermore, if a classical solution indeed loses C^1 regularity at a finite time $t = T^\sharp$, then $\inf_\xi\{Z_1, Z_2\} \to -\infty$ as $t \uparrow T^\sharp$ while Z_1 and Z_2 remain bounded from above.

3. Formation of Singularities

Let us recollect the bounds obtained so far. They are upper bound (27) and positive lower bound (41) for h; and upper bound (40) for Z_1 and Z_2. Since the solution itself is always bounded as proved in Lemma 4, the only possible singularity for a classical solution is for Z_1 or Z_2 approaching $-\infty$.

In this section, we use the comparison principle to prove that if $\inf_\xi\{Z_1, Z_2\}$ is equal to or below a threshold $-\sqrt{2\,\omega_0^\sharp}$ at some time, then it will approach $-\infty$ at some late finite time. Next, we impose this $-\sqrt{2\,\omega_0^\sharp}$ threshold as an additional lower bound on Z_1, Z_2 and prove a singularity formation with initial data which can have arbitrarily small gradients. A key and novel technique is to combine the lower and upper bounds of Z_1, Z_2 and the conservation of *physical energy* to control the positive terms in the equations for $D_1 Z_1$, $D_2 Z_2$ so that the decay of $\inf_\xi\{Z_1, Z_2\}$ is sufficient for it to reach the $-\sqrt{2\,\omega_0^\sharp}$ threshold that has been just proved. This then eventually leads to the loss of C^1 regularity in finite time.

3.1. **Comparison principle.**

First, we prove the following important comparison principle for $Z^\flat(t)$.

Lemma 7 (Strict comparison principle)**.** *Fix $T > 0$. Consider a classical solution $(h, u, v) \in C^1([0,T]\times\mathbb{R})$ to the rotating shallow water system* (6) *with C_0^1 initial data $(h_0 - 1, u_0, v_0)$ so that $\inf_\xi h_0 > 0$. Let a function $m(t) \in C^1([0,T])$ satisfy the following strict differential inequality and initial condition*

$$
\begin{cases}
\sup\limits_{\xi\in\mathbb{R}} \left\{ \sqrt{h(t,\xi)}\Big(-\dfrac{1}{2}m^2(t) + \omega_0^\sharp - \dfrac{1}{h(t,\xi)}\Big) \right\} < \dfrac{d}{dt}m(t) < 0\,, \\
Z^\flat(0) \le m(0) < 0\,.
\end{cases}
\tag{45}
$$

Then,

$$
Z^\flat(t) < m(t) \quad \textit{for any} \quad t \in (0,T]\,. \tag{46}
$$

We recall that the upper and positive lower bounds of h have been established, so the left hand side of (45) is always well-defined.

Proof. With compactly supported initial data $(h_0 - 1, u_0, v_0)$, by the bounds of h, which leads to the finite propagation speed of the solutions, we have that $Z_j(t, \cdot)$ ($j = 1, 2$) is also compactly supported, so $Z^\flat(t)$ is a well-defined continuous function as long as the C^1 solution exists.

Since $Z^\flat(0) \leq m(0)$ and the initial data has compact support, without loss of generality, there exists a $\xi' \in \mathbb{R}$ such that $Z_1(0, \xi') \leq m(0) < 0$ and $Z_1(0, \xi') \leq Z_2(0, \xi')$. Hence one has

$$\begin{aligned}
&\sqrt{h}\left(-\frac{1}{2}Z_1^2 + \bar{\gamma} Z_1(Z_2 - Z_1) + \omega_0^\sharp - \frac{1}{h}\right)(0, \xi') \\
\leq &\sqrt{h}\left(-\frac{1}{2}Z_1^2 + \omega_0^\sharp - \frac{1}{h}\right)(0, \xi') \\
\leq &\sqrt{h}(0, \xi')\left(-\frac{1}{2}m^2(0) + \omega_0^\sharp - \frac{1}{h}(0, \xi')\right).
\end{aligned}$$

This, together with the equations (38) and (45), implies $D_1 Z_1(0, \xi') < \frac{d}{dt}m(0)$ where the characteristic curve associated with $D_1 Z_1$ emits from $(0, \xi')$. Together with $Z_1(0, \xi') \leq m(0)$, this implies $Z^\flat(t) < m(t)$ for at least a finite interval of positive times. In other words, there exists a positive time $T_0 \leq T$ so that

$$Z^\flat(t) < m(t) \quad \text{for all} \quad t \in (0, T_0]\,. \tag{47}$$

If $T_0 = T$, then the Lemma is proven. So we only consider $T_0 < T$.

First, Lemmas 4 and 5 ensure the upper and positive lower bound of h over $[0, T] \times \mathbb{R}$; combine this with finite speed of propagation and the assumed compact support of $(h_0 - 1, u_0, v_0)$ to have $(h - 1, u, v)$ supported inside $[0, T] \times [-L, L]$ for a sufficiently large L. Thus, the function

$$\psi(t, \xi) := \sqrt{h(t, \xi)}\Big(-\frac{1}{2}m^2(t) + \omega_0^\sharp - \frac{1}{h(t, \xi)}\Big) \tag{48}$$

depends only on $m(t)$ for $(t, \xi) \in [0, T] \times ((-\infty, -L] \cup [L, +\infty))$, which implies that at each fixed time t, the supremum of $\psi(t, \xi)$ must be attainable inside the closed box $[0, T] \times [-L, L]$. It is then a straightforward analysis exercise to show that $\sup\limits_{\xi \in \mathbb{R}} \psi(t, \xi)$ is a continuous function of t over $[0, T]$. Combining this with (45) and the assumed continuity of $\frac{d}{dt}m(t)$ over the closed interval $[0, T]$ implies there exists a fixed $\varepsilon > 0$ so that

$$\varepsilon + \sup_{\xi \in \mathbb{R}} \psi(t, \xi) < \frac{dm}{dt}(t)\,, \quad \text{for all} \ \ t \in [0, T]\,. \tag{49}$$

Secondly, at time $T_0 < T$, by (47) and the fact that the solution is compactly supported, we assume without loss of generality that

$$Z_1(T_0, \xi_0) = Z^\flat(T_0) < m(T_0) < 0 \qquad \text{for some} \quad \xi_0 \in \mathbb{R}\,, \tag{50}$$

where the last inequality is due to the assumption that both $m(0)$ and $m'(t)$ are negative. This implies

$$\big\{-(\overline{\gamma}+\tfrac{1}{2})Z_1^2+\overline{\gamma}Z_1Z_2\big\}\big|_{(T_0,\xi_0)}\leq -\frac{1}{2}m^2(T_0).$$

Combine it with (49) to obtain

$$\varepsilon+\left\{\sqrt{h}\Big(-(\overline{\gamma}+\tfrac{1}{2})Z_1^2+\overline{\gamma}Z_1Z_2+\omega_0^\sharp-\frac{1}{h}\Big)\right\}\bigg|_{(T_0,\xi_0)}\leq\varepsilon+\psi(T_0,\xi_0)<\frac{d}{dt}m(T_0)\,. \tag{51}$$

Let $\Xi(t)$ be the solution of the following Cauchy problem for ODE

$$\begin{cases}\dfrac{d}{dt}\Xi(t)=-h^{\frac{\gamma+1}{2}}(t,\Xi(t)),\\ \Xi(T_0)=\xi_0.\end{cases} \tag{52}$$

Hence $\Big\{(t,\Xi(t))\,\Big|\,T_0\leq t\leq T\Big\}$ is the characteristic curve associated with $D_1=\partial_t-h^{\frac{\gamma+1}{2}}\partial_\xi$. Now, by the definition of Z_1,Z_2, the positive lower bound of h, the hypothesis that (h,u,v) and $m(t)$ are C^1, and the fact that the curve $(t,\Xi(t))$ is C^1 and contained in a compact region, we must have $m'(t)$, $Z_1(t,\Xi(t))$, $Z_2(t,\Xi(t))$, $\sqrt{h(t,\Xi(t))}$, and $1/h(t,\Xi(t))$ to be uniformly continuous functions of t over $[T_0,T]$. Crucially, the moduli of continuity of all these functions of t are independent of the choices of T_0, ξ_0. Therefore, with the value of ε in (51) fixed, we can find a sufficiently large and fixed integer N, so that with

$$\delta:=\frac{T-T_0}{N}\,,$$

the oscillation of $\sqrt{h}\Big(-(\overline{\gamma}+\frac{1}{2})Z_1^2+\overline{\gamma}Z_1Z_2+\omega_0^\sharp-\frac{1}{h}\Big)$ over $\Big\{(t,\Xi(t))\,\Big|\,T_0\leq t\leq T_0+\delta\Big\}$ is less than $\dfrac{\varepsilon}{2}$ and the oscillation of $m'(t)$ over $[T_0,T_0+\delta]$ is also less than $\dfrac{\varepsilon}{2}$. Then, by (51),

$$\left\{\sqrt{h}\Big(-(\overline{\gamma}+\tfrac{1}{2})Z_1^2+\overline{\gamma}Z_1Z_2+\omega_0^\sharp-\frac{1}{h}\Big)\right\}\bigg|_{(t,\Xi(t))}<\frac{d}{dt}m(t)\qquad\text{for all }t\in[T_0,T_0+\delta]\,.$$

Therefore, by the equation in (38) for Z_1 where $D_1=\partial_t-h^{\frac{\gamma+1}{2}}\partial_\xi$ and by the definition of Ξ in (52), we have

$$\frac{d}{dt}Z_1(t,\Xi(t))<\frac{d}{dt}m(t)\qquad\text{for all }t\in[T_0,T_0+\delta]\,.$$

Integrate this from T_0 to any $t\in[T_0,T_0+\delta]$ to obtain, noting (50),

$$Z_1(t,\Xi(t))-m(t)\leq Z_1(T_0,\xi_0)-m(T_0)<0\qquad\text{for all }t\in[T_0,T_0+\delta]\,,$$

which allows us to extend (47) to

$$Z^\flat(t)<m(t)\quad\text{for all}\quad t\in(0,T_0+\delta]\,,$$

for a fixed $\delta>0$. Repeat this extension N times to prove (46). $\square$

3.2. **Existence of a threshold for formation of singularity.** Now we prove Theorem 1, which shows that the loss of C^1 regularity always takes place in finite time, provided at some time t, $Z^\flat(t)$ is below the time-independent threshold $-\sqrt{2\,\omega_0^\sharp}$.

Proof of Theorem 1. It suffices to consider $T'=0$.

It follows from Lemmas 4 and 5 (the estimates (27) and (41)) that we have

$$\sup_{\xi\in\mathbb{R}} h(t,\xi)\le \theta^\sharp(t) \quad \text{and} \quad \inf_{\xi\in\mathbb{R}} h(t,\xi)\ge \Big(\frac{1}{\sqrt{\inf_\xi h_0}}+\frac{t}{2}W_0^\sharp\Big)^{-2} =: \theta^\flat(t)\,, \tag{53}$$

respectively. Apparently $\theta^\sharp \ge \theta^\flat > 0$. Now, let $m(t)$ be the solution of the following Cauchy problem

$$\frac{d}{dt}m(t)=\sqrt{\theta^\flat(t)}\Big(-\frac{1}{2}m^2(t)+\omega_0^\sharp-\frac{1}{2}\frac{1}{\theta^\sharp(t)}\Big) \tag{54}$$

and

$$m(0)=Z^\flat(0)\le -\sqrt{2\omega_0^\sharp}.$$

It is straightforward to see from above that

$$\frac{d}{dt}m(t)<0 \quad \text{and} \quad m(t)\le -\sqrt{2\omega_0^\sharp} \quad \text{for} \quad t\ge 0\,. \tag{55}$$

This together with (53) further implies that

$$\sqrt{h(t,\xi)}\Big(-\frac{1}{2}m^2(t)+\omega_0^\sharp-\frac{1}{h(t,\xi)}\Big)\le \sqrt{\theta^\flat(t)}\Big(-\frac{1}{2}m^2(t)+\omega_0^\sharp-\frac{1}{2}\frac{1}{\theta^\sharp(t)}\Big)-\frac{\theta^\flat(t)}{2\theta^\sharp(t)}\,,$$

and by (54), its right hand side is less than $\frac{d}{dt}m(t)$. Hence we have

$$\sup_\xi\Big\{\sqrt{h(t,\xi)}\Big(-\frac{1}{2}m^2(t)+\omega_0^\sharp-\frac{1}{h(t,\xi)}\Big)\Big\}<\frac{d}{dt}m(t)\,.$$

Therefore, $m(t)$ satisfies the assumptions of the comparison principle, Lemma 7, as long as it remains finite. Applying Lemma 7 yields

$$Z^\flat(t)<m(t) \qquad \text{as long as } m(t) \text{ remains finite}\,. \tag{56}$$

Next, by (55), for any $T_1>0$ such that $m(T_1)$ is finite, we have

$$m(t)\le m(T_1)<-\sqrt{2\omega_0^\sharp} \qquad \text{for} \quad t\ge T_1\,. \tag{57}$$

It follows from (54) that the following differential inequality holds

$$\frac{d}{dt}m(t)<\sqrt{\theta^\flat(t)}\Big[-\frac{1}{2}m^2(t)+\omega_0^\sharp\Big] \qquad \text{as long as } m(t) \text{ remains finite}\,.$$

Using partial fractions yields

$$\frac{dm}{m-\sqrt{2\omega_0^\sharp}}-\frac{dm}{m+\sqrt{2\omega_0^\sharp}}<-\sqrt{2\omega_0^\sharp\theta^\flat(t)}\,dt.$$

Integrate this inequality from T_1 to $t \geq T_1$ with relevant signs determined by (57),

$$\ln \frac{m(t)-\sqrt{2\omega_0^\sharp}}{m(t)+\sqrt{2\omega_0^\sharp}} - \ln \frac{m(T_1)-\sqrt{2\omega_0^\sharp}}{m(T_1)+\sqrt{2\omega_0^\sharp}} < -\sqrt{2\omega_0^\sharp}\int_{T_1}^t \sqrt{\theta^\flat(s)}\,ds.$$

Combining with the definition of $\theta^\flat$ in (53) gives

$$\ln \frac{m(t)-\sqrt{2\omega_0^\sharp}}{m(t)+\sqrt{2\omega_0^\sharp}} < \frac{2\sqrt{2\omega_0^\sharp}}{W_0^\sharp}\ln\Big(\frac{2}{\sqrt{\inf_\xi h_0}} + W_0^\sharp T_1\Big) - \frac{2\sqrt{2\omega_0^\sharp}}{W_0^\sharp}\ln\Big(\frac{2}{\sqrt{\inf_\xi h_0}} + W_0^\sharp t\Big)$$

$$+\ln \frac{m(T_1)-\sqrt{2\omega_0^\sharp}}{m(T_1)+\sqrt{2\omega_0^\sharp}}, \quad \text{for} \quad t \geq T_1\,.$$

By (57) again, the right side of the above expression evaluates to a positive value at $t = T_1$ and thus the right side as a decreasing function of t will approach 0 from above in a finite time. This implies the decreasing function $m(t)$ approaches $-\infty$ in finite time. Therefore, by (56), $Z^\flat(t)$ approaches $-\infty$ in finite time. Moreover, by Corollary 6, the only type of singularity must satisfy (13)-(14). Hence the proof of the theorem is completed. $\square$

3.3. **General initial data with small gradients.** By (40), we always have an upper bound for Z_j. In order to prove the singularity formation for general initial data, it follows from Theorem 1 that if at some time t, $Z^\flat(t) \leq -\sqrt{2\omega_0^\sharp}$, then the singularity must occur in finite time. Therefore, in order to prove the singularity formation with general initial data, we need only to focus on the case where

$$Z_1, Z_2 \in \Big(-\sqrt{2\omega_0^\sharp},\, \max\{Z_0^\sharp,\, \sqrt{2\omega_0^\sharp}\}\Big]. \tag{58}$$

Note that the condition (58) implies that

$$|Z_2 - Z_1| < G_0,$$

where the gap G_0 is defined in (17).

The nice thing about (58) is that it gives an additional bound. In particular, considering the comparison principle in Lemma 7 and especially the $-\frac{1}{h}$ term in the differential inequality in (45), we need a much sharper upper bound for h than the previously established one. To start, by the definition in (10), the gap condition (58) implies

$$\Big|\partial_\xi h^{\gamma/2}(t,\xi)\Big| = \frac{\gamma}{2}\,\frac{|Z_2 - Z_1|}{2} < \frac{\gamma}{4}\,G_0\,. \tag{59}$$

In order to turn such estimate into an upper bound of h, we utilize the well-known conservation of total physical energy for the rotating shallow water system.

For C^1_0 initial data (h_0-1,u_0,v_0) and strictly positive h, it is straightforward to show that $E(t)$ defined in (15) is invariant with respect to time, i.e.,

$$E(t)\equiv E_0\,.$$

With the potential energy density $\mathcal{Q}$ defined in (16), we have

$$\int_{-\infty}^{\infty}\mathcal{Q}(h(t,\xi))\,d\xi\le E(t)\equiv E_0\,.$$

Combine this with Proposition 10 in Appendix A to have, for fixed t,

$$\int_{-\infty}^{\infty}\big(h^{\gamma/2}(t,\xi)-1\big)^2\,d\xi\le\int_{-\infty}^{\infty}\frac{\mathcal{Q}(h(t,\xi))}{\zeta(\alpha^\sharp(t),\gamma)}\,d\xi\le\frac{E_0}{\zeta(\alpha^\sharp(t),\gamma)}\,, \tag{60}$$

where

$$\alpha^\sharp(t):=\sup_\xi(h^{\gamma/2}(t,\xi)-1) \tag{61}$$

and ζ is the function defined as follows

$$\zeta(\beta,\gamma):=\frac{1}{\gamma^2}\Big\{(\beta+1)^{-\frac{2}{\gamma}}+(\beta+1)^{-\frac{2}{\gamma}-1}\Big\}\,. \tag{62}$$

Next, we estimate h in the following lemma.

Lemma 8. *Fix $T>0$. Consider a classical solution $(h,u,v)\in C^1([0,T]\times\mathbb{R})$ of the rotating shallow water system (6) with C^1_0 initial data (h_0-1,u_0,v_0) so that $\inf_\xi h_0>0$. Assume*

$$Z^\flat(t)>-\sqrt{2\omega_0^\sharp}\quad\textit{for all}\ \ t\in[0,T]\,. \tag{63}$$

Then,

$$\sup_\xi h(t,\xi)<\Big[\mathcal{F}_\gamma^{-1}\big(G_0\,E_0\big)+1\Big]^{\frac{2}{\gamma}}\,,\quad\textit{for all}\ \ t\in[0,T]\,, \tag{64}$$

where $\mathcal{F}_\gamma^{-1}$ is the inverse of function $\mathcal{F}_\gamma$ defined by (19).

Proof. Throughout the proof, we always have uniform boundedness of h,u,v,R_1,R_2 and uniform positive lower bound of h guaranteed by Lemmas 4 and 5.

For fixed $t\in[0,T]$, by assumption (63) on lower bound and Lemma 5 on upper bound of Z_j, the gap condition (58) is satisfied and therefore estimate (59) holds. Combine this with estimate (60), definition in (61) and a simple functional inequality in Proposition 11 of Appendix A to have

$$\begin{aligned}(\alpha^\sharp(t))^3&=\sup_\xi(h^{\gamma/2}(t,\xi)-1)^3\\&\le\frac{3}{4}\,\|h^{\gamma/2}-1\|^2_{L^2}\,\big\|\partial_\xi\big(h^{\gamma/2}\big)\big\|_{L^\infty}\\&<\frac{3}{4}\,\frac{E_0}{\zeta(\alpha^\sharp(t),\gamma)}\,\frac{\gamma}{4}\,G_0\,.\end{aligned}$$

Hence

$$\mathcal{F}_\gamma(\alpha^\sharp)=\frac{16}{3\gamma}\,(\alpha^\sharp)^3\,\zeta(\alpha^\sharp,\gamma)<G_0E_0.$$

Therefore, by the monotonicity of $\mathcal{F}_\gamma$, we prove

$$\alpha^\sharp(t)= \sup_\xi (h^{\gamma/2}(t,\xi)-1) < \mathcal{F}_\gamma^{-1}(G_0\,E_0)\,, \quad \text{for all } \; t\in[0,T]\,,$$

which apparently implies (64). $\square$

We are ready to state and prove the main theorem of finite time singularity formation for the solutions with arbitrarily small initial gradients.

Theorem 9. *Under the same assumptions and notations as Lemma 8, if*

$$-\sqrt{2\omega_0^\sharp} < Z^\flat(0) < -\sqrt{2}\sqrt{\omega_0^\sharp - \Big[\mathcal{F}_\gamma^{-1}\big(G_0\,E_0\big)+1\Big]^{-\frac{2}{\gamma}}}, \tag{65}$$

then $Z^\flat(t)$ will reach $-\sqrt{2\omega_0^\sharp}$ at a finite time that is bounded by a continuous function of G_0, E_0, $\omega_0^\sharp$, and $Z^\flat(0)$.

Proof. Define

$$h_0^* := \Big[\mathcal{F}_\gamma^{-1}\big(G_0\,E_0\big)+1\Big]^{\frac{2}{\gamma}}.$$

Then, we choose $m(t)$ to be the solution of the following initial value problem for the ODE

$$\begin{cases} \dfrac{d}{dt}m(t) = \sqrt{h_0^*}\Big(-\dfrac{1}{2}m^2(t)+\omega_0^\sharp\Big) - \dfrac{1}{\sqrt{h_0^*}}\,, \\ m(0) = Z^\flat(0). \end{cases}$$

Since by (65), we have $-\sqrt{2\omega_0^\sharp} < m(0) < -\sqrt{2}\sqrt{\omega_0^\sharp - \dfrac{1}{h_0^*}}$, a straightforward calculation shows that $m(t)$ is strictly decreasing and negative, and it must reach the threshold $-\sqrt{2\omega_0^\sharp}$ at a finite time $T>0$ which continuously depends on G_0, E_0, $\omega_0^\sharp$, and $Z^\flat(0)$, namely

$$m(T) = -\sqrt{2\omega_0^\sharp} \quad \text{and} \quad m(t)\le m(T) \quad \text{for} \quad t\in[0,T]\,. \tag{66}$$

Case 1. If $Z^\flat(t) = -\sqrt{2\omega_0^\sharp}$ at some $t\in[0,T]$, then we finish the proof of the current theorem.

Case 2. Otherwise, by the first half of (65) and continuity of Z_j, we must have

$$Z^\flat(t) > -\sqrt{2\omega_0^\sharp}\,, \quad \text{for all } \; t\in[0,T]\,, \tag{67}$$

which makes the assumption (63) of Lemma 8 satisfied. Therefore, by the estimate (64) of Lemma 8 and definition of h_0^*, we have $\sup_\xi h(t,\xi) < h_0^*$ for $t\in[0,T]$. Combined with (66) so that $-\dfrac{1}{2}m^2(t)+\omega_0^\sharp \ge 0$ on $[0,T]$, this leads to

$$\sup_{\xi\in\mathbb{R}}\left\{\sqrt{h(t,\xi)}\Big(-\frac{1}{2}m^2(t)+\omega_0^\sharp - \frac{1}{h(t,\xi)}\Big)\right\} < \sqrt{h_0^*}\Big(-\frac{1}{2}m^2(t)+\omega_0^\sharp\Big) - \frac{1}{\sqrt{h_0^*}} = \frac{d}{dt}m(t)\,.$$

This implies that $m(t)$ satisfies the assumptions of the comparison principle, Lemma 7 and therefore

$$\inf_{\xi} \{Z_1, Z_2\} < m(T) = -\sqrt{2\omega_0^{\sharp}},$$

where the latter equality is due to (66). This contradicts (67). We have just shown that Case 2 is impossible. The proof of the theorem is completed. □

It is easy to see that Theorem 2 is a direct consequence of Theorems 1 and 9.

4. Klein-Gordon Equation and Global Existence

In this section, we prove Theorem 3. Differentiate the third equation in (6) with respect to t, and combine it with the second equation in (6) and (9) to obtain

$$\partial_{tt} v - \partial_\xi \left(\frac{1}{\gamma(\omega_0(\xi) - \partial_\xi v)^\gamma} \right) + v = 0. \tag{68}$$

This is a typical quasilinear Klein-Gordon equation. The well-posedness for the Cauchy problem (6) and (7) is equivalent to the well-posedness for the Cauchy problem for the Klein-Gordon equation (68).

For a general function $\omega_0(\xi)$, the linear part of the equation (68) is a Klein-Gordon operator with variable coefficients. There are very limited results for this type of equations because of lack of understanding for the associated linear operator. If $\omega_0(\xi)$ is a constant and, without loss of generality, we assume that $\omega_0(\xi) \equiv 1$, then the equation (68) can be written as

$$\partial_{tt} v - \partial_{\xi\xi} v + v = \left((\gamma+1)\partial_\xi v + \frac{(\gamma+1)(\gamma+2)}{2} (\partial_\xi v)^2 \right) \partial_{\xi\xi} v + R_4, \tag{69}$$

where

$$R_4 = \partial_{\xi\xi} v \left(\frac{1}{(1-\partial_\xi v)^{(\gamma+1)}} - 1 - (\gamma+1)\partial_\xi v - \frac{(\gamma+1)(\gamma+2)}{2} (\partial_\xi v)^2 \right) \tag{70}$$

satisfies

$$|R_4| \le C |\partial_\xi v|^3 |\partial_{\xi\xi} v| \quad \text{for} \quad |\partial_\xi v| \le \frac{1}{2}. \tag{71}$$

Hence R_4 can be regarded as quartic terms and higher order nonlinear terms.

The equation (69) is an example of quasilinear Klein-Gordon equation with constant linear coefficients and quadratic nonlinearity. It has attracted much attention in analysis since the 1980s. There are two key factors in proving global existence of small classical solutions to quasilinear Klein-Gordon equation: one is the spatial dimension n since the solutions to the linear Klein-Gordon equation have dispersive decay rate $(1+t)^{-n/2}$, and the other is the lowest degree of nonlinearity and sometimes also the type of nonlinearity. For brevity, we recall only results for general nonlinearities of degree as low as two, unless otherwise noted. When the spatial dimension is at least four, the global existence of small classical solutions was proved in [19]. The breakthrough for study on three dimensional quasilinear Klein-Gordon equation was made by Klainerman [20] and Shatah [32] independently by using the vector field approach and the normal form method, respectively. The results for two dimensional semilinear Klein-Gordon equation were established in [26, 27] by combining the vector field approach and the normal form method together.

Note that the equation (69) is a quasilinear Klein-Gordon equation with quadratic nonlinearity in one spatial dimension. Since the $(1+t)^{-1/2}$ dispersive decay rate for one dimensional linear Klein-Gordon equation is fast enough to easily control the quartic and higher order nonlinearity, the R_4 term in (69) would cause little trouble in the analysis. But such decay rate is too slow for one to accomplish a direct analysis on general quadratic or cubic terms. Actually, it is conjectured (see for instance [9]) that for Klein-Gordon equation with general quadratic and cubic nonlinear terms, the classical solutions with small and smooth initial data might develop finite time singularities. Then, some null conditions on the structure of main parts of the nonlinear terms, which for (69) are the quadratic and cubic terms, are required for the global existence of classical solutions with small initial data. In [9], Delort introduced one such null condition and then obtained the global existence result subject to such null condition by performing delicate analysis with the tools of normal form and vector field, and with the hyperbolic coordinate transformation (see also [10]).

Now, we shall check that the null condition raised in [9] is in fact satisfied by (69). We first recap the definition of null condition following the steps of [9, (1.2)-(1.9) in pages 6,7], which only invloves the quadratic and cubic nonlinear terms

$$F = Q(v, \partial_t\partial_x v, \partial_x^2 v; \partial_t v, \partial_x v) + P(v, \partial_t\partial_x v, \partial_x^2 v; \partial_t v, \partial_x v)$$

where Q is a quadratic homogeneous polynomial and P a cubic homogeneous polynomial, and the combined power of $(\partial_t\partial_x v, \partial_x^2 v)$ in each monomial term is at most 1. Decompose Q

$$Q(v, \partial_t\partial_x v, \partial_x^2 v; \partial_t v, \partial_x v) = Q'(v, \partial_t v, \partial_x v) + Q''(v, \partial_t\partial_x v, \partial_x^2 v; \partial_t v, \partial_x v)\,,$$

where Q'' is linear in $(\partial_t\partial_x v, \partial_x^2 v)$. Apparently, the notation is such that a single prime $'$ (resp. double primes $''$) indicates the highest order of derivatives is less than (resp. equal to) 2. Then, in [9, (1.4)], further decomposition is introduced as follows with the lower-index k indicating the power of $(\partial_t v, \partial_x v)$

$$\begin{aligned} Q'(v, \partial_t v, \partial_x v) &= Q_0'(v) + iQ_1'(v; -i\partial_t v, -i\partial_x v) - Q_2'(-i\partial_t v, -i\partial_x v), \\ Q''(v, \partial_t\partial_x v, \partial_x^2 v; \partial_t v, \partial_x v) &= Q_0''(v, -\partial_t\partial_x v, -\partial_x^2 v) + iQ_1''(-\partial_t\partial_x v, -\partial_x^2 v; -i\partial_t v, -i\partial_x v), \end{aligned}$$

where $i = \sqrt{-1}$, Q_0' is homogenous of degree 2 in v, Q_1' is linear in v and in $(\partial_t v, \partial_x v)$, Q_2' is homogenous of degree 2, Q_0'' is linear in v and in $(\partial_t\partial_x v, \partial_x^2 v)$, Q_1'' is linear in $(\partial_t\partial_x v, \partial_x^2 v)$ and in $(\partial_t v, \partial_x v)$.

Similarly, in [9, (1.5)-(1.6)], following the same notational conventions, the cubic nonlinearity is decomposed as

$$P(v,\partial_t\partial_x v,\partial_x^2 v;\partial_t v,\partial_x v)=P'(v,\partial_t v,\partial_x v)+P''(v,\partial_t\partial_x v,\partial_x^2 v;\partial_t v,\partial_x v),$$

where P' and P'' are homogeneous of degree 3, and P'' is linear in $(\partial_t\partial_x v,\partial_x^2 v)$. Furthermore,

$$\begin{aligned}P'(v,\partial_t v,\partial_x v)&=\sum_{k=0}^{3} i^k P'_k(v;-i\partial_t v,-i\partial_x v),\\ P''(v,\partial_t\partial_x v,\partial_x^2 v;\partial_t v,\partial_x v)&=\sum_{k=0}^{2} i^k P''_k(v,-\partial_t\partial_x v,-\partial_x^2 v;-i\partial_t v,i\partial_x v),\end{aligned}$$

where the degree of $(\partial_t v,\partial_x v)$ in P'_k and P''_k is homogeneously k.

Next in [9, (1.7)], the functions of two variables, $q'_k(\omega_0,\omega_1), q''_k(\omega_0,\omega_1), p'_k(\omega_0,\omega_1), p''_k(\omega_0,\omega_1)$ are introduced as the result of substituting $(1,-\omega_0\omega_1,-\omega_1^2;i\omega_0,i\omega_1)$ for $(v,\partial_t\partial_x v,\partial_x^2 v;\partial_t v,\partial_x v)$ in Q'_k, Q''_k, P'_k, P''_k respectively. Also denote $p_k = p'_k + p''_k$. Then, set $\omega_0(y) = \frac{1}{\sqrt{1-y^2}}$ and $\omega_1(y) = \frac{-y}{\sqrt{1-y^2}}$ to define

$$\begin{aligned}\Phi(y) =& p_1(\omega_0(y),\omega_1(y)) + 3p_3(\omega_0(y),\omega_1(y))\\ &+ q'_1(q'_0+q'_2)(\omega_0(y),\omega_1(y)) - q''_1(q''_0+2q'_2)(\omega_0(y),\omega_1(y))\,.\end{aligned}\tag{72}$$

The null condition in [9, Definition 1.1 in page 7] is basically

$$\Phi(y)\equiv 0\,.$$

Now, for our purpose of validating this null condition for (69), we replace our notations of ξ derivatives by x derivatives and denote the quadratic and cubic nonlinearities as

$$\begin{aligned}Q(v,\partial_t\partial_x v,\partial_x^2 v;\partial_t v,\partial_x v)&=(\gamma+1)\,\partial_x v\,\partial_x^2 v\,,\\ P(v,\partial_t\partial_x v,\partial_x^2 v;\partial_t v,\partial_x v)&=\frac{(\gamma+1)(\gamma+2)}{2}\,(\partial_x v)^2\,\partial_x^2 v\,.\end{aligned}$$

It is then straightforward to follow the above recipe to show that all single primed functions are identically zero and, except for Q''_1, P''_2, all other Q''_k and P''_k are also zero. This means the only non-zero quantity in the definition (72) of $\Phi(y)$ is q''_1. Therefore, we have the null condition $\Phi(y)\equiv 0$ validated. Hence it follows from [9, Theorem 1.2] and [10, Theorem 1.2] that there exists a global classical solution for the Cauchy problem for (69) as long as the initial data are compactly supported and sufficiently small in a high order Sobolev space. Note that the initial data in theorems of [9, 10] are in terms of $(v,\partial_t v)$, which is a natural choice for the Cauchy problem of the Klein-Gordon equation (69). Therefore, under the smallness assumptions for (u_0,v_0) as that in Theorem 3, which are related to $(v,\partial_t v)$ by the third equation in (6), we prove the global existence of classical solutions for one dimensional rotating shallow water system. Also note that the $|\partial_\xi v|<\frac{1}{2}$ condition of (71) remains true for all positive times for sufficiently small data due to the following reasons: it is true for large times by [9, Theorem 1.10] in which the hyperbolic coordinate $T \geq T_0$ with T_0 a positive constant, and for local times using the argument of [9, Proposition 1.4], in particular [9, (1.16)]. This finishes the proof of Theorem 3.

Appendix A. Two elementary propositions

In this appendix, we present two elementary propositions which are used in Section 3.

Proposition 10. Let $\gamma \geq 1$. Given any two constants α and β satisfying $-1 < \alpha \leq \beta$, we have

$$\alpha^2 \leq \frac{\mathcal{Q}((\alpha+1)^{\frac{2}{\gamma}})}{\zeta(\beta,\gamma)}\,, \tag{73}$$

where $\mathcal{Q}$ and ζ are defined in (16) and (62), respectively.

Proof. Define

$$q(\alpha) := \mathcal{Q}((\alpha+1)^{\frac{2}{\gamma}}) - \alpha^2\,\zeta(\beta,\gamma)\,.$$

By the definition of $\mathcal{Q}$ and straightforward differentiation, we have

$$\begin{aligned} q'(\alpha) &= \frac{2}{\gamma}(\alpha+1)^{\frac{2}{\gamma}-1}\cdot\frac{1}{\gamma}\Big\{\Big[(\alpha+1)^{\frac{2}{\gamma}}\Big]^{\gamma-2} - \Big[(\alpha+1)^{\frac{2}{\gamma}}\Big]^{-2}\Big\} - 2\alpha\,\zeta(\beta,\gamma) \\ &= \frac{2\alpha}{\gamma^2}\Big\{(\alpha+1)^{-\frac{2}{\gamma}} + (\alpha+1)^{-\frac{2}{\gamma}-1} - (\beta+1)^{-\frac{2}{\gamma}} - (\beta+1)^{-\frac{2}{\gamma}-1}\Big\}. \end{aligned}$$

Since $\alpha \in (-1,\beta]$, it is apparent from the above that $q'(\alpha)\alpha \geq 0$. Therefore, $q(\alpha) \geq q(0) = 0$. In other words, the estimate (73) is proved. □

Proposition 11. Given a compactly supported function $g(\xi) \in C^1(\mathbb{R})$, one has

$$\|g^3\|_{L^\infty} \leq \frac{3}{4}\,\|g\|_{L^2}^2\,\|g'\|_{L^\infty}\,. \tag{74}$$

Proof. This is a special case of the Gagliardo-Nirenberg interpolation inequality, but we prove it for completeness.

For any $\xi \in \mathbb{R}$, one has

$$\min\{\|g\|_{L^2(-\infty,\xi)}, \|g\|_{L^2(\xi,\infty)}\} \leq \frac{1}{2}\|g\|_{L^2(\mathbb{R})}\,.$$

Without loss of generality, one assumes $\|g\|_{L^2(-\infty,\xi)} = \min\{\|g\|_{L^2(-\infty,\xi)}, \|g\|_{L^2(\xi,\infty)}\}$. Hence,

$$\big|g^3(\xi)\big| = \Big|\int_{-\infty}^{\xi}\frac{d}{d\xi}g^3(\xi)\,d\xi\Big| \leq 3\,\|g\|_{L^2((-\infty,\xi))}^2\,\|g'\|_{L^\infty((-\infty,\xi))} \leq \frac{3}{4}\,\|g\|_{L^2(\mathbb{R})}^2\,\|g'\|_{L^\infty(\mathbb{R})}. \tag{75}$$

Thus we have the desired inequality (74). □

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