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# An improved local blow-up condition for Euler–Poisson equations with attractive forcing

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ABSTRACT

We improve the recent result of Chae and Tadmor (2008) [10] proving a one-sided threshold condition which leads to a finite-time breakdown of the Euler–Poisson equations in arbitrary dimension $n$.



## 1. Introduction

The pressure-less Euler–Poisson (EP) equations in dimension $n \geq 1$ are

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \tag{1.1a}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1}(\rho - c), \tag{1.1b}$$

governing the unknown density $\rho = \rho(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$ and velocity $\mathbf{u} = \mathbf{u}(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ subject to initial conditions $\rho(0, x) = \rho_0(x)$ and $\mathbf{u}(0, x) = \mathbf{u}_0(x)$. They involve two constants: (i) a fixed background state $c \geq 0$ — typical cases include the case of zero background, $c = 0$, or the case of a nonzero background given by the average mass, $c = \int \rho(t, x)\mathrm{d}x = \int \rho_0(x)\mathrm{d}x$; and (ii) a constant $k$ which parameterizes the repulsive $k > 0$ or attractive $k < 0$ forcing, governed by the Poisson potential $\Delta^{-1}(\rho - c)$. The EP system appears in numerous applications including semiconductors and plasma physics ($k > 0$) and the collapse of stars due to self gravitation ($k < 0$) [1–4]. In particular, the pressureless EP model becomes relevant in interstellar clouds where gravitional forces dominate pressure gradient, [5], for example, or in the context of the Euler–Monge–Ampère systems and their quasi-neutral limits to the *incompressible* Euler equations [6].

This paper is restricted to the *attractive case*, $k < 0$. We begin by setting $c = 1, k = -1$ in (1.1a), (1.1b) to arrive at the unit-free EP system,

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.2a}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \Delta^{-1}(\rho - 1). \tag{1.2b}$$

Our discussion remains valid for the general physical parameters $c \geq 0, k < 0$ upon a simple rescaling and limiting arguments, outlined in Corollary 1.1 below for $c > 0$ and Corollary 1.2 for the case of zero background $c = 0$.

We are concerned here with the persistence of $C^1$ regularity for solutions of the attractive EP system. Our Main theorem reveals a *pointwise* criterion on the initial data, a so-called critical threshold criterion [7–9], that leads to finite time blow-up of $\nabla \mathbf{u}$. It quantifies the balance between the two term $\operatorname{div} \mathbf{u}$ and $\rho$, which govern two competing mechanisms that dictate the $C^1$ regularity of EP flows. Our result also stands out as a generalization of several existing results [7,10,11,9] for which further discussion is given after the Main theorem and its corollary.

**Main Theorem 1.1.** *Consider the $n$-dimensional, attractive Euler–Poisson system (1.2a), (1.2b) subject to initial data $\rho_0$, $\mathbf{u}_0$. Then, the solution will lose $C^1$ regularity at a finite time $t = t_c < \infty$, if there exists a non-vacuum initial state $\rho_0(\bar{x}) > 0$ with vanishing initial*

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*vorticity,* $\nabla \times \mathbf{u}_0(\bar{x}) = 0$*, at some* $\bar{x} \in \mathbb{R}^n$ *such that the following sup-critical condition is fulfilled,*

$$\operatorname{div} \mathbf{u}_0(\bar{x}) < \operatorname{sgn}(\rho_0(\bar{x}) - 1)\sqrt{nF(\rho_0(\bar{x}))}, \tag{1.3a}$$

*where*

$$F(\rho) := \begin{cases} 1 + \dfrac{2\rho}{n-2} - \dfrac{n\rho^{2/n}}{n-2}, & n \neq 2, \\ 1 - \rho + \rho \ln \rho, & n = 2. \end{cases} \tag{1.3b}$$

*In particular,* $\min_x \operatorname{div} \mathbf{u}(t, x) \to -\infty$ *and* $\max_x \rho(t, x) \to \infty$ *as* $t \uparrow t_c$*.*

**Proof.** Combine Lemmas 3.1 and 4.2, while noting that the curve

$$\operatorname{div} \mathbf{u} = \operatorname{sgn}(\rho - 1)\sqrt{nF(\rho)},$$

is the separatrix along the boundary of the blow-up region $\Omega = \Omega_1 \cup \Omega_2$ defined in (4.3) and illustrated in Fig. 4.1. □

We note in passing that, by classical arguments, the force-free Euler system $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0$ exhibits finite time blow-up if and only if there exists at least one *negative* eigenvalue of $\nabla \mathbf{u}_0(\bar{x})$. In the above theorem, however, finite-time blow-up can occur solely depending on the initial profile of $\operatorname{div} \mathbf{u}_0$ and $\rho_0$ regardless of individual eigenvalues of $\nabla \mathbf{u}_0$.

We also note that, by rescaling $\rho$ to $\rho/c$, $x$ to $\sqrt{-kc}\,x$ and $t$ to $\sqrt{-kc}\,t$, the Main theorem immediately applies to the EP system (1.1a), (1.1b) with physical parameters. Since the EP system with $k < 0$ models the collapse of interstellar cloud, the following corollary reveals a pointwise condition for mass concentration, $\rho \to \infty$, which interestingly preludes the birth of new stars.

**Corollary 1.1.** *Consider the Euler–Poisson system (1.1a), (1.1b) with* $c > 0$, $k < 0$ *subject to initial data* $\rho_0$, $\mathbf{u}_0$*. Then, the solution will lose* $C^1$ *regularity at a finite time* $t_c < \infty$*, if there exists a non-vacuum initial state* $\rho_0(\bar{x}) > 0$ *with a vanishing initial vorticity,* $\nabla \times \mathbf{u}_0(\bar{x}) = 0$*, such that the super-critical condition is fulfilled,*

$$\operatorname{div} \mathbf{u}_0(\bar{x}) < \operatorname{sgn}(\rho_0(\bar{x}) - c)\sqrt{-nkcF\left(\frac{\rho_0(\bar{x})}{c}\right)} \tag{1.4}$$

*where* $F(\cdot)$ *is given in (1.3b). In particular,* $\min_x \operatorname{div} \mathbf{u}(t, x) \to -\infty$ *and* $\max_x \rho(t, x) \to \infty$ *as* $t \uparrow t_c$*.*

In the limiting regime as $c \to 0+$, condition (1.4) converges to a super-critical condition which is summarized by the following result, the proof of which is given in Section 5.

**Corollary 1.2.** *Consider the n-dimensional Euler–Poisson system (1.1a), (1.1b) with* $c = 0$, $k < 0$ *subject to initial data* $\rho_0$, $\mathbf{u}_0$*. Assume a vanishing initial vorticity everywhere,* $\nabla \times \mathbf{u}_0 \equiv 0$*. Then, the solution will lose* $C^1$ *regularity at a finite time* $t_c < \infty$*, if either* (i) $n = 1, 2$ *or* (ii) $n \geq 3$ *and there exists a non-vacuum initial state* $\rho_0(\bar{x}) > 0$ *such that*

$$\operatorname{div} \mathbf{u}_0(\bar{x}) < \sqrt{-\frac{2nk\rho_0(\bar{x})}{n-2}}, \quad n \geq 3. \tag{1.5}$$

In other words, the pressureless and vorticity-free one- and two-dimensional attractive Euler–Poisson systems with zero background ($c = 0$), inevitably collapse to singularity at a finite time. On the other hand, the complete characterization of finite-time breakdown in higher dimensions remains open, even for $c = 0$.

The concept of Critical Threshold and associated methodology is originated and developed in a series of papers by Engelberg, Liu and Tadmor [7], Liu and Tadmor [9,8] and more. It first appears in [7] regarding pointwise criteria for $C^1$ solution regularity of 1D EP system. The key argument in that paper is based on the convective derivative along particle paths $' = \partial_t + \mathbf{u} \cdot \nabla$. It makes it possible to obtain a 2-by-2 ODE system for $u_x$ and $\rho$ along particle paths — the so-called Lagrangian formulation. Phase plane analysis is then employed to study the finiteness of the ODE solutions and therefore $C^1$ regularity of the PDE solution. Similar results stay valid for Euler–Poisson systems with geometric symmetry in higher dimensions [3,8]. To treat genuinely multi-D cases, Liu and Tadmor introduce in [8] the method of spectral dynamics which relies on the ODE system governing eigenvalues of

$$M := \nabla \mathbf{u},$$

which is the velocity gradient matrix, along particle paths. They identify if-and-only-if, pointwise conditions for global existence of $C^1$ solutions to *restricted* Euler–Poisson systems. Chae and Tadmor [10] further extend the Critical Threshold argument to multi-D full Euler–Poisson systems (1.2a), (1.2b) with attractive forcing $k < 0$. Their result, however, offers a blow-up region $\nabla \times \mathbf{u}_0 = 0$, $\operatorname{div} \mathbf{u}_0 < -\sqrt{-nkc}$ which is only a subset of the blow-up region in (1.4). This subset is to the left of the solid line $d \leq d^- := -\sqrt{-nkc}$ depicted in Fig. 4.1. Finally, a recent paper by Tadmor and Wei [12] reveals the critical threshold phenomena in the 1D Euler–Poisson system with pressure.

When tracking other results on the well-posedness of Euler–Poisson equations, we find them commonly relying on (the vast family of) energy methods and thus fundamentally differ from our pointwise results obtained via the Lagrangian approach. With a repulsive force $k > 0$, we refer to [13,14] for the global existence of classical solutions with small data and [15] for the nonexistence of global solutions. With attractive force $k < 0$, see [1] for local regularity of classical solutions and [16,17] for nonexistence results. Discussions on weak solutions of Euler–Poisson systems can be found in e.g. [18–20]. We also refer to [21–25] and references therein for steady-state solutions. The study of the Euler–Poisson system with damping relaxation can be found in e.g. [26–28].

The rest of this paper is organized as follows. In Section 2, we follow the idea of [10] to derive along particle paths an ODE system governing the dynamics of eigenvalues for $S := \frac{1}{2}(M + M^\top)$. This is a variation of the spectral dynamics for $M$ introduced in [8]. We then derive in Section 3 a closed $2 \times 2$ ODE system (3.1) at the cost of turning one equation into inequality. By the comparison principle, this inequality is in favor of blow-up. Thus, with the inequality sign being replaced with an equality sign, a modified ODE system is used to yield sub-solutions and to study a blow-up scenario for the original system. Section 4, devoted to the modified system, reveals the Critical Threshold for such a system. Consequently, a pointwise blow-up condition for the original system is identified. Finally, in Section 5 we prove Corollary 1.2 regarding the Euler–Poisson system with zero background using techniques developed in previous sections.

## 2. Spectral dynamics

We examine the gradient matrix $M = \nabla \mathbf{u}$ and its symmetric part, $S = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right)$. Both matrices are used to study the spectral dynamics of Euler systems (see e.g. [8] for $M$ and [10] for $S$). The relation between the spectra of $M$ and $S$ is described in the following.

**Proposition 2.1.** *Let* $\{\lambda_M\}$ *denote the eigenvalues of* $M$ *and* $\{\lambda_S\}$ *for* $S$*. Then*

$$\sum_{\lambda_M} \lambda_M = \sum_{\lambda_S} \lambda_S = \operatorname{div} \mathbf{u}, \tag{2.1}$$

$$\sum_{\lambda_M} \lambda_M^2 = \sum_{\lambda_S} \lambda_S^2 - \frac{1}{2}|\boldsymbol{\omega}|^2. \tag{2.2}$$

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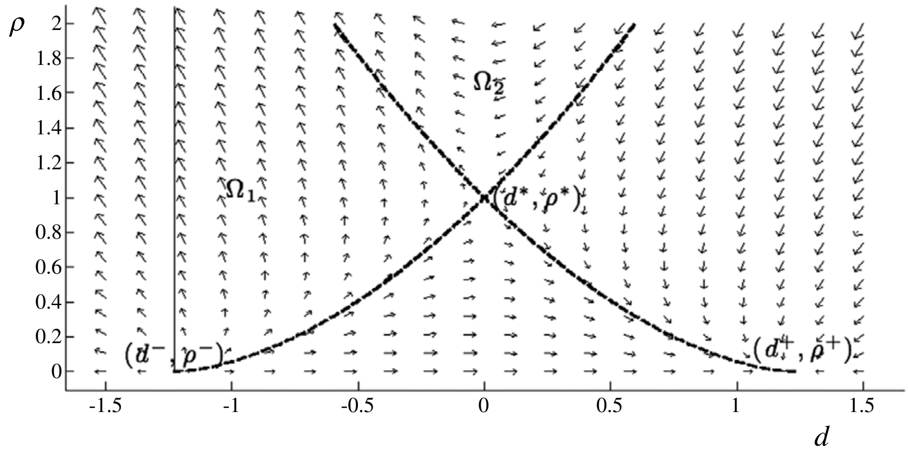


**Fig. 4.1.** Phase plane of (4.1) with blow-up region $\Omega_1 \cup \Omega_2$ which extends the Chae–Tadmor region [10] $d \le d^-$.

*Here,* $\omega$ *is the* $\frac{n(n-1)}{2}$ vorticity *vector which consists of the off-diagonal entries of* $A := \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^\top\right)$.

**Proof.** Use identity $M = S + A$ and the skew-symmetry of $A$,

$$\sum_{\lambda_M} \lambda_M = \mathrm{tr}(M) = \mathrm{tr}(S + A) = \mathrm{tr}(S) = \sum_{\lambda_S} \lambda_S.$$

Squaring the last identity we have $M^2 = S^2 + A^2 + AS + SA$ and therefore,

$$\sum_{\lambda_M} \lambda_M^2 = \mathrm{tr}(M^2) = \mathrm{tr}(S^2 + A^2 + AS + SA) = \sum_{\lambda_S} \lambda_S^2 + \mathrm{tr}(A^2).$$

Note that $AS + SA$ is skew-symmetric and thus traceless. A simple calculation yields $\mathrm{tr}(A^2) = -\frac{1}{2}|\omega|^2$. □

Following [8], we turn to study the dynamics of $M$ along particle paths. Take the gradient of (1.2b) to find

$$M' + M^2 \equiv M_t + u \cdot \nabla M + M^2 = -R(\rho - 1), \tag{2.3}$$

where $R$ stands for the *Riesz matrix*, $R = \{R_{ij}\} := \{\partial_{x_i x_j}\Delta^{-1}\}$.

The trace of (2.3) then yields that the divergence, $d := \mathrm{tr}(M)$, is governed by

$$d' = -\sum_{\lambda_M} \lambda_M^2 - (\rho - 1),$$

and in view of (2.2),

$$d' = -\sum_{\lambda_S} \lambda_S^2 + \frac{1}{2}|\omega|^2 - (\rho - 1). \tag{2.4}$$

We now make the first observation regarding the invariance of the vorticity $\omega$: taking the skew-symmetric part of the $M$- equation (2.3),

$$A' + AS + SA = 0. \tag{2.5}$$

It follows that if the initial vorticity vanishes, $\omega_0(\bar{x}) \mapsto \nabla\times\mathbf{u}_0(\bar{x}) = 0$, then by (2.5), $\omega \mapsto \nabla \times \mathbf{u}$ vanishes along the particle path which emanates from $\bar{x}$. This allows us to decouple the vorticity and divergence dynamics, and (2.4) implies

$$d' = -\sum_{\lambda_S} \lambda_S^2 - (\rho - 1), \qquad \nabla \times \mathbf{u} = 0. \tag{2.6}$$

Finally, we use Cauchy–Schwartz $\sum \lambda_S^2 \ge \frac{1}{n}\left(\sum \lambda_S\right)^2 = \frac{1}{n}d^2$ and the fact that all $\lambda_S$ are real (due to the symmetry of $S$), to deduce the *inequality*,

$$d' \le -\frac{1}{n}d^2 - (\rho - 1). \tag{2.7a}$$

This, together with the mass equation (1.2a) which can be written along particle path

$$\rho' = -d\rho, \tag{2.7b}$$

give us the desired closed system which dominates $(\rho, d)$ along particle paths.

**Remark 2.1.** The approach pursued in this paper will be based on the *inequality* (2.7a) and is therefore limited to derivation of a finite time breakdown. To argue the global regularity, one needs to study the underlying *equality* (2.6), and to this end, to study the trace $\sum \lambda_S^2$. In the two-dimensional case, for example, one can use $\sum \lambda_S^2 = d^2/2 + \eta^2/2$ to replace (2.7a) with

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 - (\rho - 1), \qquad \eta := \lambda_{S,2} - \lambda_{S,1}.$$

In this framework, global 2D regularity is dictated by the dynamics of the *spectral gap*, $\eta = \lambda_{S,2} - \lambda_{S,1}$, which in turn requires the dynamics of the Riesz transform $R(\rho - 1)$.

## 3. A comparison principle with a majorant system

The blow-up analysis, driven by the inequalities (2.7),

$$d' \le -\tfrac{1}{n}d^2 - (\rho - 1), \tag{3.1a}$$

$$\rho' = -d\rho. \tag{3.1b}$$

is carried out by standard comparison with the majorant system

$$e' = -\tfrac{1}{n}e^2 - (\zeta - 1), \tag{3.2a}$$

$$\zeta' = -e\zeta. \tag{3.2b}$$

The following proposition guarantees the monotonicity of the solution operator associated with (3.1).

**Lemma 3.1.** *The following monotone relation between system* (3.1) *and system* (3.2) *is invariant forward in time,*

$$\begin{cases} d(0) < e(0) \\ 0 < \zeta(0) < \rho(0) \end{cases} \text{ implies } \begin{cases} d(t) < e(t) \\ 0 < \zeta(t) < \rho(t) \end{cases} \quad \text{for } t \ge 0, \tag{3.3}$$

*as long as all solutions remain finite on the time interval* $[0, t]$.

**Proof.** Invariance of positivity of $\zeta$ is a direct consequence of (3.2b) and finiteness of $e$. The rest can be proved by contradiction. Suppose $t_1$ is the earliest time when (3.3) is violated. Then,

$$\zeta(t_1) = \zeta(0)\exp\left(-\int_0^{t_1} e(t)\mathrm{d}t\right) < \rho(0)\exp\left(-\int_0^{t_1} d(t)\mathrm{d}t\right) = \rho(t_1). \tag{3.4}$$

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Therefore, we are left with only one possibility, namely, $e(t_1) = d(t_1)$. Subtracting (3.1a) from (3.2a),

$$(e-d)' \geq -\frac{1}{n}(e^2-d^2) - (\zeta - \rho), \tag{3.5}$$

and by (3.4), we find that at $t = t_1$,

$$RHS \text{ of } (3.5)_{|t=t_1} = 0 - [\zeta(t_1) - \rho(t_1)] > 0.$$

However, this contradicts the negativity of the expression on the left of (3.5), since $e(t) - d(t) > 0$ for all $t < t_1$ and vanishes at $t = t_1$ which imply that

$$LHS \text{ of } (3.5)_{|t=t_1} = \big(e(t_1) - d(t_1)\big)' \leq 0. \quad \square$$

In the next section, we employ phase plane analysis on the modified system (3.2). When translated in terms of the original system (3.1), however, such analysis can only yield blow-up results and is insufficient for global existence results. In other words, estimate (3.3) is only useful for proving $d \searrow -\infty$, the key mechanism for blow-up of $C^1$ solutions.

## 4. Stability analysis of the majorant system

We shall prove the blow-up of the majorant system (3.2), $e(t) \to -\infty$ as $t \uparrow t_c$, which in turn, by Lemma 3.1 implies $d(t) \to -\infty$. Abusing notations, we express the majorant system in terms of the original variables $(e, \zeta) \mapsto (d, \rho)$:

$$d' = -\frac{1}{n}d^2 - (\rho - 1), \tag{4.1a}$$

$$\rho' = -d\rho. \tag{4.1b}$$

The (in-)stability analysis of (4.1) hinges on the path invariants of this system. To this end, we use the same $q$-transformation employed in [29,9]: setting $q := d^2$ and differentiate along the path $\{(t, X(a,t)) \mid X_t(a,t) = u(t, X(a,t)), X(a,0) = a\}$, we find

$$\frac{dq}{d\rho} = 2d\frac{d'}{\rho'} = \frac{2}{n\rho}q + 2\left(1 - \frac{1}{\rho}\right),$$

which yields

$$\frac{d}{d\rho}\left(q\rho^{-\frac{2}{n}}\right) = 2(1-\rho^{-1})\rho^{-\frac{2}{n}}.$$

Upon integration, we arrive at the following key observation.

**Lemma 4.1.** *The majorant system (4.1) is equipped with the path invariant,*

$$I(d(t), \rho(t)) = I(d_0, \rho_0),$$

*along each path $(t, x(t))$ initiated with a non-vacuum state $(d_0, \rho_0 > 0)$. Here,*

$$\begin{aligned} I(d,\rho) &:= d^2\rho^{-\frac{2}{n}} - 2\int_1^\rho (1-r^{-1})r^{-\frac{2}{n}}\,dr \\ &= \rho^{-\frac{2}{n}}\left(d^2 - nF(\rho)\right), \end{aligned} \tag{4.2}$$

*where $F(\cdot)$ is specified in (1.3b).*

It is a simple calculation to show that the majorant system (4.1) admits three distinct critical points (see Fig. 4.1):

$$(d^*, \rho^*) := (0, 1), \quad (d^\pm, \rho^\pm) := (\pm\sqrt{n}, 0).$$

and that $(0, 1)$ is a saddle point, $(-\sqrt{n}, 0)$ a nodal source and $(\sqrt{n}, 0)$ a nodal sink. The separatrix is given by the zero level set $I(d, \rho) = 0$. Moreover, the right branch of the separatrix, $d = \sqrt{nF(\rho)}$ connects critical points $(0, 1)$ and $(\sqrt{n}, 0)$ while the left branch, $d = -\sqrt{nF(\rho)}$ connects $(0, 1)$ and $(-\sqrt{n}, 0)$.

By inspection of the phase plane in Fig. 4.1, we postulate the following invariant region of finite-time blow-up for the modified system (4.1),

$$\Omega = \Omega_1 \cup \Omega_2 = \{(d,\rho) \mid d < \mathrm{sgn}(\rho - 1)\sqrt{nF(\rho)}\} \tag{4.3a}$$

where

$$\Omega_1 := \{(d,\rho) \mid I(d,\rho) > 0 \text{ and } d < 0 \text{ and } \rho > 0\}, \tag{4.3b}$$

$$\Omega_2 := \{(d,\rho) \mid I(d,\rho) < 0 \text{ and } \rho > 1\}. \tag{4.3c}$$

**Lemma 4.2.** *Consider the modified system (4.1), equipped with initial data $(d_0, \rho_0)$. If $(d_0, \rho_0) \in \Omega$, then $\mathrm{div}\,\mathbf{u} \to -\infty$ and $\rho \to \infty$ at a finite time.*

**Proof.** We begin by recalling (1.3b), consult (4.2),

$$F(\rho) = \frac{2}{n}\rho^{\frac{2}{n}}\int_1^\rho (1-r^{-1})r^{-\frac{2}{n}}\,dr.$$

Clearly, $F(1) = F'(1) = 0$ and a simple calculation shows that $F''(\rho) = \frac{2}{n}\rho^{\frac{2}{n}-2}$, which implies that $F(\rho)$ is a strictly convex function of positive $\rho$ and attains its only minimum at $\rho = 1$,

$$F(\rho) \geq F(1) = 0. \tag{4.4}$$

We shall also utilize the invariance of (4.2)

$$d^2 - nF(\rho) = \rho^{\frac{2}{n}}I_0, \qquad I_0 = I(d_0, \rho_0). \tag{4.5}$$

We now turn to discuss the two possible blow-up scenarios, depending whether the initial data $(d_0, \rho_0)$ belong to the blow-up regions $\Omega_1$ or $\Omega_2$ given in (4.3).

*Case* #1. Assume that $(d_0, \rho_0) \in \Omega_1$ so that the invariant $I$ remains a *positive* constant

$$I > 0.$$

In this case, $d$ remains negative, for otherwise, setting $d = 0$ in (4.5) would result in $F(\rho) = -\rho^{\frac{2}{n}}I/n < 0$, violating (4.4). Thus, (4.5) and (4.4) yield an upper bound,

$$d \leq -\rho^{\frac{1}{n}}\sqrt{I}.$$

Then, by (4.1b), we have a Riccati type of equation $\rho' \geq \sqrt{I}\rho^{1+\frac{1}{n}}$ for which the solution exhibits blow-up $\rho \to +\infty$ and the divergence $d = \mathrm{div}\,\mathbf{u}$ approaches $-\infty$ at a finite time due to (4.5).

*Case* #2. Assume that $(d_0, \rho_0) \in \Omega_2$ so that the invariant $I$ remains a *negative* constant

$$I < 0.$$

In this case, $\rho - 1$ remains positive, for otherwise setting $\rho = 1$ in (4.5) would result in $F(1) = (d^2 - I)/n > 0$ in contradiction to (4.4). Now, for $\rho > 1$ we have

$$F(\rho) = \frac{2}{n}\rho^{2/n}\int_1^\rho \left(1 - \frac{1}{r}\right)\frac{1}{r^{2/n}}dr \leq \frac{2}{n}\rho^{2/n}(\rho - 1).$$

This together with (4.5) yield

$$\frac{2}{n}\rho^{2/n}(\rho-1) \geq F(\rho) = \frac{1}{n}\left(d^2 - \rho^{2/n}I\right) \geq -\frac{1}{n}\rho^{2/n}I$$

and the lower bound, $\rho - 1 \geq -I/2$ follows. Thus, by (4.1a), we end up with a Riccati type of equation

$$d' \leq -\frac{d^2}{n} + \frac{I}{2}.$$

Since the invariant $I$ remains a negative constant, the solution exhibits blow-up $d = \mathrm{div}\,\mathbf{u} \to -\infty$ at a finite time even if initially $d_0 > 0$. The density $\rho$ also approaches $\infty$ in finite time due to (4.5). $\square$

The last step of proving the Main theorem is just to combine the comparison principle in Lemma 3.1 with the above lemma. We notice that $\Omega$ is an open set and thus given any initial data $(d_0, \rho_0) \in \Omega$ for the original system, we can always find $\varepsilon > 0$ and initial data $(d_0 + \varepsilon, \rho_0 - \varepsilon) \in \Omega$ for the modified system. This

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latter initial data will lead to a finite time blow-up of the modified system and therefore, by Lemma 3.1, initial data $(d_0, \rho_0) \in \Omega$ will lead to finite time blow-up of the original system.

## 5. Critical threshold for zero background

We now turn to the attractive Euler–Poisson system (1.1a), (1.1b) with zero background $c = 0$ and prove Corollary 1.2. For simplicity, we only show the case with $k = -1$ since a straightforward rescaling argument, $x \to \sqrt{-k}\,x$ and $t \to \sqrt{-k}\,t$, will cover the case for general $k < 0$.

**Proof of Corollary 1.2.** Following the same calculation that leads to the majorant system (4.1a), (4.1b), we arrive at a similar ODE system for the case $c = 0$, $k = -1$,

$$d' = -\frac{1}{n}d^2 - \rho, \tag{5.1a}$$

$$\rho' = -d\rho. \tag{5.1b}$$

Then, as an analogue to the invariant (4.2), we find the corresponding invariant,

$$I(d,\rho) := d^2\rho^{-\frac{2}{n}} - 2\int_a^\rho r^{-\frac{2}{n}}\,\mathrm{d}r.$$

By choosing the constant

$$a = \begin{cases} +\infty, & n = 1, \\ 1, & n = 2, \\ 0, & n \geq 3, \end{cases}$$

we have

$$I(d,\rho) = \begin{cases} d^2\rho^{-2} + 2\rho^{-1}, & n = 1 \\ d^2\rho^{-1} - 2\ln\rho, & n = 2 \\ d^2\rho^{-\frac{2}{n}} - \dfrac{2n}{n-2}\rho^{1-\frac{2}{n}}, & n \geq 3. \end{cases} \tag{5.2}$$

Using the positivity of $\rho$ and $d^2$ in (5.2), we have $I \geq 2\rho^{-1} > 0$ for $n = 1$ and $I \geq -2\ln\rho$ for $n = 2$. Both estimates imply that $\rho$ is bounded from below by a positive constant. In the case of $n \geq 3$, the sup-critical condition (1.5) implies $I < 0$. Thus, by (5.2), we have $0 > I \geq -\frac{2n}{n-2}\rho^{1-\frac{2}{n}}$ which, again, implies $\rho$ is greater than a positive constant.

Therefore, by (5.1a), $d$ satisfies a differential inequality

$$d' \leq -\frac{d^2 + \alpha}{n}$$

with positive constant $\alpha$. Obviously, $d(t)$ approaches $-\infty$ at a time no later than $\frac{n\pi}{2\sqrt{\alpha}}$. □

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