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On the classical solutions of two dimensional inviscid rotating shallow water system

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ABSTRACT

We prove global existence and asymptotic behavior of classical solutions for two dimensional inviscid rotating shallow water system with small initial data subject to the zero relative vorticity condition. One of the key steps is a reformulation of the problem into a symmetric quasilinear Klein–Gordon system with quadratic nonlinearity, for which the global existence of classical solutions is then proved with combination of the vector field approach and the normal form method. We also probe the case of general initial data and reveal a lower bound for the lifespan that is almost inversely proportional to the size of the initial relative vorticity.

1. Introduction and main results

The system of rotating shallow water (RSW) equations is a widely adopted 2D approximation of the 3D incompressible Euler equations and the Boussinesq equations in the regime of large scale geophysical fluid motion (cf. [18]). It is also regarded as an important extension of the compressible Euler equations with additional rotating force.

The rotating shallow water system is of the form

$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla h + \mathbf{u}^{\perp} = 0, \tag{1.2}$$

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where $h = h(t, x_1, x_2)$ and $\mathbf{u} = (u_1(t, x_1, x_2), u_2(t, x_1, x_2))^T$ denote the total height and velocity field of the fluids, respectively, and $\mathbf{u}^\perp := (-u_2, u_1)^T$ corresponds to the rotating force. For mathematical convenience, all physical parameters are scaled to the unit (cf. [15] for detailed discussion on scaling).

We introduce the perturbations $(\rho, \mathbf{u}) := (h - 1, \mathbf{u})$ of the background solution $(h, \mathbf{u}) = (1, \mathbf{0})$ and arrive at

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) + \nabla \cdot \mathbf{u} = 0, \tag{1.3}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \rho + \mathbf{u}^\perp = 0, \tag{1.4}$$

subject to initial data

$$\rho(0, \cdot) = \rho_0, \qquad \mathbf{u}(0, \cdot) = \mathbf{u}_0. \tag{1.5}$$

An important feature of the RSW system is that the relative vorticity[1] $\theta := \nabla \times \mathbf{u} - \rho = (\partial_1 u_2 - \partial_2 u_1) - \rho$ is convected by $\mathbf{u}$,

$$\partial_t \theta + \nabla \cdot (\theta \mathbf{u}) = 0. \tag{1.6}$$

Indeed, $\nabla \times$ (1.4)–(1.3) readily leads to (1.6). The linearity of (1.6) then suggests that $\theta \equiv 0$ be an invariant with respect to time (as long as $\mathbf{u} \in C^1$), i.e.

$$\theta_0 \equiv 0 \quad \Leftrightarrow \quad \theta(t, \cdot) \equiv 0 \quad \Leftrightarrow \quad \nabla \times \mathbf{u} \equiv \rho. \tag{1.7}$$

Before stating the main theorems, we fix some notations. For $1 \leqslant p \leqslant \infty$, let L^p denote the standard L^p space on R^2. For $l \geqslant 0$ and $s \geqslant 0$, define the weighted Sobolev norm associated with the space $H^{l,s}$ as

$$\|v\|_{H^{l,s}} := \left\| \left(1 + |x|^2\right)^{s/2} (1 - \Delta)^{l/2} v \right\|_{L^2}. \tag{1.8}$$

Also, let $H^l := H^{l,0}$ denote the standard Sobolev space.

Theorem 1.1. *Consider the RSW system* (1.3) *and* (1.4) *with initial data* (1.5) *which satisfies* $\mathbf{u}_0 = (u_{1,0}, u_{2,0})^T \in H^{k+2,k}$ *for* $k \geqslant 52$ *and zero relative vorticity condition*

$$\rho_0 = \partial_1 u_{2,0} - \partial_2 u_{1,0}.$$

Then, there exists a universal constant $\delta_0 > 0$ *such that the RSW system admits a unique classical solution* $(\rho, \mathbf{u})$ *for all time, provided that the initial data satisfy*

$$\|\mathbf{u}_0\|_{H^{k+2,k}} = \delta < \delta_0.$$

Moreover, there exists a free solution $\mathbf{u}^+(t, \cdot)$ *of linear Klein–Gordon equations such that*

$$\left\|\mathbf{u}(t, \cdot) - \mathbf{u}^+(t, \cdot)\right\|_{H^{k-15}} + \left\|\partial_t \mathbf{u}(t, \cdot) - \partial_t \mathbf{u}^+(t, \cdot)\right\|_{H^{k-16}} \leqslant C(1+t)^{-1},$$

where $\mathbf{u}^+(t, \cdot) := (\cos(1 - \Delta)^{1/2} t)\mathbf{u}_0^+ + ((1 - \Delta)^{-1/2} \sin((1 - \Delta)^{1/2} t))\mathbf{u}_1^+$ *for some* $\mathbf{u}_0^+ \in H^{k-15}$ *and* $\mathbf{u}_1^+ \in H^{k-16}$.

[1] Our definition here for the relative vorticity is a little different from that in geophysical fluid dynamics.

The proof is a straightforward combination of Lemma 2.1, Theorem 4.1 and Theorem 4.3 below. Before we discuss the key ideas for the proof, several remarks are in order.

Remark 1.1. Theorem 1.1 shows fundamentally different lifespan of the classical solutions for the RSW system in comparison with the compressible Euler systems. Note that the relative vorticity $\theta = \nabla \times \mathbf{u} - \rho$ in the RSW equations plays a very similar role as vorticity $\nabla \times \mathbf{u}$ in the compressible Euler equations. Correspondingly, RSW solutions with zero relative vorticity θ is an analogue of irrotational solutions for the compressible Euler equations. However, the life span for 2D compressible Euler equations with zero vorticity was proved to be bounded from below [22] and from above [19] by $O(1/\delta^2)$. Here, δ indicates the size of the initial data. Our result for 2D RSW system, on the other hand, is global in time due to the additional rotating force.

Remark 1.2. Recent studies have shown the stabilizing effects of rotation in terms of lifespan of classical solutions in various settings. In [13], the authors obtained critical threshold for the initial data of a two dimensional pressureless model with rotating force and, in particular, proved global existence for subcritical initial data. In [5], it was shown that rotating force prolongs lifespan when the pressure is present and dominated by the rotating force. We also refer to [1] for discussion of RSW equations in periodic domains subject to balanced rotating and pressure gradient forces. Along the line of these results, Theorem 1.1 confirms the stabilizing effect of rotation in the global existence of classical solutions for the full, balanced RSW system.

Remark 1.3. The regularity requirement for initial data in Theorem 1.1, i.e. $\mathbf{u}_0 \in H^{k+2,k}$ with $k > 52$, serves to shorten technical calculations. It is not optimal but neverless covers such important scenarios as compactly supported, smooth initial data.

One of key ingredients of the proof is to treat the RSW system as a system of quasilinear Klein–Gordon equations (cf. Lemma 2.1). Such reformulation allows us to adapt the fruitful ideas on nonlinear Klein–Gordon equations in recent decades. The global existence for three dimensional quasilinear Klein–Gordon equations with quadratic nonlinearity was proved independently in [12] using the vector fields approach and in [21] using the normal form method. In the two dimensional case, global existence of classical solutions was proved in [16] for semilinear, scalar equations, where the authors combined the vector fields approach and the normal form method. For other related results on two dimensional Klein–Gordon equations, see e.g. [9,6,17] and references therein. For applications of the Klein–Gordon equations in fluid equations, we refer to [8] on global existence of three dimensional Euler-Poisson system for irrotational flows. Note that the irrotationality condition used in [8] plays a counterpart of the zero-relative-vorticity condition in our result.

For general initial data, we have the following theorem on the lifespan of classical solutions.

Theorem 1.2. *Consider the RSW system* (1.3) *and* (1.4), *with initial data* (1.5) *satisfying* $(\rho_0, \mathbf{u}_0) \in H^{k+1,k}$ *with* $k \geqslant 52$. *Let* δ *and* ε *denote the size of the initial data,*

$$\delta = \left\| (\rho_0, \mathbf{u}_0) \right\|_{H^{k+1,k}},$$

and the size of the initial relative vorticity,

$$\varepsilon = \left\| (\partial_1 u_{2,0} - \partial_2 u_{1,0}) - \rho_0 \right\|_{H^2},$$

respectively. Then, there exists a universal constant $\delta_0 > 0$ *such that, for any* $\delta \leqslant \delta_0$, *the RSW system admits a unique classical solution* $(\rho, \mathbf{u})$ *for*

$$t \in \left[0, C_1 \varepsilon^{-\frac{1}{1+C_2\delta}}\right]. \tag{1.9}$$

Here, C_1 *and* C_2 *are constants independent of* δ *and* ε.

Theorem 1.2 confirms the key role that relative vorticity plays in the studies of geophysical fluid dynamics [18]. In fact, having two uncorrelated scales ε and δ in (1.9) allows us to solely let the size of the initial relative vorticity $\varepsilon \to 0$ and achieve a very long lifespan of classical solutions, regardless of the total size of initial data.

The proof of Theorem 1.2, given in Section 5, treats the full RSW system as perturbation to the zero relative vorticity one and utilizes the standard energy methods for symmetric hyperbolic PDE systems. The dispersive estimates of Theorem 1.1 is crucial in controlling the total energy growth.

Regarding improvement and generalization of Theorem 1.1 and 1.2, we make the following remarks.

Remark 1.4. An obstruction to getting better estimates of the lower bound of lifespan with general initial data (i.e. nonzero initial relative vorticity) is the rapid growth of the vortical mode. This was illustrated in e.g. [2] for quasigeostrophic equation, a closed related model for rotating shallow water equations.

Remark 1.5. Theorems 1.1 and 1.2 are also true for two dimensional compressible Euler equations with rotating force and general pressure law (e.g., $p(\rho) = A\rho^\gamma$ for $\gamma \geqslant 1$). The proofs remain largely the same except for the symmetrization part and associated energy estimates.

We also note that the study of hyperbolic PDE systems with small initial data is closely related to the singular limit problems with large initial data, see [15,1] and references therein for results on the particular case of inviscid RSW equations. A collection of open problems and recent progress on viscous shallow water equations and related models can be found in [4,3] and references therein.

The structure of the rest of the paper is outlined as follows. In Section 2, we reformulate the RSW system into a symmetric hyperbolic system of first order PDEs. Under the zero relative vorticity condition, it is further transformed into a system of symmetric quasilinear Klein–Gordon equations. Section 3 is devoted to the local well-posedness of the RSW equations with general initial data and zero relative vorticity initial data. Section 4 contains the proof of Theorem 1.1 in a series of lemmas, inspired by the results from [7,16]. The proof of Theorem 1.2 on general initial data can be found in Section 5. Appendix A contains the proof of a technical proposition used in Section 4.

2. Reformulation of the RSW system

In order to obtain local well-posedness for RSW system, we first symmetrize the system (1.3) and (1.4). This will also be used for proving global existence, where we need to reduce (1.3) and (1.4) to a symmetric quasilinear Klein–Gordon system.

Introduce a symmetrizer $m := 2(\sqrt{1+\rho} - 1)$ such that $\rho = m + \frac{1}{4}m^2$, then (1.3) and (1.4) are transformed into a symmetric hyperbolic PDE system,

$$\partial_t m + \mathbf{u} \cdot \nabla m + \frac{1}{2} m \nabla \cdot \mathbf{u} + \nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{2} m \nabla m + \nabla m + \mathbf{u}^\perp = 0. \tag{2.2}$$

The following lemma asserts that, under the invariant (1.7), the above system amounts to a system of Klein–Gordon equations with symmetric quasilinear terms.

Lemma 2.1. *Under the invariant* (1.7), $\nabla \times \mathbf{u} = \rho$, *and transformation* $m = 2(\sqrt{\rho+1} - 1)$, *the solution to the RSW system* (1.3) *and* (1.4) *satisfies the following symmetric system of quasilinear Klein–Gordon equations for* $\mathbf{U} := (m, u_1, u_2)^T$,

$$\partial_{tt}\mathbf{U} - \Delta \mathbf{U} + \mathbf{U} = \sum_{i,j=1}^{2} A_{ij}(\mathbf{U})\partial_{ij}\mathbf{U} + \sum_{j=1}^{2} A_{0j}(\mathbf{U})\partial_{0j}\mathbf{U} + R(\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}), \tag{2.3}$$

where linear functions A_{ij} and A_{0j} map R^3 *vectors to* symmetric 3×3 *matrices and satisfy* $A_{ij} = A_{ji}$. *The remainder term R depends linearly on the tensor product* $\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}$ *with* $\tilde{\mathbf{U}} := (\mathbf{U}^T, \partial_t \mathbf{U}^T, \partial_1 \mathbf{U}^T, \partial_2 \mathbf{U}^T)$.

Here and below, for notational convenience, we use both ∂_t and ∂_0 to represent time derivatives.

Proof. Rewrite (2.1), (2.2) into a matrix-vector form,

$$\partial_t \mathbf{U} + \sum_{a=1,2} \left(u_a I + \frac{1}{2} m J_a \right) \partial_a \mathbf{U} = \mathcal{L}(\mathbf{U}), \tag{2.4}$$

where

$$J_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{L}(\mathbf{U}) := -\begin{pmatrix} \nabla \cdot \mathbf{u} \\ \nabla m + \mathbf{u}^{\perp} \end{pmatrix}. \tag{2.5}$$

By taking time derivative on the above system, we have

$$\partial_{tt} \mathbf{U} + N(\mathbf{U}) = \mathcal{L}^2(\mathbf{U}),$$

where the nonlinear term

$$\begin{aligned} N(\mathbf{U}) &= \partial_t \sum_{a=1,2} \left(u_a I + \frac{1}{2} m J_a \right) \partial_a \mathbf{U} + \mathcal{L}\left(\sum_{a=1,2} \left(u_a I + \frac{1}{2} m J_a \right) \partial_a \mathbf{U} \right) \\ &:= N_1 + N_2. \end{aligned} \tag{2.6}$$

The $\mathcal{L}^2$ term, using the calculus identity $\nabla(\nabla \cdot \mathbf{u}) - \nabla^{\perp}(\nabla \times \mathbf{u}) = \Delta \mathbf{u}$, is

$$\mathcal{L}^2(\mathbf{U}) = \begin{pmatrix} \nabla \cdot (\nabla m + \mathbf{u}^{\perp}) \\ \nabla(\nabla \cdot \mathbf{u}) + (\nabla m + \mathbf{u}^{\perp})^{\perp} \end{pmatrix} = \begin{pmatrix} (\Delta - 1)m - (\nabla \times \mathbf{u} - m) \\ (\Delta - 1)\mathbf{u} + \nabla^{\perp}(\nabla \times \mathbf{u} - m) \end{pmatrix}. \tag{2.7}$$

Since $\rho = m + \frac{1}{4} m^2$, we have

$$\mathcal{L}^2(\mathbf{U}) = (\Delta - 1)\mathbf{U} + \begin{pmatrix} -(\nabla \times \mathbf{u} - \rho + \frac{1}{4} m^2) \\ \nabla^{\perp}(\nabla \times \mathbf{u} - \rho + \frac{1}{4} m^2) \end{pmatrix}. \tag{2.8}$$

Using (1.7), it yields

$$\mathcal{L}^2(\mathbf{U}) = (\Delta - 1)\mathbf{U} + \frac{1}{4} \begin{pmatrix} -m^2 \\ \nabla^{\perp}(m^2) \end{pmatrix} := (\Delta - 1)\mathbf{U} + N_3. \tag{2.9}$$

Note that we've revealed the Klein–Gordon structure in (2.3), it suffices to show that the other terms N_1, N_2, N_3 can all be split into a symmetric second order part and a remainder lower order part as given in (2.3).

- The N_1 term in (2.6). It is easy to see that the lower order terms (with less than second order derivatives) in N_1 are quadratic in $\tilde{\mathbf{U}}$, i.e. linear in $\tilde{\mathbf{U}} \otimes \tilde{\mathbf{U}}$. The terms with second order derivatives are

$$\sum_{a=1,2}\left(u_a I + \frac{1}{2} m J_a\right)\partial_t\partial_a \mathbf{U}$$

where matrices I, J_a are all symmetric.

- The N_2 term in (2.6). Observe that the linear operator $\mathcal{L}$ partially comes from a linearization of the nonlinear terms in (2.4) and it indeed can be represented as

$$\mathcal{L}(\mathbf{U}) = \sum_{a=1,2} J_a\partial_a \mathbf{U} + K\mathbf{U}$$

with constant matrix K. Thus, manipulate the N_2 term,

$$N_2 = \sum_{a=1,2} J_a\partial_a\left(\sum_{b=1,2}\left(u_b I + \frac{1}{2} m J_b\right)\partial_b \mathbf{U}\right) + K\left(\sum_{b=1,2}\left(u_b I + \frac{1}{2} m J_b\right)\partial_b \mathbf{U}\right)$$
$$= \sum_{a=1,2}\sum_{b=1,2}\left(u_b J_a + \frac{1}{2} m J_a J_b\right)\partial_a\partial_b \mathbf{U} + \text{ quadratic terms of } \tilde{\mathbf{U}}.$$

The quasilinear terms above have the desired symmetric structure since for each index pair (a, b), the coefficient of $\partial_a\partial_b\mathbf{U} = \partial_b\partial_a\mathbf{U}$ is

$$\frac{1}{2}\left(u_b J_a + \frac{1}{2} m J_a J_b\right) + \frac{1}{2}\left(u_a J_b + \frac{1}{2} m J_b J_a\right)$$

which, by the definition of J_a, is symmetric.

- The N_3 term in (2.7). By definition, this term has no second order derivatives and is quadratic in $\tilde{\mathbf{U}}$.

This finishes the proof for the lemma. □

We end this section with the following remark:

Remark 2.1. The Klein–Gordon structure is hinged on the rotating force which corresponds to the $\mathbf{u}^{\perp}$ in the RSW system. If rotation was absent, the shallow water equations should be transformed into a system of nonlinear wave equations instead of nonlinear Klein–Gordon equations.

3. Local well-posedness

As in the previous section, let $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$. Define the vector fields

$$\Gamma := \{\Gamma_j\}_{j=1}^{6} = \{\partial_0, \partial_1, \partial_2, L_1, L_2, \Omega_{12}\}, \tag{3.1}$$

where

$$L_j := x_j\partial_t + t\partial_j, \quad j = 1, 2; \qquad \Omega_{12} := x_1\partial_2 - x_2\partial_1.$$

We abbreviate

$$\partial^{\alpha} = \partial_t^{\alpha_1}\partial_1^{\alpha_2}\partial_2^{\alpha_3} \quad \text{for } \alpha = (\alpha_1, \alpha_2, \alpha_3),$$

and

$$\Gamma^\beta = \Gamma_1^{\beta_1} \cdots \Gamma_6^{\beta_6} \quad \text{for } \beta = (\beta_1, \dots, \beta_6).$$

The local well-posedness of (2.1) and (2.2) and their regularity are contained in the following theorem.

Theorem 3.1.

(i) *Let* $(m_0, \mathbf{u}_0) \in H^n$ *with* $n \geqslant 3$. *Then, there exists a* $T > 0$ *depending only on* $\|(m_0, \mathbf{u}_0)\|_{H^3}$ *such that* (2.1) *and* (2.2) *admit a unique solution* $\mathbf{U} = (m, u_1, u_2)^T$ *on* $[0, T]$ *satisfying*

$$\mathbf{U} \in \bigcap_{j=0}^{n} C^j\big([0, T], H^{n-j}\big). \tag{3.2}$$

(ii) *Under the assumptions in* (i), *also assume* $\rho_0 = \partial_1 u_{2,0} - \partial_2 u_{1,0}$. *Then,*

$$\rho(t, \cdot) = \partial_1 u_2(t, \cdot) - \partial_2 u_1(t, \cdot) \quad \text{for } t \in [0, T], \tag{3.3}$$

and $\mathbf{U} = (m, u_1, u_2)^T$ *satisfies the Klein–Gordon system* (2.3). *If the initial data belong to the weighted Sobolev space* (*cf. Definition* (1.8)) *such that* $\mathbf{U}_0 \in H^{k+1,k}$ *with* $k \geqslant 3$, *then the above solution* $\mathbf{U}$ *satisfies*

$$\Gamma^\alpha \mathbf{U}, \qquad \Gamma^\alpha \partial \mathbf{U} \in C\big([0, T], L^2\big), \qquad \big(1 + |x|\big)\Gamma^\beta \mathbf{U} \in C\big([0, T], L^\infty\big), \tag{3.4}$$

for any multi-indices $|\alpha| \leqslant k$ *and* $|\beta| \leqslant k - 3$.

Proof. The proof of (i) follows from the standard local well-posedness and regularity theory for symmetric hyperbolic system, cf. [10,14].

For part (ii), (3.3) comes from the derivation of (1.7). Then, it follows from Lemma 2.1 that **U** solves (2.3). Finally, the proof of (3.4) is based on the arguments in [11,20]. Note that, using (2.1), (2.2), one has

$$\mathbf{U}(0, \cdot) \in H^{k+1,k} \quad \text{and} \quad \partial_t^l \mathbf{U}(0, \cdot) \in H^{k+1-l,k}$$

as long as $\mathbf{u}_0 \in H^{k+2,k}$ and $\rho_0 = \nabla \times \mathbf{u}_0$. □

4. Global existence and asymptotic behavior of solutions with zero relative vorticity

Throughout this section, we focus on the solutions with zero relative vorticity – cf. (1.7). Theorem 3.1, part (ii), suggests that the RSW system be treated as a system of quasilinear Klein–Gorden equations. To this end, it is convenient to introduce the following generalized Sobolev norms associated with vector fields Γ defined in (3.1),

$$|\mathbf{U}|_{l,d}(t) := \sum_{|\alpha| \leqslant l} \big\| \big(1 + t + |x|\big)^{-d} \Gamma^\alpha \mathbf{U}(t, x) \big\|_{L_x^\infty},$$

$$\|\mathbf{U}\|_{H_\Gamma^l}(t) := \sum_{|\alpha| \leqslant l} \big\| \Gamma^\alpha \mathbf{U}(t, x) \big\|_{L_x^2}.$$

To extend the local solution of (2.1) and (2.2) globally in time, it suffices to prove a global *a priori* estimates for the solutions to (2.3). We start with defining a functional (see e.g. [16]) measuring the size of the solution at time $t \geqslant 0$,

$$X(t) := \sup_{s\in[0,t]} \Big\{ |\mathbf{U}|_{k-25,-1}(s) + \|\mathbf{U}\|_{H_\Gamma^{k-9}}(s) + \|\partial\mathbf{U}\|_{H_\Gamma^{k-9}}(s) \\ + (1+s)^{-\sigma}\|\mathbf{U}\|_{H_\Gamma^k}(s) + (1+s)^{-\sigma}\|\partial\mathbf{U}\|_{H_\Gamma^k}(s)\Big\}, \tag{4.1}$$

here, pick any fixed $\sigma \in (0,1/2)$ and $k \geqslant 52$.

We then state and prove the following global existence result regarding any symmetric quasilinear system of Klein–Gordon equations in 2D. Two key lemmas used in the proof will be given and proved immediately after this.

Theorem 4.1. *For any $k \geqslant 52$, there exists a universal constant δ_0 such that the system admits a unique classical solution for all times if*

$$\|\mathbf{U}_0\|_{H^{k+1,k}} + \|\partial_t \mathbf{U}_0\|_{H^{k+1,k}} = \delta < \delta_0.$$

In particular, $X(t) \leqslant C\delta$ holds uniformly for all positive times.

Proof. By the definition of $X(t)$ and local existence (3.4) of Theorem 3.1, there exists T such that

$$X(T) \leqslant 4C_1\delta. \tag{4.2}$$

Here, we choose constant C_1 to be greater than all constants appearing in Lemma 4.1 and 4.2 below. Then, choose δ to be sufficiently small so that the assumptions of Lemma 4.1 and 4.2 are satisfied, which in turn implies

$$X(T) \leqslant 2C_1\delta + 32C_1^3\delta^2.$$

Impose one more smallness condition on δ so that $X(T) \leqslant 3C_1\delta$ in the above estimate. Finally, by the continuity argument, we can extend T in (4.2) to infinity, i.e., have $X(t) \leqslant 4C_1\delta$ uniformly for all positive times. □

The following lemmas provide estimates on the lower order norms and highest order norms of $X(t)$, respectively. The quadratic term $X^2(t)$, rather than linear term, on the right-hand side of these estimates guarantees that we can extend such estimates globally as long as $X(t)$ stays sufficiently small.

Lemma 4.1. *Assume $\|\mathbf{U}_0\|_{H^{k+1,k}} \leqslant 1$, $X(t) \leqslant 1$. Then, the solution $\mathbf{U}$ of (2.3) satisfies*

$$|\mathbf{U}|_{k-25,-1}(t) + \|\mathbf{U}\|_{H_\Gamma^{k-9}}(t) + \|\partial\mathbf{U}\|_{H_\Gamma^{k-9}}(t) \leqslant C\big(\|\mathbf{U}_0\|_{H^{k+1,k}} + \|\partial_t\mathbf{U}_0\|_{H^{k+1,k}} + X^2(t)\big) \tag{4.3}$$

as long as the classical solution exists. Here, the constant C is independent of δ and t.

Before we give the proof, we have the following remark:

Remark 4.1. The dispersive estimate in (4.3) is mainly a consequence of linear Klein–Gordon structure, which is due to rotation as mentioned in Remark 2.1. If the rotation was absent, $\mathbf{U}$ should satisfy a system of nonlinear wave equations and the decay rate in its L^∞ type norms should be $O(t^{-1/2})$ instead of $O(t^{-1})$ as suggested by the $|\mathbf{U}|_{k-25,-1}(t)$ term of (4.3).

Proof. We start the proof with defining the bilinear form associated with kernel $Q(y,z)$,

$$[G, Q, H](x) := \int_{\mathrm{R}^2\times\mathrm{R}^2} G^T(y) Q(x-y, x-z) H(z)\,dy\,dz$$
$$= \frac{1}{(2\pi)^4} \int_{\mathrm{R}^2\times\mathrm{R}^2} e^{i(\xi+\eta)\cdot x} \hat{G}^T(\xi) \hat{Q}(\xi,\eta) \hat{H}(\eta)\,d\xi\,d\eta.$$

Here, $G(\cdot)$, $H(\cdot)$ are any (2×1)-vector-valued functions defined on R^2 and $Q(\cdot,\cdot)$ is (2×2)-matrix-valued distribution defined on $\mathrm{R}^2\times\mathrm{R}^2$. Fourier transform is denoted by ˆ in R^2 and in $\mathrm{R}^2\times\mathrm{R}^2$.

The same notation will be used for scalar-valued functions

$$[g, q, h](x) := \int_{\mathrm{R}^2\times\mathrm{R}^2} g(y) q(x-y, x-z) h(z)\,dy\,dz.$$

We follow the construction of [21] to transform (2.3) in terms of the new variable

$$\mathbf{V} = (V_1, V_2, V_3)^T = \mathbf{U} + \mathbf{W} = \mathbf{U} + (W_1, W_2, W_3)^T \tag{4.4}$$

where

$$W_k := \sum_{i,j=1}^{3} \left[\begin{pmatrix} U_i \\ \partial_t U_i \end{pmatrix}, Q_k^{ij}, \begin{pmatrix} U_j \\ \partial_t U_j \end{pmatrix} \right], \quad k = 1, 2, 3. \tag{4.5}$$

The kernels Q_k^{ij} are to be determined later so that the new variable $\mathbf{V}$ satisfies a Klein–Gordon system with *cubic* nonlinearity for which estimate (4.3) will be proved using techniques from [16,7].

Without loss of generality, we will demonstrate the proof using V_1, U_1, W_1, Q_1^{ij} associated with the mass equation. From now on, the subscript “1” is neglected for simplicity.

Step 1. We claim that there exist kernels Q^{ij} in (4.5) such that $V = U + W$ satisfies the following Klein–Gordon equation with cubic and quartic nonlinearity,

$$(\partial_{tt} - \Delta + 1) V = S \tag{4.6}$$

where

$$S := \sum_{\substack{|\alpha|+|\beta|+|\gamma|\leqslant 4 \\ \max\{|\alpha|,|\beta|,|\gamma|\}\leqslant 3}} \sum_{a,b,c=1}^{3} \left[\partial^\alpha U_a \partial^\beta U_b, q^{abc}_{\alpha\beta\gamma}, \partial^\gamma U_c \right]$$
$$+ \sum_{\substack{|\alpha|+|\beta|+|\gamma|+|\zeta|\leqslant 4 \\ \max\{|\alpha|,|\beta|,|\gamma|,|\zeta|\}\leqslant 3}} \sum_{a,b,c,d=1}^{3} \left[\partial^\alpha U_a \partial^\beta U_b, q^{abcd}_{\alpha\beta\gamma\zeta}, \partial^\gamma U_c \partial^\zeta U_d \right] \tag{4.7}$$

with $q^{abc}_{\alpha\beta\gamma}$, $q^{abcd}_{\alpha\beta\gamma\zeta}$ being linear combinations of the entries of all Q_k^{ij}'s. Moreover, all the Q_k^{ij}'s satisfy the growth condition

$$\left|D^N \hat{Q}(\xi,\eta)\right| \leqslant C_N\left(1+|\xi|^4+|\eta|^4\right) \tag{4.8}$$

for any nonnegative integer N.

Indeed, substitute V on the left-hand side of (4.6) with $V = U + W$ where W is defined in (4.5) for $k = 1$ and U satisfies the first equation of (2.3),

$$(\partial_{tt} - \Delta + 1)V = (\partial_{tt} - \Delta + 1)U + (\partial_{tt} - \Delta + 1)W. \tag{4.9}$$

By (2.3), the U terms on the right-hand side of (4.9) amount to

$$(\partial_{tt} - \Delta + 1)U = \sum_{i,j=1}^{3} \left[\begin{pmatrix} U_i \\ \partial_t U_i \end{pmatrix}, P^{ij}, \begin{pmatrix} U_j \\ \partial_t U_j \end{pmatrix}\right] \tag{4.10}$$

with $\hat{P}^{ij}(\xi,\eta)$ being 2×2 matrices of polynomials with degree less than or equal to 2. By (4.5), the W terms on the right-hand side of (4.9) are of the form

$$(\partial_{tt} - \Delta + 1)W = \sum_{i,j=1}^{3} \left[\begin{pmatrix} U_i \\ \partial_t U_i \end{pmatrix}, \mathcal{A}Q^{ij}, \begin{pmatrix} U_j \\ \partial_t U_j \end{pmatrix}\right] + S, \tag{4.11}$$

where S satisfies (4.7) and linear transform $\mathcal{A}$ is defined as

$$\widehat{\mathcal{A}Q}(\xi,\eta) := 2\begin{pmatrix} 0 & |\xi|^2+1 \\ -1 & 0 \end{pmatrix} \hat{Q} \begin{pmatrix} 0 & -1 \\ |\eta|^2+1 & 0 \end{pmatrix} + (2\xi \cdot \eta - 1)\hat{Q}. \tag{4.12}$$

Combining (4.9)–(4.12), we find that, for proving (4.6)–(4.8), it suffices to show that there exist solutions $\hat{Q}^{ij}(\xi,\eta)$ to

$$\hat{P}^{ij}(\xi,\eta) + \widehat{\mathcal{A}Q}^{ij}(\xi,\eta) \equiv 0 \tag{4.13}$$

that satisfy the growth condition (4.8). This part of the calculation only involves basic linear algebra and calculus so we neglect the details.

Step 2. We apply the decay estimate of Georgiev in [7] to obtain the L^∞ estimate for V and therefore U with $O(t^{-1})$ decay in time.

Theorem 4.2. *(See [7, Theorem 1].) Suppose $u(t,x)$ is a solution of*

$$(\partial_{tt} - \Delta + 1)u = f(t,x).$$

Then, for $t \geqslant 0$, we have

$$\begin{aligned}\left|\left(1+t+|x|\right)u(t,x)\right| &\leqslant C\sum_{n=0}^{\infty}\sum_{|\alpha|\leqslant 4}\sup_{s\in(0,t)}\phi_n(s)\left\|\left(1+s+|y|\right)\Gamma^\alpha f(s,y)\right\|_{L^2_y} \\ &\quad + C\sum_{n=0}^{\infty}\sum_{|\alpha|\leqslant 5}\left\|\left(1+|y|\right)\phi_n(y)\Gamma^\alpha u(0,y)\right\|_{L^2_y}.\end{aligned}$$

Here, $\{\phi_n\}_{n=0}^{\infty}$ is a Littlewood–Paley partition of unity,

$$\sum_{n=0}^{\infty}\phi_n(s)=1,\quad s\geqslant 0;\qquad \phi_n\in C_0^{\infty}(\mathrm{R}),\qquad \phi_n\geqslant 0\quad \textit{for all } n\geqslant 0,$$
$$\operatorname{supp}\phi_n=\left[2^{n-1},2^{n+1}\right]\quad \textit{for } n\geqslant 1,\qquad \operatorname{supp}\phi_0\cap\mathrm{R}_+=(0,2].$$

Apply Γ^α on the Klein–Gordon equation (4.6) and use the commutation properties of the vector fields to obtain $(\partial_{tt}-\Delta+1)\Gamma^\alpha V=\Gamma^\alpha S$ so that by Theorem 4.2,

$$\bigl|\bigl(1+t+|x|\bigr)\Gamma^\alpha V(t,x)\bigr|\leqslant C\sum_{n=0}^{\infty}\sum_{|\beta|\leqslant|\alpha|+4}\sup_{s\in(0,t)}\phi_n(s)\bigl\|\bigl(1+s+|y|\bigr)\Gamma^\beta S(s,y)\bigr\|_{L_y^2}$$
$$+C\sum_{n=0}^{\infty}\sum_{|\beta|\leqslant|\alpha|+5}\bigl\|\bigl(1+|y|\bigr)\phi_n(y)\Gamma^\beta V(0,y)\bigr\|_{L_y^2}.$$

Since $V=U+W$, we immediately have estimates for $|(1+t+|x|)\Gamma^\alpha U|$. After taking summation over all α's with $|\alpha|\leqslant k-25$, we arrive at

$$|U|_{k-25,-1}(t)\leqslant|W|_{k-25,-1}+C\left(\sum_{n=0}^{\infty}\sum_{|\beta|\leqslant k-21}\sup_{s\in(0,t)}\phi_n(s)\bigl\|\bigl(1+s+|y|\bigr)\Gamma^\beta S(s,y)\bigr\|_{L_y^2}\right.$$
$$\left.+\bigl\|U(0,y)\bigr\|_{H^{k+1,k}}+\sum_{|\beta|\leqslant k-20}\bigl\|\bigl(1+|y|\bigr)\Gamma^\beta W(0,y)\bigr\|_{L_y^2}\right).\tag{4.14}$$

To obtain estimate on each term, we use the following proposition, the proof of which is given in Appendix A.

Proposition 4.1. *Let Fourier transform of the scalar function $q:\mathrm{R}^2\times\mathrm{R}^2\mapsto\mathrm{R}$ satisfy the growth condition (4.8). Let $f(t,x)$, $g(t,x)$, $h(t,x)$ be functions with sufficient regularity. Consider $p=\infty$ (respectively $p=2$). Then, at each $t\geqslant 0$, for $a=|\beta|+8$ (respectively $a=|\beta|+6$) and $b=\lceil\frac{|\beta|}{2}\rceil+7$ where $\lceil\frac{|\beta|}{2}\rceil$ is the maximal integer less than or equal to $\frac{|\beta|}{2}$, one has*

$$\bigl\|\bigl(1+t+|x|\bigr)\Gamma^\beta[f,q,g]\bigr\|_{L_x^p}\leqslant C\bigl(\|f\|_{H_\Gamma^a}|g|_{b,-1}+|f|_{b,-1}\|g\|_{H_\Gamma^a}\bigr),\tag{4.15}$$

$$\bigl\|\bigl(1+t+|x|\bigr)\Gamma^\beta[f,q,gh]\bigr\|_{L_x^p}$$
$$\leqslant C(1+t)^{-1}\bigl(\|f\|_{H_\Gamma^a}|g|_{b,-1}|h|_{b,-1}+|f|_{b,-1}\bigl(\|g\|_{H_\Gamma^a}|h|_{\lceil\frac{a}{2}\rceil,-1}+|g|_{\lceil\frac{a}{2}\rceil,-1}\|h\|_{H_\Gamma^a}\bigr)\bigr).\tag{4.16}$$

Apply Proposition 4.1 (with $p=\infty$) on the first term, and (with $p=2$) on the second and fourth terms of the right-hand side of (4.14) and use the definition of W in (4.5) and S in (4.6),

$$|U|_{k-25,-1}(t)\leqslant CX^2(t)+C\sum_{n=0}^{\infty}\sup_{s\in(0,t)}\phi_n(s)(1+s)^{-1}\bigl(X^3(t)+X^4(t)\bigr)$$
$$+C\bigl(\|U_0\|_{H^{k+1,k}}+\|\mathbf{U}_0\|_{H^{k+1,k}}^2\bigr).$$

We finish the L^∞ estimate part of (4.3) using the fact that

$$\sum_{n=0}^{\infty} \sup_{s\in(0,t)} \phi_n(s)(1+s)^{-1} < \frac{5}{2}$$

and the assumptions $\|\mathbf{U}_0\|_{H^{k+1,k}} \leqslant 1$ and $X(t) \leqslant 1$.

Step 3. We derive the L^2 estimate part regarding the terms $\|\mathbf{U}\|_{H_\Gamma^{k-9}}(t) + \|\partial\mathbf{U}\|_{H_\Gamma^{k-9}}(t)$ by a very similar approach as in Step 2. In fact, apply Γ^α on the Klein–Gordon equation (4.6) and use the commutation properties of the vector fields to obtain $(\partial_{tt} - \Delta + 1)\Gamma^\alpha V = \Gamma^\alpha S$. Then, we take the inner product of this equation with $\partial_t \Gamma^\alpha V$ and sum over all α with $|\alpha| \leqslant k-9$ to obtain

$$\big\|V(t,x)\big\|_{H_\Gamma^{k-9}} + \big\|\partial V(t,x)\big\|_{H_\Gamma^{k-9}} \leqslant C\int_0^t \big\|S(s,x)\big\|_{H_\Gamma^{k-9}}\,ds + C\big(\big\|V(0,x)\big\|_{H_\Gamma^{k-9}} + \big\|\partial V(0,x)\big\|_{H_\Gamma^{k-9}}\big).$$

Since $V = U + W$, we have

$$\begin{aligned}
&\big\|U(t,x)\big\|_{H_\Gamma^{k-9}} + \big\|\partial U(t,x)\big\|_{H_\Gamma^{k-9}} \\
&\quad\leqslant \big(\big\|W(t,x)\big\|_{H_\Gamma^{k-9}} + \big\|\partial W(t,x)\big\|_{H_\Gamma^{k-9}}\big) + C\int_0^t \big\|S(s,x)\big\|_{H_\Gamma^{k-9}}\,ds \\
&\qquad + C\big(\big\|U(0,x)\big\|_{H_\Gamma^{k-9}} + \big\|\partial U(0,x)\big\|_{H_\Gamma^{k-9}} + \big\|W(0,x)\big\|_{H_\Gamma^{k-9}} + \big\|\partial W(0,x)\big\|_{H_\Gamma^{k-9}}\big) \\
&\quad =: I + II + III.
\end{aligned}$$

The estimates of the I, II, III terms above follow closely to that in (4.14), where we use Proposition 4.1 repeatedly and the fact that $k \geqslant 52$. First,

$$I \leqslant (1+t)^{-1} \sum_{|\beta|\leqslant k-9} \big\|\big(1+t+|x|\big)\Gamma^\beta W(t,x)\big\|_{L_x^2}.$$

Using (4.5) and Proposition 4.1, we have

$$I \leqslant C(1+t)^{-1}\big\|\mathbf{U}(t,x)\big\|_{H_\Gamma^k}\big|\mathbf{U}(t,x)\big|_{k-25,-1} \leqslant C(1+t)^{-1+\sigma}X^2(t).$$

Second, since S contains at most third order derivatives of $\mathbf{U}$, we get

$$II \leqslant \int_0^t (1+s)^{-1} \sum_{|\beta|\leqslant k-9} \big\|\big(1+s+|x|\big)\Gamma^\beta S(s,x)\big\|_{L_x^2}\,ds.$$

It follows from (4.7) and Proposition 4.1 that one has

$$II \leqslant C \int_0^t (1+s)^{-2} \left\| \mathbf{U}(s,x) \right\|_{H^k_\Gamma} \left(\left| \mathbf{U}(s,x) \right|^2_{k-25,-1} + \left| \mathbf{U}(s,x) \right|^3_{k-25,-1} \right)$$

$$\leqslant C \int_0^t (1+s)^{-2+\sigma} \left(X^3(s) + X^4(s) \right) ds.$$

Finally, we have

$$III \leqslant C \left(\left\| U(0,y) \right\|_{H^{k+1,k}} + \left\| \mathbf{U}(0,y) \right\|^2_{H^{k+1,k}} \right).$$

These estimates finish the proof of Lemma 4.1 given assumptions $\|\mathbf{U}_0\|_{H^{k+1,k}} \leqslant 1$ and $X(t) \leqslant 1$. □

Note that in the above estimates for I and II, we use the k-th order norms to bound all lower order norms. In order to get the global *a priori* estimate, we have to close the estimates for the highest order norms. For the RSW system, this is achieved by the energy estimates on the highest order Sobolev norms $\| \cdot \|_{H^k_\Gamma}$, where its symmetric structure shown in Lemma 2.1 plays a crucial role. Furthermore, our deliberate choice of growth for the highest order norm helps handle the $O(t^{-1})$ decay rate which is not integrable.

Lemma 4.2. *Assume* $\|A_{ij}(\mathbf{U})\|_{L^\infty} \leqslant 1/4$. *Then, the solution* $\mathbf{U}$ *of* (2.3) *satisfies*

$$(1+t)^{-\sigma} \left(\|\mathbf{U}\|_{H^k_\Gamma}(t) + \|\partial \mathbf{U}\|_{H^k_\Gamma}(t) \right) \leqslant C \left(\|\mathbf{U}_0\|_{H^{k+1,k}} + \|\partial_t \mathbf{U}_0\|_{H^{k+1,k}} + X^2(t) \right)$$

as long as the classical solution exists. Here, constant C is independent of δ and t.

Proof. The proof of this lemma combines the ideas in [9,16,6] for energy estimates for Klein–Gordon equations.

Define an energy functional

$$F(t) := \frac{1}{2} \sum_{|\alpha| \leqslant k} \left(\|\partial_t \Gamma^\alpha \mathbf{U}\|^2_{L^2} + \|\nabla \Gamma^\alpha \mathbf{U}\|^2_{L^2} + \|\Gamma^\alpha \mathbf{U}\|^2_{L^2} \right)(t)$$

$$+ \frac{1}{2} \sum_{|\alpha| \leqslant k} \sum_{i,j=1}^{2} \left\langle A_{ij}(\mathbf{U}) \partial_i \Gamma^\alpha \mathbf{U}, \partial_j \Gamma^\alpha \mathbf{u} \right\rangle (t),$$

where $\langle , \rangle$ is the L^2 inner product defined by $\langle f, g \rangle = \int_{\mathrm{R}^2} f^T g \, dx$. Clearly, by the commutation property of Γ, ∂ and the assumption $\|A_{ij}(\mathbf{U})\|_{L^\infty} \leqslant \frac{1}{4}$, we have

$$C_1 \sqrt{F(t)} \leqslant \|\mathbf{U}\|_{H^k_\Gamma}(t) + \|\partial \mathbf{U}\|_{H^k_\Gamma}(t) \leqslant C_2 \sqrt{F(t)}, \tag{4.17}$$

which ensures the equivalence of $\sqrt{F(t)}$ and $\|\mathbf{U}\|_{H^k_\Gamma}(t) + \|\partial \mathbf{U}\|_{H^k_\Gamma}(t)$.

Applying Γ^α to (2.3) and taking the L^2 inner product on the resulting system with $\partial_t \Gamma^\alpha \mathbf{U}$ (i.e. $\partial_0 \Gamma^\alpha \mathbf{U}$), it follows from Leibniz's rule that

$$\big\langle (\partial_{tt} - \Delta + 1)\Gamma^\alpha \mathbf{U}, \partial_t \Gamma^\alpha \mathbf{U}\big\rangle = \sum_{i,j=1}^{2} \big\langle A_{ij}(\mathbf{U})\partial_{ij}\Gamma^\alpha \mathbf{U}, \partial_0 \Gamma^\alpha \mathbf{U}\big\rangle + \sum_{j=1}^{2}\big\langle A_{0j}\partial_{0j}\Gamma^\alpha \mathbf{U}, \partial_0\Gamma^\alpha \mathbf{U}\big\rangle + \sum_{|\beta|+|\gamma|\leqslant|\alpha|} \big\langle R^{\beta\gamma}\big(\Gamma^\beta \tilde{\mathbf{U}} \otimes \Gamma^\gamma \tilde{\mathbf{U}}\big), \partial_0 \Gamma^\alpha \mathbf{U}\big\rangle,$$

where all $R^{\beta\gamma}$'s are linear functions. Here $\tilde{\mathbf{U}} = (\mathbf{U}^T, \partial_t \mathbf{U}^T, \partial_1 \mathbf{U}^T, \partial_2 \mathbf{U}^T)$.

Upon integrating by parts, one has

$$\begin{aligned} &\frac{1}{2}\partial_t\big(\|\partial_t \Gamma^\alpha \mathbf{U}\|_{L^2}^2 + \|\nabla \Gamma^\alpha \mathbf{U}\|_{L^2}^2 + \|\Gamma^\alpha \mathbf{U}\|_{L^2}^2\big) \\ &\quad = -\sum_{i,j=1}^{2} \big\langle A_{ij}\partial_i \Gamma^\alpha \mathbf{U}, \partial_0\partial_j \Gamma^\alpha \mathbf{U}\big\rangle - \big\langle \partial_j A_{ij}(\mathbf{U})\partial_i \Gamma^\alpha \mathbf{U}, \partial_0 \Gamma^\alpha \mathbf{U}\big\rangle \\ &\qquad - \frac{1}{2}\sum_{j=1}^{2}\big\langle \partial_j A_{0j}\partial_0 \Gamma^\alpha \mathbf{U}, \partial_0 \Gamma^\alpha \mathbf{U}\big\rangle + \sum_{|\beta|+|\gamma|\leqslant|\alpha|} \big\langle R^{\beta\gamma}\big(\Gamma^\beta \tilde{\mathbf{U}} \otimes \Gamma^\gamma \tilde{\mathbf{U}}\big), \partial_0 \Gamma^\alpha \mathbf{U}\big\rangle. \end{aligned} \tag{4.18}$$

Since A_{ij} and A_{0j} are symmetric matrices, we have

$$\big\langle A_{ij}\partial_i \Gamma^\alpha \mathbf{U}, \partial_0 \partial_j \Gamma^\alpha \mathbf{U}\big\rangle = -\frac{1}{2}\partial_t\big\langle A_{ij}\partial_i \Gamma^\alpha \mathbf{U}, \partial_j \Gamma^\alpha \mathbf{U}\big\rangle + \frac{1}{2}\big\langle \partial_0 A_{ij}\partial_i \Gamma^\alpha \mathbf{U}, \partial_j \Gamma^\alpha \mathbf{U}\big\rangle.$$

Summing for all indices α with $|\alpha| \leqslant k$ in (4.18) and using the fact that $\min\{|\beta|, |\gamma|\} \leqslant k/2 < k - 25$ in the last terms on the right-hand side of (4.18), one obtains

$$\partial_t F(t) \leqslant C|\mathbf{U}|_{k-25,0} F(t).$$

Thus,

$$\partial_t \sqrt{F(t)} \leqslant C(1+t)^{-1}|\mathbf{U}|_{k-25,-1}(1+t)^{\sigma}(1+t)^{-\sigma}\sqrt{F(t)}.$$

By the virtue of (4.17) and the definition of $X(t)$ in (4.1), we have

$$\begin{aligned} \|\mathbf{U}\|_{H^k_\Gamma} + \|\partial \mathbf{U}\|_{H^k_\Gamma} &\leqslant C\big(\sqrt{F(t)} - \sqrt{F(0)}\big) + C\sqrt{F(0)} \\ &\leqslant C\int_0^t (1+s)^{-1} X(s)(1+s)^{\sigma} X(s)\,ds + C\|\mathbf{U}_0\|_{H^{k+1,k}} \\ &\leqslant C(1+t)^{\sigma} X^2(t) + C\|\mathbf{U}_0\|_{H^{k+1,k}}, \end{aligned}$$

which finishes the proof of Lemma 4.2. □

The proofs of these lemmas also help reveal the asymptotic behavior of U in the following theorem.

Theorem 4.3. *Under the same assumptions as in Theorem* 4.1, *there exists a free solution* $\mathbf{U}^+$ *of linear Klein–Gordon system such that*

$$\left\|\mathbf{U}(t,\cdot)-\mathbf{U}^+(t,\cdot)\right\|_{H^{k-15}}+\left\|\partial_t\mathbf{U}(t,\cdot)-\partial_t\mathbf{U}^+(t,\cdot)\right\|_{H^{k-16}}\leqslant C(1+t)^{-1}, \tag{4.19}$$

where

$$\mathbf{U}^+(t,\cdot):=\cos\bigl((1-\Delta)^{1/2}t\bigr)\mathbf{U}_0^+ +(1-\Delta)^{-1/2}\sin\bigl((1-\Delta)^{1/2}t\bigr)\mathbf{U}_1^+ \tag{4.20}$$

for some initial data $\mathbf{U}_0^+\in H^{k-15}$ *and* $\mathbf{U}_1^+\in H^{k-16}$.

Proof. Recall the definitions of $\mathbf{V}$ in (4.4), $\mathbf{W}$ in (4.5) and S in (4.6) and (4.7). Without loss of generality, switch the notations in (4.6) to boldface $\mathbf{V}$ and $\mathbf{S}$ while cubic nonlinearity of $\mathbf{S}$ remains valid. Then, Theorem 4.1 and Proposition 4.1 imply that

$$\left\|\mathbf{S}(s,\cdot)\right\|_{H^{k-15}}\leqslant C(1+s)^{-2}. \tag{4.21}$$

Therefore, we can define

$$\mathbf{U}_0^+:=\mathbf{V}(0,\cdot)-\int_0^\infty(1-\Delta)^{-1/2}\sin\bigl((1-\Delta)^{1/2}s\bigr)\mathbf{S}(s,\cdot)\,ds\in H^{k-15} \tag{4.22}$$

and

$$\mathbf{U}_1^+:=\partial_t\mathbf{V}(0,\cdot)+\int_0^\infty\cos\bigl((1-\Delta)^{1/2}s\bigr)\mathbf{S}(s,\cdot)\,ds\in H^{k-16}. \tag{4.23}$$

Then, applying the Duhamel's principle on (4.6) yields

$$\begin{aligned}\mathbf{V}(t,\cdot)&=\cos\bigl((1-\Delta)^{1/2}t\bigr)\mathbf{V}(0,\cdot)+(1-\Delta)^{-1/2}\sin\bigl((1-\Delta)^{1/2}t\bigr)\partial_t\mathbf{V}(0,\cdot)\\&\quad+\int_0^t(1-\Delta)^{-1/2}\sin\bigl((1-\Delta)^{1/2}(t-s)\bigr)\mathbf{S}(s,\cdot)\,ds.\end{aligned} \tag{4.24}$$

It follows from (4.20), (4.22), (4.23), and (4.24) that we have

$$\mathbf{V}(t,\cdot)-\mathbf{U}^+(t,\cdot)=\int_t^\infty(1-\Delta)^{-1/2}\sin\bigl((1-\Delta)^{1/2}(t-s)\bigr)\mathbf{S}(t,\cdot)\,ds.$$

Therefore, by (4.21),

$$\left\|\mathbf{V}(t,\cdot)-\mathbf{U}^+(t,\cdot)\right\|_{H^{k-15}}+\left\|\partial_t\mathbf{V}(t,\cdot)-\partial_t\mathbf{U}^+(t,\cdot)\right\|_{H^{k-16}}\leqslant C(1+t)^{-1}.$$

Finally, applying Theorem 4.1 and Proposition 4.1 to $\mathbf{W}$ defined in (4.5) gives

$$\left\|\mathbf{W}(t,\cdot)\right\|_{H^{k-15}} + \left\|\partial_t \mathbf{W}(t,\cdot)\right\|_{H^{k-16}} \leqslant C(1+t)^{-1}.$$

We then conclude that $\mathbf{U} - \mathbf{U}^+ = \mathbf{V} - \mathbf{U}^+ + \mathbf{W}$ also decays like $C(1+t)^{-1}$ as in (4.19). □

5. Lifespan of classical solutions with general initial data

The proof of Theorem 1.2 combines standard energy methods with the estimates from Section 4, in particular, the $O(t^{-1})$ decay rate of L^∞ type norms.

We start with extracting the zero-relative-vorticity part $(\rho_0^K, \mathbf{u}_0^K)$ from the general initial data $(\rho_0, \mathbf{u}_0)$ (supscript "K" here stands for Klein–Gordon). This can be achieved by L^2 projection of $(\rho_0, \mathbf{u}_0)$ onto the function space with zero relative vorticity, but we find that it is easier to use the complementary projection,

$$\rho_0 - \rho_0^K := (\Delta - 1)^{-1}(\partial_1 u_{2,0} - \partial_2 u_{1,0} - \rho_0),$$
$$\mathbf{u}_0 - \mathbf{u}_0^K := \left(\partial_2\left(\rho_0 - \rho_0^K\right), \partial_1\left(\rho_0 - \rho_0^K\right)\right).$$

Using the notations from Theorem 1.2, we have

$$\left\|(\rho_0, \mathbf{u}_0) - \left(\rho_0^K, \mathbf{u}_0^K\right)\right\|_{H^3} \leqslant C\|\partial_1 u_{2,0} - \partial_2 u_{1,0} - \rho_0\|_{H^2} = C\varepsilon, \tag{5.1}$$

$$\left\|\left(\rho_0^K, \mathbf{u}_0^K\right)\right\|_{H^{k+1,k}} \leqslant C\left\|(\rho_0, \mathbf{u}_0)\right\|_{H^{k+1,k}} = C\delta. \tag{5.2}$$

Let $m^K := 2(\sqrt{1+\rho^K} - 1)$, then $\mathbf{U}^K := (m^K, u_1^K, u_2^K)^T$ solves the symmetrized RSW system (2.1), (2.2) as well as the Klein–Gordon system (2.3). By choosing δ in (5.2) to be sufficiently small, we have $\mathbf{U}_0^K$ satisfy the assumptions in Theorem 4.1, so that there exists a unique global solution $\mathbf{U}^K$ to (2.1) and (2.1) associated with the initial data $\mathbf{U}_0^K$. In addition, the following estimate holds true

$$\left|\mathbf{U}^K\right|_{W^{k-25,\infty}} \leqslant \frac{C\delta}{1+t}. \tag{5.3}$$

Now it remains to estimate the difference $\mathbf{E} := \mathbf{U} - \mathbf{U}^K$. To this end, we write the symmetrized RSW system (2.1), (2.2) into the following compact form

$$\partial_t \mathbf{U} + \sum_{j=1}^{2} B_j(\mathbf{U})\partial_j \mathbf{U} = \mathcal{L}(\mathbf{U}), \tag{5.4}$$

with symmetric matrices

$$B_j(\mathbf{U}) := u_j I + \frac{1}{2} m J_j$$

and J_j, $\mathcal{L}$ defined in (2.5).

Since both $\mathbf{U} = \mathbf{U}^K + \mathbf{E}$ and $\mathbf{U}^K$ satisfy the same system (5.4), straightforward calculation shows that $\mathbf{E}$ satisfies

$$\partial_t \mathbf{E} + \sum_{j=1}^{2} B_j(\mathbf{E})\partial_j \mathbf{E} + \sum_{j=1}^{2} B_j\big(\mathbf{U}^K\big)\partial_j \mathbf{E} + \sum_{j=1}^{2} B_j(\mathbf{E})\partial_j \mathbf{U}^K = \mathcal{L}(\mathbf{E}),$$

subject to initial data $\mathbf{E}_0 = (2\sqrt{1+\rho_0} - 2\sqrt{1+\rho_0^K}, \mathbf{u}_0 - \mathbf{u}_0^K)$.

Employing the standard energy method, we have $e(t) := \|\mathbf{E}(t,\cdot)\|_{H^3}$ satisfy an energy inequality,

$$e'(t) \leqslant Ce(t)\big(|\nabla \mathbf{E}|_{L^\infty} + \big|\mathbf{U}^K\big|_{W^{4,\infty}}\big).$$

We then use Sobolev inequalities and estimate (5.3) to further the above inequality as

$$e'(t) \leqslant Ce(t)\left(e(t) + \frac{\delta}{1+t}\right).$$

Dividing it with $e^2(t)$ gives

$$\big(-e^{-1}(t)\big)' \leqslant C\left(1 + \frac{\delta}{1+t}e^{-1}(t)\right)$$

which is linear in terms of $e^{-1}(t)$. We finally arrive at

$$e(t) \leqslant (1+t)^{C\delta}\left(e^{-1}(0) - \frac{C}{1+C\delta}\big[(1+t)^{1+C\delta} - 1\big]\right)^{-1}.$$

By (5.1), the initial value is bounded by $e(0) \leqslant C\varepsilon$. Thus, $e(t)$ remains bounded as long as

$$t < \left(\frac{1+C\delta}{C\varepsilon} + 1\right)^{\frac{1}{1+C\delta}} - 1$$

which is of the same order as (1.9) in Theorem 1.2 under the smallness assumption on δ and the fact that $\varepsilon \leqslant \delta$.

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Appendix A. Proof of Proposition 4.1

We first claim the following estimate on kernel $q(y,z)$.

Proposition A.1. *Let $q(y,z)$ satisfy the growth condition* (4.8)*, i.e.*

$$\big|D^N \hat{q}(\xi,\eta)\big| \leqslant C_N\big(1 + |\xi|^4 + |\eta|^4\big) \tag{A.1}$$

for any $N \geqslant 0$. Then, for

$$q_1 := (1-\Delta_y)^{-3}(1-\Delta_z)^{-3}q(y,z) \tag{A.2}$$

the following estimate holds true

$$\big(1+|y|+|z|\big)^l q_1(y,z) \in L^1\big(\mathrm{R}_y^2 \times \mathrm{R}_z^2\big),$$

for any $l \geqslant 0$.

Proof. By (A.1), the following growth condition for q_1 holds:

$$\big|\hat{q}_1(\xi,\eta)\big| = \frac{|\hat{q}(\xi,\eta)|}{\big(1+|\xi|^2\big)^3\big(1+|\eta|^2\big)^3} \leqslant C\big(1+|\xi|^2\big)^{-1}\big(1+|\eta|^2\big)^{-1}.$$

By induction, we have

$$\big|D^N\hat{q}_1(\xi,\eta)\big| \leqslant C\big(1+|\xi|^2\big)^{-1}\big(1+|\eta|^2\big)^{-1} \in L^2\big(\mathrm{R}_\xi^2 \times \mathrm{R}_\eta^2\big)$$

for any integer $N \geqslant 0$. Therefore, applying the Plancherel Theorem, we have $(1+|y|^N+|z|^N)q_1(y,z) \in L^2_{yz}$, which readily implies

$$\big(1+|y|+|z|\big)^l q_1(y,z) = \big(1+|y|+|z|\big)^{l-N}\cdot\big(1+|y|+|z|\big)^N q_1(y,z) \in L^1\big(\mathrm{R}_y^2 \times \mathrm{R}_z^2\big)$$

for suitably large N. □

To prove Proposition 4.1, we note that (4.16) is a direct consequence of (4.15) and therefore it suffices to prove (4.15) alone. We then show the following estimate that serves as a slightly stronger version of (4.15): for any kernel $q(y,z)$ satisfying the growth condition (A.1), there exists a constant C independent of f, g such that

$$\big\|\big(1+t+|x|\big)\Gamma^\beta[f,q,g]\big\|_{L_x^p} \leqslant C\sum_{i+j=|\beta|}\min\big\{\|f\|_{H_\Gamma^{i+\gamma}}|g|_{j+6,-1},\ |f|_{i+6,-1}\|g\|_{H_\Gamma^{j+\gamma}}\big\} \tag{A.3}$$

where $\gamma = 6$ if $p = 2$ or $\gamma = 8$ if $p = \infty$.

We prove (A.3) by induction.

Step 1. Set $|\beta| = 0$. Upon integrating by parts, the left-hand side of (A.3) becomes

$$\big\|\big(1+t+|x|\big)[f,q,g]\big\|_{L_x^p} = \big\|\big(1+t+|x|\big)[f_1,q_1,g_1]\big\|_{L_x^p} \tag{A.4}$$

where q_1 is defined in (A.2) and

$$f_1(t,y) := (1-\Delta_y)^3 f(t,y), \qquad g_1(t,z) := (1-\Delta_z)^3 g(t,z).$$

It follows from (A.4) that we have

$$\begin{aligned}
&\big\|\big(1+t+|x|\big)[f,q,g]\big\|_{L_x^p} \\
&\quad = \bigg\|\big(1+t+|x|\big)\int\limits_{\mathrm{R}^2\times\mathrm{R}^2} f_1(t,x-y)q_1(y,z)g_1(t,x-z)\,dy\,dz\bigg\|_{L_x^p}
\end{aligned}$$

$$\leqslant \left\| \int_{\mathrm{R}^2\times\mathrm{R}^2} \left|\left(1+t+|x-y|\right) f_1(t,x-y) q_1(y,z) g_1(t,x-z)\right| dy\,dz \right\|_{L^p_x}$$
$$+ \left\| \int_{\mathrm{R}^2\times\mathrm{R}^2} \left| f_1(t,x-y) y q_1(y,z) g_1(t,x-z)\right| dy\,dz \right\|_{L^p_x}$$
$$\leqslant \left|\left(1+t+|y|\right) f_1(t,y)\right|_{L^\infty_y} \left\| q_1(y,z)\right\|_{L^1_{yz}} \left\| g_1(t,z)\right\|_{L^p_z}$$
$$+ \left| f_1(t,y)\right|_{L^\infty_y} \left\| y q_1(y,z)\right\|_{L^1_{yz}} \left\| g_1(t,z)\right\|_{L^p_z},$$

where we used Young's inequality.

Combed with Proposition A.1 (and Sobolev inequality if $p=\infty$), the above estimate leads to

$$\left\|\left(1+t+|x|\right)[f,q,g]\right\|_{L^p_x} \leqslant C|f_1|_{0,-1}\|g_1\|_{L^p} \leqslant C|f|_{6,-1}\|g\|_{H^\gamma_\Gamma}.$$

The same estimate holds if we switch f and g. Thus, we proved (A.3) for $|\beta|=0$.

Step 2. Suppose (A.3) is true for all $(n-1)$-th order vector fields. Now pick any n-th order vector field $\Gamma^\beta := \Gamma^{\beta'}\Gamma^1$ where $|\beta'|=n-1$ and $\Gamma^1 \in \{\partial_t, \partial_1, \partial_2, t\partial_1 + x_1\partial_t, t\partial_2 + x_2\partial_t, x_1\partial_2 - x_2\partial_1\}$. By product rule and the definition of normal forms (4.5), for any $\partial \in \{\partial_t, \partial_1, \partial_2\}$, we have

$$\partial[f,q,g] = [\partial f,q,g] + [f,q,\partial g],$$
$$t\partial[f,q,g] = [t\partial f,q,g] + [f,q,t\partial g],$$
$$x_i\partial[f,q,g] = \left(\left[x_i\partial f(t,x), q(y,z), g(t,x)\right] + \left[\partial f(t,x), y_i q(y,z), g(t,x)\right]\right)$$
$$+ \left(\left[f(t,x), q(y,z), x_i\partial g(t,x)\right] + \left[f(t,x), z_i q(y,z), \partial g(t,x)\right]\right),$$

which immediately implies that

$$\Gamma^\beta[f,q,g] = \Gamma^{\beta'}\left(\left[\Gamma^1 f,q,g\right] + \left[f,q,\Gamma^1 g\right]\right)$$
$$+ \Gamma^{\beta'}\sum_{i=0}^{2}\sum_{j=1}^{2} C_{ij}\left([\partial_i f, y_j q, g] + [f, z_j q, \partial_i g]\right). \tag{A.5}$$

Here, the kernels are $q(y,z)$, $y_j q(y,z)$, $z_j q(y,z)$, all satisfying the growth condition (A.1). Therefore, the inductive hypothesis is true and we apply (A.3) with $|\beta'|=n-1$ on (A.5) to conclude that (A.3) also holds for $|\beta|=n$. This finishes the proof for (A.3).

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