HABILITATIONSSCHRIFT

Integrable systems and differential geometry

Dr. James D.E. Grant

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Included Papers:


Curriculum Vitae
Summary

The work presented in this Habilitationsschrift centres on the interaction between integrable systems theory and differential geometry.

There are two main themes to the work presented here. The first concerns the study of the properties of integrable systems of partial differential equations that arise in a geometrical setting. The main examples in this context are the self-dual Yang-Mills equations and the equations that govern hyper-complex, self-dual Ricci-flat or Grassmann manifolds. Such topics are the main themes of papers [1, 2, 4, 6, 7]. The second main theme concerns the application of techniques from differential geometry and global analysis in the construction and study of certain integrable systems. These are the main topics of papers [3, 5].

\footnote{Numbered citations refer to the papers contained in this thesis. Please see the Table of Contents on Page 2.}
In classical finite-dimensional Hamiltonian mechanics, there is a generally accepted notion of what constitutes an “integrable system”. Specifically, let \((M, \omega)\) be a \(2n\)-dimensional, symplectic manifold and \(H \in C^\infty(M)\) the Hamiltonian of the system. If there exist \(n\) conserved quantities that are in involution, and functionally independent on \(M \setminus S\) (where \(S\) is of measure zero), then the system is defined to be integrable\(^2\). Moreover, the set \(M \setminus S\) is fibred by \(n\)-dimensional tori, and Hamiltonian motion generated by the function \(H\) is simply linear motion on these tori.

When one moves from ordinary differential equations to partial differential equations, there are various inequivalent “definitions” of what an integrable system of partial differential equations should be. These generally centre around the system in question having one or more of the following properties:

- the system admits an infinite-dimensional symmetry algebra;
- the system admits an inverse scattering transform;
- there is an associated linear problem;
- the system admits a Lax pair formulation;
- the system has soliton solutions;

In essence, rather than there being a universal definition of the concept of integrability, one recognises integrable systems by their characteristic features and behaviour (see the Prefaces in \([Gue97, HSW99]\)).

There are two large classes of differential equations that are generally regarded as being integrable. The first are particular types of harmonic map equations, whereas the second are the self-dual Yang-Mills equations on self-dual four-manifolds. Since the harmonic map equations may be constructed as a symmetry reduction of the self-dual Yang-Mills equations\(^3\), we will concentrate on the latter. It has then been proposed by Ward [War90] (see also [MW96]) that the self-dual Yang-Mills equations and, in particular, the twistor theoretic nature of

\(^2\)In the sense of Liouville.

\(^3\)In fact, the self-dual Yang-Mills equations may also be constructed from the harmonic map equations, at the expense of considering harmonic maps into infinite-dimensional manifolds and groups.
these equations [War77] should be looked on as being the underlying characteristic of a set of differential equations that implies its integrability. In particular, almost all known integrable systems arise as symmetry reductions of the self-dual Yang-Mills equations on particular self-dual four-manifolds.

A substantial proportion of the work contained in this thesis investigates integrable systems that arise from the consideration of self-dual systems in four-dimensions, and generalisations of such systems that arise in higher dimensions. The general philosophy is that existence of a (suitable) twistor space formulation of the problem indicates that one is dealing with an integrable system.

In Paper 1, with I.A.B. Strachan (Glasgow), I studied the equations that arise when one considers hyper-complex structures on four-dimensional manifolds, and hyper-symplectic structures that arise in \((- - + +)\) signature. Hyper-complex manifolds (in \(4n\)-dimensions, for \(n = 1, 2, \ldots \)) have a concomittant twistor space which is simply an \(S^2\) bundle over the manifold. Based on work in my Ph.D. thesis [Gra93], we developed an approach to four-dimensional hypercomplex manifolds based on the existence of a set of vector fields that obey special commutation relations. These commutation relations may then be interpreted as Lax equations, therefore exhibiting the integrability of the system in one of the senses mentioned above. We derived a number of different field equations for such hypercomplex manifolds. One of these equations is in involution form, and is particularly well-suited to the study of the symmetry properties and the associated hierarchy of conservation laws.

A more general class of manifolds is considered in Paper 2. Here, I investigated some natural connections that arise between right-flat \((p, q)\) Grassmann structures and integrable systems. In particular, such geometrical structures (which again have natural associated twistor constructions [BE91]) naturally give rise to conditions that may be formulated in Lax form, where one has a “Lax \(p\)-tuple” of linear differential operators, depending a spectral parameter that lives in \((q - 1)\)-dimensional complex projective space. Generally, the differential operators contain partial derivatives with respect to the spectral parameter. Such structures therefore lead to a natural generalisation of the Lax pair formulation of conventional integrable systems. Note that an alternative generalisation of the work in Paper 1 has been considered by David Calderbank in his work on integrable background geometries [Cal].

Papers 4, 6 and 7 should be viewed as an extended single piece of work. These papers begin the exploration of potential connections between integrable systems.

\footnote{The notable exceptions at this point being the KP and Davey-Stewartson hierarchies of equations.}
theory and topology. In particular, the moduli spaces of solutions to the self-dual Yang-Mills equations on four-manifolds play a central rôle in Donaldson’s work on the structure of smooth four-manifolds (see, e.g., [DK90, FU91]). The motivation for the work contained in these papers is to determine whether the integrable systems aspects of the self-dual Yang-Mills equations may give insight into the structure of such moduli spaces. In particular, one might hope that the symmetries of the self-dual Yang-Mills equations would give rise to geometrical structures on the moduli spaces, and corresponding group actions that preserve such structures. In Papers 4 and 6, we consider the action of symmetries of the self-dual Yang-Mills equations on the moduli spaces of solutions to these equations on \( \mathbb{R}^4 \) the curvature of which is \( L^2 \) (so-called instanton solutions). Generally, the action of the symmetry group does not preserve the \( L^2 \) nature of the curvature, and one of the main results of Paper 6 is that, when we restrict to transformations that do preserve this condition, then the group orbits turn out to be rather small. In particular, in the case of the one-instanton moduli space, the orbits of the symmetry group of the self-dual Yang-Mills equations are only one-dimensional orbits on the five-dimensional moduli space. In Paper 7, we studied reducible connections on open subsets of \( \mathbb{R}^4 \). In this case, it was found that all reducible connections on open subsets of \( \mathbb{R}^4 \) lie in the orbit of the flat connection on \( \mathbb{R}^4 \) under the action of the non-local symmetry group of the self-dual Yang-Mills equations. In addition, the reducible connections lie within a larger class of connections that bear many similarities to harmonic maps of finite type (see, e.g., [BP94, BP95, Gue97]).

The main conclusion of Papers 6 and 7 is that instanton solutions to the self-dual Yang-Mills equations on \( \mathbb{R}^4 \) and reducible connections on open subsets of \( \mathbb{R}^4 \) have very different properties under the action of the symmetries of the self-dual Yang-Mills equations. The fact that reducible and irreducible connections play a distinct rôle in Donaldson’s approach to four-manifold topology suggests that there may be a rather deep link between the global properties of integrable systems and topological field theory. Such potential links are currently under investigation.

The remaining papers included in this thesis (Papers 3 and 5) also lie on the interface between integrable systems theory and differential geometry, with a slightly different emphasis.

\footnote{For example, in unpublished work, I showed that when one considers holomorphic (hence harmonic) maps \( \mathbb{C}P^1 \to \mathbb{C}P^n \), then the construction of [MP97] may be used to construct projective structures on the space of holomorphic maps \( \mathbb{C}P^1 \to \mathbb{C}P^2 \) and more general structures for \( n > 2 \). The automorphism group of these structures coincides precisely with the transformations that arise from the action of “dressing transformations” on harmonic maps.}

\footnote{Recall that reducible and irreducible connections play rather distinct rôles in Donaldson theory.}
In Paper 3, with E. Musso (L’Aquila), I studied constrained variational problems defined on the space of integral curves of a Frenet system in a homogeneous space, $G/K$. The approach used involves the application of the theory of exterior differential systems, in particular Pfaffian systems, to variational problems as developed by Cartan [Car71] and formalised by Griffiths [Gri83]. In particular, we show that if a Lagrangian is $G$-invariant and coisotropic, then the extremal curves can be found by quadratures. The proof relies on the reduction theory of Hamiltonian systems with symmetries and a concrete geometric description of the Marsden-Weinstein reduced spaces in terms of the phase portraits of the system. Our approach unifies various known examples of constrained variational problems, such as the total squared curvature functional, the projective, conformal and pseudo-conformal arc-length functionals, the Delaunay and the Poincaré variational problems.

Finally, in Paper 5, with M. Dunajski (Cambridge) and I.A.B. Strachan (Glasgow), I considered the use of deformations of Lie algebra homomorphisms to construct deformations of dispersionless integrable systems arising as symmetry reductions of anti–self–dual Yang–Mills equations with a gauge group $\text{Diff}(S^1)$. It is known that certain dispersionless integrable systems may be constructed as limiting forms of dispersive integrable systems. In recent years, there has been some interest in essentially reversing such limiting processes, and attempting to construct dispersive integrable systems from an original dispersionless integrable system by some sort of “quantisation” procedure. Such a quantisation procedure has been developed in the work of Kupershmidt [Kup90]. In this approach one formulates the dispersionless system in terms of an underlying Poisson structure. One then replaces the Poisson brackets that occur with brackets derived from the Moyal product. Such an approach has been successfully applied to, for example, the dKP and $SU(\infty)$ Toda equations in $2+1$ dimensions, and to Plebanski’s first heavenly equation [Str92] in four dimensions.

In Paper 5, we constructed deformations of multidimensional integrable systems which are based on the algebra of vector fields, $\text{diff}(\Sigma)$, of $\Sigma$, where $\Sigma \cong S^1$ or $\mathbb{R}$. Since this algebra admits no non-trivial deformations, we adopt an alternative method to perturbing our system. This is based on the approach of Ovsienko and Roger [OR98] where a homomorphism from $\text{diff}(\Sigma)$ to the Poisson algebra on $T^*\Sigma$ can be used to construct non-trivial deformations. We shall use this idea to construct integrable deformations of various equations associated to the algebra $\text{diff}(\Sigma)$. To complete the circle, the most interesting example of such a system that we study is a system that arises when one considers symmetry reductions of the equations for hyper-complex manifolds in dimension 4. Such manifolds, we recall, were the topic of Paper 1.
Bibliography


Paper 1

Hypercomplex integrable systems,
with I.A.B. Strachan,
Hypercomplex integrable systems

James D E Grant and I A B Strachan
Department of Mathematics, University of Hull, Hull HU6 7RX, UK
E-mail: j.d.grant@maths.hull.ac.uk and i.a.strachan@maths.hull.ac.uk

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Abstract. In this paper we study hypercomplex manifolds in four dimensions. Rather than using an approach based on differential forms, we develop a dual approach using vector fields. The condition on these vector fields may then be interpreted as Lax equations, exhibiting the integrability properties of such manifolds. A number of different field equations for such hypercomplex manifolds are derived, one of which is in Cauchy–Kovaleskaya form which enables a formal general solution to be given. Various other properties of the field equations and their solutions are studied, such as their symmetry properties and the associated hierarchy of conservation laws.

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1. Introduction

The study of hyperKähler geometries has developed in two distinct directions. Starting with the work of Calabi [1] there has been a purely geometric vein, where such manifolds are constructed and studied geometrically without reference to any defining set of field equations. The second vein starts with such field equations—systems of differential equations, and uses the solutions of such systems to construct the manifolds.

To show how these two approaches are connected it is necessary to restrict one’s attention to four dimensions, where the hyperKähler condition is equivalent to the existence of a metric with an anti-self-dual Weyl tensor and vanishing Ricci tensor. Such metrics were shown by Penrose to have a corresponding twistor space, and conversely, that from such a twistor space one may reconstruct the metric. Such a structure appears at first sight to be special to four dimensions where there is the notion of self-duality, but it was shown in [2] that hyperKähler metrics have a corresponding twistor space in any $4N$-dimensional space.

With more recent work, initiated by Ward [3] (see also [4]), these two threads may be seen to be intimately interwoven. The existence of a (suitable) twistor space indicates that one is dealing with an integrable system. Thus any set of field equations for a hyperKähler metric provides one with an example of a multidimensional integrable system. From a solution to this integrable system one may construct the associated twistor space (and vice versa), whose properties may be studied, and hence properties of the metric, without recourse to the particular differential equation, whose precise form depends on the particular coordinate representation being used.
The study of hypercomplex manifolds has had a similar history. In [5] Finley and Plebański studied field equations which were derived from the following condition on the self-dual two-form $\Sigma^i$:

$$d\Sigma^i = \alpha \wedge \Sigma^i, \quad i = 1, 2, 3,$$

though without mention of the associated complex geometry. Similarly, Boyer [6] studied the geometric aspects, but did not write down field equations for such metrics. In neither case was the link with integrable systems made. The aim of this paper is to show that one may write down systems of differential equations, the solutions of which define hypercomplex manifolds (here we will restrict our attention to four dimensions, though many of the ideas will generalize to $4N$-dimensions). Since hypercomplex manifolds also have associated twistor spaces (in four dimensions the hypercomplex condition implies, though is not implied by, the anti-self-duality of the Weyl tensor [6]) the field equations for these systems will be examples of multicomponent, multidimensional integrable systems.

Since hyperKähler metrics are obviously Kähler they may be written in terms of a single function, the Kähler potential $\Omega$:

$$g = \frac{\partial^2 \Omega}{\partial x^i \partial \tilde{x}^j} \, dx^i \, d\tilde{x}^j, \quad i, j = 1, 2.$$

In four dimensions the hyperKähler conditions result in the differential equation

$$\Omega_{x^1 \tilde{x}^1} \Omega_{x^2 \tilde{x}^2} - \Omega_{x^1 \tilde{x}^2} \Omega_{x^2 \tilde{x}^1} = 1$$

known as Plebański’s equation. In this form it is hard (though not impossible) to apply ideas from the theory of integrable systems, which are best suited to evolutionary-type equations. In [7] the first author showed how, by performing a suitable Legendre transformation one may obtain an equation in evolutionary, or Cauchy–Kovaleskaya form, namely

$$\psi_{tt} = \psi_{zt} \psi_{xy} - \psi_{yz} \psi_{xt}$$

(1)

and the second author [8] showed how to construct the associated integrable hierarchy, based on the study of the generalized symmetries of this equation.

In this paper the following two-component generalization of (1) will be studied:

$$g_{tt} = \{g, g_x\} + \{g_z, h\},$$

$$h_{tt} = \{g, h_x\} + \{h_z, h\}.$$  \hspace{1cm} (2)

From the purely integrable systems aspect, considered in section 3, one may view this system as resulting from dropping the volume preserving condition on the vector fields which appear in the Lax pair used to construct (1). This approach will be followed in section 3. However, the system has a more geometric interpretation: solutions define hypercomplex metrics. This aspect will be considered in the next section. It should be pointed out that these hypercomplex structures are not the most general possible, but a particular subclass. The more general case will be considered in a future paper.

2. Geometrical description

2.1. Hypercomplex geometry

In this section, we wish to investigate a special subclass of four-dimensional hypercomplex structures. In particular, assume we are working on a four-manifold $M$, and that we have a local basis for the tangent space, in the form of a set of four linearly independent vector fields,
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\{e_i : i = 1, \ldots, 4\}. We wish to study the geometrical structures which arise if we assume that these vector fields obey the Lie-bracket relations:

\[ [e_1, e_2] + [e_3, e_4] = 0, \quad [e_1, e_3] + [e_4, e_2] = 0, \quad [e_1, e_4] + [e_2, e_3] = 0. \tag{3} \]

We will show that in this case, the manifold \( M \) is (locally) hypercomplex. Recall that a manifold \( M \) of dimension \( n = 4m \) is hypercomplex if it admits three integrable complex structures, \( I, J, K \), which obey the quaternion multiplication relations:

\[ I^2 = J^2 = K^2 = -\text{Id}_{T^*M}, \quad I \circ J = K, \quad J \circ K = I, \quad K \circ I = J. \]

Such structures imply that the bundle of linear frames \( L(M) \) reduces from a \( GL(4m, \mathbb{R}) \) bundle to a \( GL(m, \mathbb{H}) \) bundle. In the special case of four dimensions, the frame bundle reduces to \( GL(4, \mathbb{H}) \cong \mathbb{H}^* \cong \mathbb{R}^6 \times SU(2) \) bundle. Since \( SU(2) \) is a subgroup of \( SO(4) \), this means that in four dimensions, a hypercomplex structure automatically defines a conformal structure. In particular, there is a metric, unique up to a scale, with respect to which all the complex structures are Hermitian:

\[ g(I X, I Y) = g(J X, J Y) = g(K X, K Y) = g(X, Y), \]

where \( X, Y \) are arbitrary sections of \( T M \). The structures \( I, J, K \) along with a representative metric in the conformal structure, \( g \), define a hyper-Hermitian structure. Each metric in this conformal structure has an anti-self-dual Weyl tensor, with respect to the canonical orientation defined by any of the complex structures.

In the case we wish to study, we have a set of vector fields which satisfy the relations of equation (3). If we define the dual basis \( \{e^i\} \) for \( T^*M \), then we will show that the vector fields define three integrable complex structures which obey the relations given in equation (4) and that the metrics with respect to which all these structures are Hermitian are conformal to the metric:

\[ g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4. \]

The plan of the section is as follows. In the next section, we prove the assertions made above concerning the existence of hypercomplex structures, and a compatible conformal structure. Next the relation between this approach, based on vector fields, and the more usual approach based on forms is given. The vector field approach is useful for two reasons: firstly it enables field equations to be derived easily, and secondly it makes the connection with integrable systems more transparent. We then consider various coordinate versions of these equations.

**Theorem 2.1.** On a four-dimensional manifold \( M \), if there exist four linearly independent, non-vanishing vector fields \( \{e_i : i = 1, 2, 3, 4\} \) on a manifold \( M \) which obey the Lie-bracket relations

\[ [e_1, e_2] + [e_3, e_4] = 0, \quad [e_1, e_3] + [e_4, e_2] = 0, \quad [e_1, e_4] + [e_2, e_3] = 0, \tag{4} \]

then the manifold is locally hypercomplex. Moreover, there exists a unique conformal structure, defined by the representative metric:

\[ g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4 \tag{5} \]

with respect to which all of the complex structures are Hermitian.
Proof. Consider maps \((I, J, K)\) on \(TM\) defined by
\[
I(e_1) = +e_2, \quad I(e_2) = -e_1, \quad I(e_3) = +e_4, \quad I(e_4) = -e_3,
\]
\[
J(e_1) = +e_3, \quad J(e_2) = -e_4, \quad J(e_3) = -e_1, \quad J(e_4) = +e_2,
\]
\[
K(e_1) = +e_4, \quad K(e_2) = +e_3, \quad K(e_3) = -e_2, \quad K(e_4) = -e_1.
\]
To impose that these structures are integrable, we consider the complexification of the tangent space \(T_{\mathbb{C}}M = TM \otimes \mathbb{C}\). Consider first the structure \(I\). The almost complex structure leads to a direct sum decomposition of \(T_{\mathbb{C}}M\) as \(T_{\mathbb{R}}^1 \oplus T_{\mathbb{R}}^2\) where \(v \in T^{(1,0)}\) if \(I(v) = iv\), and \(v \in T^{(0,1)}\) if \(I(v) = -iv\). The complex structure is integrable iff the space \(T^{(1,0)}\) is closed under the Lie bracket. In the case of the structure \(I\), \(T^{(1,0)}\) is spanned by the vector fields \([e_1 + ie_2, e_3 + ie_4]\), and the only non-trivial Lie bracket we must consider is
\[
[e_1 + ie_2, e_3 + ie_4] = [e_1, e_3] + [e_4, e_2] + i([e_1, e_4] + [e_2, e_3]).
\]
For the right-hand side to lie in \(T^{(1,0)}\) implies a number of algebraic relations which, together with the analogous equations for \(J\) and \(K\) and the quaternionic relations, implies that
\[
[e_1, e_2] + [e_3, e_4] = -A_2 e_1 + A_1 e_2 - A_4 e_3 + A_3 e_4,
\]
\[
[e_1, e_3] + [e_4, e_2] = -A_3 e_1 + A_4 e_2 + A_1 e_3 - A_2 e_4,
\]
\[
[e_1, e_4] + [e_2, e_3] = -A_4 e_1 - A_3 e_2 + A_2 e_3 + A_1 e_4
\]
for some set of functions \(\{A_1, A_2, A_3, A_4\}\). Thus by construction the corresponding metric (5) is hypercomplex. It will be useful in what follows to combine these functions into a one-form \(A = A_1 e^1\) and regard this as a connection on \(M\).

Unlike the conditions for a metric to be hyperKähler, the hypercomplex conditions are invariant under conformal changes of the metric. If \(g \mapsto e^{2\Lambda}g\) then the corresponding transformation for the connection \(A\) is
\[
A \mapsto A - d\Lambda.
\]
The subclass of hypercomplex structures examined in this paper are defined by the conformally invariant condition \(dA = 0\), i.e. the connection defined by \(A\) is flat. Thus locally one may define, for this subclass of hypercomplex structures, the conformal factor so that \(A = 0\). This fixes the conformal structure.

It is a straightforward exercise using the explicit form of the complex structures given above to show that metric \(g\) given above is Hermitian with respect to each of the complex structures, and that it is the unique symmetric tensor (up to a rescaling) with this property.

The hypercomplex condition determines a metric up to a conformal factor. The zero curvature condition on the connection \(A\) may be used to fix the conformal factor. This determines the metric uniquely, up to trivial transformations. In terms of a null tetrad in which
\[
g = e^1 \otimes e^2 + e^3 \otimes e^4
\]
the conditions (4) become
\[
[e_1, e_2] + [e_1, e_4] = 0, \quad [e_1, e_3] = 0, \quad [e_2, e_4] = 0.
\]
This form will be used in later sections and is also used in the following example.

Example. Consider the following vector fields
\[
e_1 = \partial_w \quad e_2 = (1 + |w|^2)\partial_{\bar{w}} + \bar{z}w\partial_r,
\]
\[
e_3 = \partial_z \quad e_4 = (1 + |z|^2)\partial_{\bar{z}} + zw\partial_r.
\]
It is easy to verify that these satisfy the conditions (6). Hence, by the above theorem, the corresponding Hermitian hypercomplex metric on $\mathbb{C}^2$ is conformal to
\[
g = (1 + |z|^2) \, dw \, d\overline{w} + (1 + |z|^2) \, dz \, d\overline{z} - z \overline{w} \, dz - \overline{z} \, w \, d\overline{w}.
\]
Note, this metric is locally conformal to the Fubini–Study metric on $\mathbb{CP}^2$. However this construction does not extend from $\mathbb{C}^2$ to $\mathbb{CP}^2$. If it did it would contradict Boyer’s [6] classification of compact hyper-Hermitian manifolds. Other examples, with tri-holomorphic Killing vectors, have been constructed in [9].

For the metric to be hyperKähler the corresponding Kähler forms defined by
\[
\Omega_I(X, Y) = g(IX, Y),
\]
\[
\Omega_J(X, Y) = g(JX, Y),
\]
\[
\Omega_K(X, Y) = g(KX, Y),
\]
must be closed, or equivalently, $\nabla I = \nabla J = \nabla K = 0$. It was shown in [10] that these conditions are equivalent to the vector fields $e_i$ being volume preserving, that is
\[
\mathcal{L}_{e_i} \omega = 0
\]
where $\omega$ is some volume form.

### 2.2. Dual description

The starting point for the study of four-dimensional hypercomplex manifolds has traditionally been the equation
\[
d \Sigma^i = \alpha \wedge \Sigma^i, \quad i = 1, 2, 3
\]
where the $\Sigma^i$ are self-dual two-forms on the manifold $M$. In this paper we have so far used a dual description, using vector fields rather than forms. This section is intended to bridge the gap between these two approaches. It is first necessary to fix some notation. The connection one-forms are defined by
\[
de^i + \Gamma^i_j \wedge e^j = 0,
\]
so, in components, $\Gamma^i_j = \Gamma^i_{jk} e^k$. The skew-symmetric parts of $\Gamma^i_{jk}$ are related to the structure functions defined by the Lie bracket $[e_j, e_k] = c_{jk}^i e_i$ by
\[
\Gamma^i_{jk} = \frac{1}{2} c_{jk}^i.
\]
This formula enables one to connect these two approaches.

**Proposition 2.1.** The connection between equations (5) and (7) is given by the formulae
\[
\alpha = A - \chi
\]
where
\[
A = A_i e^i,
\]
\[
\chi = c_{ij} e^i.
\]

**Proof.** Consider the self-dual two-form $\Sigma^1 = e^1 \wedge e^2 + e^3 \wedge e^4$. Then
\[
d \Sigma^1 = d(e^1 \wedge e^2 + e^3 \wedge e^4)
\]
\[
= -\Gamma_{[a}^i b] e^a \wedge e^b + \Gamma_{[a}^{2 b]} e^a \wedge e^b \wedge e^1
\]
\[
-\Gamma_{[a}^{3 b]} e^a \wedge e^b \wedge e^4 + \Gamma_{[a}^{4 b]} e^a \wedge e^b \wedge e^3.
\]
\[
\frac{1}{2} c_{ab} e^a \wedge e^b \wedge \epsilon^2 + \frac{1}{2} c_{ab} e^a \wedge e^b \wedge \epsilon^1 - \frac{1}{2} c_{ab} e^a \wedge e^b \wedge \epsilon^4 \\
+ \frac{1}{2} c_{ab} e^a \wedge e^b \wedge \epsilon^3 \\
= \epsilon^1 \wedge \epsilon^2 \wedge (c_{ab} e^b + A_3 \epsilon^3 + A_4 \epsilon^4) + \epsilon^3 \wedge \epsilon^4 \wedge (c_{ab} e^b + A_1 \epsilon^1 + A_2 \epsilon^2) \\
= (\epsilon^1 \wedge \epsilon^2 + \epsilon^3 \wedge \epsilon^4) \wedge (\Lambda - \chi), \\
= (\Lambda - \chi) \wedge \Sigma^1.
\]

The manipulations for the remaining self-dual forms are identical. \[\Box\]

The hypercomplex condition on the metric is invariant under conformal changes of the metric. Thus one needs to calculate the transformation properties of these forms under such a change.

**Lemma 2.1.** Under the conformal change \(g \mapsto e^{2\Lambda} g\) the forms \(\alpha, A, \chi\) transform as

\[
\alpha \mapsto \alpha + 2d\Lambda, \\
A \mapsto A - d\Lambda, \\
\chi \mapsto \chi - 3d\Lambda.
\]

**Proof.** The proof of this is entirely straightforward, following directly from the definition of the various forms. It is interesting to note that from these one may construct two linearly independent conformally invariant two-forms. \[\Box\]

Locally, if \(d\alpha = 0\) then a conformal factor may be found so the resulting metric is hyperKähler. The metrics studied in this paper come from the conformally invariant condition \(dA = 0\). This leaves another conformally invariant condition \(d\chi = 0\), the significance of which will be investigated elsewhere.

### 2.3. Local coordinate representations

The aim of this section is to give some local coordinate descriptions of the equations (4). The main point that we make use of is that equations (4) may be interpreted as the integrability conditions for null planes in the complexified tangent space of the manifold, and that we can introduce coordinates which naturally describe these surfaces.

With this approach in mind, we consider the complexified tangent bundle, \(T_c T^* M = TM \otimes \mathbb{C}\). From the vector fields \(\{e_i\}\), we define the complex basis for \(T_c M\):

\[
u = e_1 + ie_2, \\
w = e_3 + ie_4.
\]

In the case where we are considering the complexification of a real Riemannian structure, we would impose the reality condition that \(u\) and \(v\) would be related by complex conjugation, as would \(w\) and \(x\). More generally, we require the existence of a real structure on \(T_c M\) to define real slices of different signatures. Since we will be interested in the metrics of both Riemannian and ultra-hyperbolic signatures, we will generally assume, for the moment, that the vector fields \(u, v, w, x\) are complex, and simply impose the relevant reality conditions later.
In terms of the vector fields $u, v, w, x$, the relations (4) take the form:

\[
[u, v] = 0, \\
[u, w] + [v, x] = 0, \\
[v, x] = 0.
\]

The first of these equations states that at each point (on a suitable local neighbourhood of) $M$, the plane spanned by the vector fields $u$ and $w$ is integrable. It follows that we can introduce coordinates $(t, y)$ on each of these surfaces with

\[
u = \partial_t, \quad w = \partial_y.
\]

If we now consider the space with $t$ and $y$ held constant, the last of equations (11) tells us that this space forms an integrable plane as well. As such, we can introduce coordinates $(x, z)$ on these planes, and the coordinates $(t, x, y, z)$ should give a suitable coordinate system on some local region of $M$. Note that the coordinates $(x, z)$ are not fully determined by these conditions. The different choices of coordinates correspond to different coordinate expressions for the vector fields $v$ and $x$, and in fact there are several geometrically distinct coordinate systems that we wish to investigate.

2.3.1. Case I. The first case we consider is where we take the coordinates $(x, z)$ to be null. This is the analogue of the usual complex coordinate description of Kähler metrics. Using the second of equations (11), we introduce functions $a$ and $b$ and let $v$ and $x$ take the form

\[
v = a_x \partial_x - b_z \partial_z, \quad x = -a_z \partial_x + b_y \partial_z.
\]

This is the generalization of the form used in [11] for the case of half-flat metrics. The functions $a$ and $b$ must now satisfy the equation

\[
\{a, a_x\} = \{b, a_z\}, \quad \{a, b_x\} = \{b, b_z\}, \quad (9)
\]

where we have defined the Poisson bracket by

\[
\{f, g\} = f_y g_z - f_z g_y.
\]

Equations (9) are a generalization of the first heavenly equation which describes half-flat metrics [12]. To see this, we note that the vectors in (4) define a metric which is conformal to a half-flat metric if the vectors $e_i$ are divergence free with respect to some volume element $\omega$ [10]. Taking $\omega = dt \wedge dx \wedge dy \wedge dz$, we find that there exists a function $\Omega$ such that $a = \Omega_z$, $b = \Omega_x$. We can therefore integrate equations (9) once and rescale our coordinates $(y, z)$ such that $\Omega$ satisfies the equation

\[
\{\Omega_x, \Omega_z\} = 1,
\]

which is the first heavenly equation [12].

In this case, we may reconstruct the metric $g$, which takes the form:

\[
g = 4(b_x a_y - a_z b_x)^{-1}\left[dt \otimes (a_z dz + b_z dx) + dy \otimes (a_x dz + b_y dx)\right].
\]

We see that in this form, the coordinates $(t, x, y, z)$ are all null, and are tailored to the geometry of the spheres worth of integrable null planes through each point of the space. If the functions and the coordinates are real, the metric will be of ultra-hyperbolic signature.
2.3.2. Case II. Here we introduce functions $\phi$ and $\psi$ and take
\[ v = \partial_t + \phi_y \partial_t - \psi_y \partial_y, \quad x = \partial_t - \phi_t \partial_t + \psi_t \partial_t. \]
The functions $\phi$ and $\psi$ must now satisfy the equations
\[ \phi_{tx} + \phi_{yz} + \{ \phi_t, \phi \} + \{ \psi, \phi_y \} = 0, \]
\[ \psi_{tx} + \psi_{yz} + \{ \psi_t, \phi \} + \{ \psi, \psi_y \} = 0. \]
In terms of these coordinates and functions, the metric is
\[ g = 4 (dt + \phi_t dz - \phi_t dx) \otimes dx + (dy - \psi_t dz + \psi_t dx) \otimes dz. \]
In this case, the coordinates $(t, y)$ are again null labelling some of the null planes in the space. The coordinates $(x, z)$ are not null, however, and label the ‘rate of change’ of the null planes (see [13] for more on the geometrical interpretation of these coordinates.)

These equations have been previously investigated in connection with anti-self-dual structures [5], and are a direct generalization of the second heavenly equation for half-flat metrics [12].

2.3.3. Case III. Finally, we consider the analogue of the expansion used to reduce the half-flat case to evolution form [7]. We therefore introduce functions $g$ and $h$ such that
\[ v = \partial_t + g_y \partial_x - h_y \partial_z, \quad x = -g_t \partial_x + h_t \partial_z. \]
We find that the functions $f$ and $g$ must satisfy the equations
\[ g_{tt} + \{ g_t, g \} + \{ h, g_z \} = 0, \]
\[ h_{tt} + \{ h_t, g \} + \{ h, h_z \} = 0. \]
\[ (10) \]
In this coordinate system, the metric takes the local form
\[ g = 4 \Delta^{-1} (dr \otimes (g_t dz + h_t dx) + dy \otimes (g_z dz + h_z dx) - \Delta^{-1} (g_t dz + h_t dx)^2) \]
with $\Delta = (h_t g_x - g_t h_z)$. In this system, the coordinates $(t, y)$ still span null planes, however the geometrical interpretation of the $(x, z)$ coordinates is not particularly clear. The main advantage of this coordinate system, however, is that the equations of motion are in evolution form. This form of the equations is therefore the natural starting point for the study of symmetry algebras [7], and the associated integrable hierarchy [8]. We shall therefore concentrate on this form of the equations from now on.

3. Integrable description

In this section the system (2) will be studied, viewing it as an integrable system and hence applying various known results from the theory of integrable systems to it. In particular a Lax pair will be given for the system, a hierarchy of conservation laws constructed and the Lie-point symmetry structure calculated.

3.1. Lax pair

A characteristic feature of an integrable system is the ability to express it as the compatibility condition for an otherwise over-determined linear system. To obtain the system (2) in such a way consider the following vector fields on complexified tangent bundle, $T_c M = T M \otimes \mathbb{C}$ of a four manifold $M$:
\[ u = \partial_t, \quad w = \partial_x, \quad v = \partial_t + g_y \partial_x - h_y \partial_z, \quad x = -g_t \partial_x + h_t \partial_z. \]
With these define the new vector fields
\[ L_0 = u - \lambda x, \]
\[ L_1 = w + \lambda v, \]
where \( \lambda \in \mathbb{CP}^1 \) is an auxiliary parameter. The compatibility conditions for the otherwise over-determined linear system \( L_0 \Psi = L_1 \Psi = 0 \) result in the following system of equations:
\[
\begin{align*}
[u, w] &= 0, \\
[u, v] + [w, x] &= 0, \\
[v, x] &= 0.
\end{align*}
\label{eq:11}
\]
With the explicit vector fields given above these reduce to the system (2):
\[
\begin{align*}
g_{tt} &= \{g, g_t\} + \{g_z, h\}, \\
h_{tt} &= \{g, h_t\} + \{h_z, h\},
\end{align*}
\label{eq:12}
\]
where, for convenience, the Poisson bracket \( \{f, g\} = f_y g_z - f_z g_y \) has been used. In the
previous section the geometry underlying this construction was given; solutions define a
hypercomplex metric on the manifold \( M \). Here we just consider the system as an example of
a four-dimensional integrable system and study it thus.

One interesting reduction of this system is to impose the condition \( g_z = h_t \). One may
solve this constraint by introducing a function \( \psi \) such that \( h = \psi_y, g = \psi_z \). With this it is
possible to integrate (2) and obtain a single evolution equation (1). With this constraint the
basic vector fields \( e_i = (u, v, w, x) \) become volume preserving, that is
\[ \mathcal{L}_e \omega = 0 \]
where \( \omega \) is the volume form \( \omega = u \wedge v \wedge w \wedge x \). In this case the metric is hyperKähler rather
than hypercomplex, with \( \psi \) being related to the Kähler potential by a Legendre transformation.
Conversely, one may regard the system (2) as a generalization of (1) when one relaxes the
volume preserving condition on the vector fields in the associated Lax pair.

3.2. Formal solutions

The system (1) is in Cauchy–Kovaleskaya form, so their formal solution may be written as a
power series in the \( t \)-variable:
\[
\begin{align*}
h &= \sum_{n=0}^{\infty} h_n(x, y, z) t^n, \\
g &= \sum_{n=0}^{\infty} g_n(x, y, z) t^n
\end{align*}
\]
and the differential equations reduce to recursion relations between the coefficients \( h_n, g_n \).
Thus a formal solution may be derived from the coefficients \( (g_0, h_0, g_1, h_1) \), or equivalently,
in terms of the initial data \( (g, h, g_t, h_t)|_{t=0} \) on the \( t = 0 \) hypersurface in \( M \). This shows that
the general solution depends on four arbitrary functions of three variables.

One simple, but explicit, solution may be obtained from the ansatz
\[
\begin{align*}
g &= t y + G(t, x, z), \\
h &= t.
\end{align*}
\]
With this the non-linearities in (12) disappear and one is left with a linear equation for \( G \),
which after a simple change of variable, is just the three-dimensional Laplace equation. Other
simple solutions may be obtained by taking known hypercomplex metrics and re-expressing
them in the above form. Some examples of solutions obtained in this way will be given later.
3.3. Symmetry structure

While symmetry techniques may be applied to any system of differential equations, the Lie-point symmetries of integrable systems have a particularly rich structure compared with non-integrable systems. Indeed, possible integrable systems may often be identified by an increase in the dimension of the Lie-algebra of symmetries, as compared with nearby non-integrable systems.

Let \( x = (x_1, \ldots, x_p) \) and \( u = (u^1, \ldots, u^q) \) be sets of independent and dependent variables, and consider a set of differential equations of degree \( k \), given by

\[
1_i(x, u^{(k)}) = 0, \quad i = 1, \ldots, m.
\]

Lie-point symmetries are generated by the vector field

\[
v = \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{a=1}^{q} \phi_a(x, u) \frac{\partial}{\partial u^a},
\]

where the coefficients are determined by the criterion

\[
pr^{(k)}v(\Delta)|_{\Delta=0} = 0
\]

where \( pr^{(k)}v \) is the \( k \)th prolongation of the vector field \( v \). These ideas, and notation, are standard, see [14].

To apply such a procedure to the system (2) it is convenient to convert the system from a second-order system in two independent variables to a first-order system in four independent variables by introducing potentials and integrability conditions. Therefore let:

\[
A = g_t, \quad B = g_y, \quad C = h_t, \quad D = h_y,
\]

in which (2) becomes

\[
\begin{align*}
A_y - B_t &= 0, \\
C_y - D_t &= 0, \\
A_x + A_y B - B_x A + C B_z - D A_z &= 0, \\
C_x + C_y B - D_x A + C D_z - C_z D &= 0.
\end{align*}
\]

Since these are first order the calculation of the first prolongation of the vector field is easy, the evolutionary form of the system also giving a distinguished variable \( t \) to eliminate in the course of the calculations.

The result is that we get five families of symmetries, the vector fields which generate these families being (where \( k \) is a constant, \( \phi, \psi \) are functions of the coordinate \( y \), and \( a_1, a_2 \) are functions of coordinates \( x, z \)):

\[
\begin{align*}
v^1[k] &= k(-t \partial_t + 2A \partial_A + B \partial_B + 2C \partial_C + D \partial_D), \\
v^2[\phi] &= -\phi \partial_t + A \phi_y \partial_B + C \phi_y \partial_D, \\
v^3[\psi] &= -t \psi_y \partial_t - \psi \partial_y + \psi_y (A \partial_A + B \partial_B + C \partial_C + D \partial_D) + t \psi_{yy} (A \partial_B + C \partial_D), \\
v^4[a_1] &= -a_1 \partial_t - (a_1 A - a_1 z C) \partial_A - (a_1 B - a_1 z D) \partial_B, \\
v^5[a_2] &= -a_2 \partial_z - (a_2 C - a_2 A) \partial_C - (a_2 D - a_2 B) \partial_D.
\end{align*}
\]

The first of these generators simply generates a scaling symmetry of the equations. Therefore, if \( A, B, C, D \) constitute a solution of the equations, then so do \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) defined by

\[
\begin{align*}
\tilde{A}(t, x, y, z) &= e^{2k} A(e^k t, x, y, z), \\
\tilde{B}(t, x, y, z) &= e^k B(e^k t, x, y, z), \\
\tilde{C}(t, x, y, z) &= e^{2k} C(e^k t, x, y, z), \\
\tilde{D}(t, x, y, z) &= e^k D(e^k t, x, y, z).
\end{align*}
\]
Similarly, the second generator generates a translation in the $t$ coordinate along with a redefinition of fields, so that in this case
\[
\tilde{A}(t, x, y, z) = A(t + \phi, x, y, z), \\
\tilde{B}(t, x, y, z) = B(t + \phi, x, y, z) + \phi A(t, x, y, z), \\
\tilde{C}(t, x, y, z) = C(t + \phi, x, y, z), \\
\tilde{D}(t, x, y, z) = D(t + \phi, x, y, z) + \phi y A(t + \phi, x, y, z).
\]
These are the only symmetries that it is possible to generally exponentiate explicitly. The non-zero commutators for this Lie algebra are
\[
[v^1[k], v^2[\phi]] = v^2[k\phi], \\
[v^2[\phi], v^3[\psi]] = v^3[\psi\phi_y - \phi\psi_y], \\
[v^3[\psi], v^4[\chi]] = v^4[\chi\psi_y - \psi\chi_y], \\
[v^4[a], v^5[b]] = v^4[a, b - ab], \\
[v^5[a], v^5[b]] = v^5[a, b - ab].
\]
where $k, l$ are arbitrary constants, $\psi, \phi$ are arbitrary functions of $y$, and $a, b$ are arbitrary functions of $x$ and $z$.

From the structure of these commutators one may decompose the Lie algebra $L$ generated by these vector fields into a direct sum $L = L_1 \oplus L_2$, where
\[
L_1 = \{v^1[k], v^2[\phi], v^3[\psi]\}, \\
L_2 = \{v^4[a], v^5[b]\}.
\]
The sub-Lie-algebra $L_1$ decomposes further as a semidirect product $L_1 = S \rtimes R$ where
\[
S = \{v^3[\psi]\}, \\
R = \{v^1[k], v^3[\phi]\}.
\]
The subalgebra $L_2$ is isomorphic to vector fields on a two-dimensional surface and corresponds to coordinate transformation in the $x,z$-variables. This is to be expected, since in the hyperKähler case the vectors are all divergence free and the symmetries turn out to be related to symplectic diffeomorphisms of two-dimensional planes. In our case, we have simply dropped the divergence-free condition from the vector fields, and the symmetry group becomes related to the larger group of diffeomorphisms, since there is no natural symplectic structure any more. Similarly, the vector fields $v^2$ generate coordinate transformations and the vector fields $v^1$ generate constant rescalings of the metric. The only vector fields which generate genuinely new metrics are those in $S$.

3.4. Conservation laws

In this section a hierarchy of conservation laws of the form
\[
g^{\mu\nu} \nabla_\mu j_\nu^{(n)} = 0, \quad n = 0, 1, \ldots
\]
will be constructed. This expression is clearly covariant, but in the calculations it will be necessary to use a particular form of the metric and the associated field equations. Explicitly we consider metrics of the form
\[
g = 2\Delta^{-2} [dt \otimes (a_t dz + b_t dx) + dy \otimes (a_y dz + b_y dx)]
\]
where $\Delta = (a_t b_y - a_y b_t)$ and with $a$ and $b$ being solutions of the field equations (9). The conformal factor in (14) has been fixed so that $\det g_{ij} = 1$; such a fixing does not change the hypercomplex or Hermitian properties of the metric. One obvious extension of these results would be to introduce the notion of a conformally invariant conservation law.

The starting point of this construction, a generalization of a procedure first applied to nonlinear $\sigma$-models [15], is the solution $\Psi$ to the Lax pair

$$\left[ \lambda \partial_t + a_t \partial x - b_t \partial z \right] \Psi = 0, \quad \lambda \in \mathbb{CP}^1.$$

Expanding $\Psi$ as a power series in $\lambda$, i.e. as $\Psi = \sum_{n=0}^{\infty} \lambda^n \Psi_n$ and equating coefficients yields the following equations for the $\Psi_0$-term:

$$[a_t \partial_t - b_t \partial_z] \Psi_0 = 0,$$

$$[a_t \partial_t - b_t \partial_z] \Psi_0 = 0,$$

and the recursion relations

$$\partial_t \Psi_n = (-a_t \partial_t + b_t \partial_z) \Psi_{n+1},$$

$$\partial_y \Psi_n = (-a_y \partial_t + b_y \partial_z) \Psi_{n+1}.$$ (16)

The first set of equations imply that $\Psi_0 = \Psi_0(t, y)$ and so we take the seed solution to be

$$\Psi_0 = \begin{pmatrix} t \\ y \end{pmatrix}$$

(here we have assembled two independent solutions into a vector). This seed solution will generate, via the recursion relations (17), the full solution to the Lax pair (15). This function defines the twistor surfaces in the corresponding twistor space. Another family of conservation laws may be obtained starting from the expansion $\tilde{\Psi} = \sum_{n=0}^{\infty} \lambda^{-n} \tilde{\Psi}_n$ and the relationship between $\Psi$ and $\tilde{\Psi}$ on the equator of $\mathbb{CP}^1$ defines the twistor space, via a patching construction [13, 16].

**Proposition 3.1.** The currents $j^{(n)}_\mu$ defined by

$$j^{(n)}_t = 0, \quad j^{(n)}_x = \Delta \frac{1}{2} \partial_t \Psi_{n+1},$$

$$j^{(n)}_y = 0, \quad j^{(n)}_z = \Delta \frac{1}{2} \partial_y \Psi_{n+1},$$

are conserved.

**Proof.** With the particular metric (14)

$$g^{\mu\nu} \nabla_\mu j^{(n)}_\nu = \partial_t [-a_t \partial_t \Psi_{n+1} + b_t \partial_z \Psi_{n+1}] + \partial_y [a_y \partial_t \Psi_{n+1} - b_t \partial_z \Psi_{n+1}],$$

$$= \partial_t [\partial_t \Psi_n] - \partial_y [\partial_z \Psi_n],$$

$$= 0.$$

This proof uses the condition $\det g_{ij} = 1$, so the Christoffel symbols $\Gamma^\mu_{ij} = 0$. □

4. Comments

Underlying the integrability of the multidimensional systems presented here is the existence of a twistor space. This paper has, though, only concentrated on the field equations and the
associated Lax pairs with little mention of the properties of the corresponding twistor space—in terms of a double fibration

\[
\begin{array}{ccc}
\text{[Lax pair]} & \\
\searrow & & \nearrow \\
\text{[hypercomplex manifold]} & & \text{[twistor space]}
\end{array}
\]

we have said little about the structure of the right-hand side. Such twistor spaces have the special property that they fibre over \( \mathbb{C}P^1 \) [6], unlike those for more general anti-self-dual Weyl spaces or scalar-flat Kähler spaces. This is manifested in the simple \( \lambda \)-dependence in the Lax pairs for hypercomplex manifolds—the Lax pairs for scalar-flat Kähler [17] and general anti-self-dual Weyl spaces [4,18] involve \( \partial_\lambda \)-terms. The hypercomplex manifolds studied here come from the conformally invariant condition \( dA = 0 \), and so one would expect the corresponding twistor space to exhibit certain extra properties. General hypercomplex manifolds (without this condition) will be studied in the sequel to this paper, this also containing the connection between the approach developed here and the Obata connection.

One characteristic feature of integrable systems is the existence of an associated hierarchy. Such hierarchies may be constructed by studying the generalized symmetry structure of the original systems of equations [14]. Such hierarchies have been constructed for hyperKähler metrics in [16]. It remains to see how such ideas may be extended to the hypercomplex systems studied here.

Further connections between hypercomplex systems and twistor theory have been investigated by Dunajski [19].

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**Appendix**

While many of the ideas in this paper will generalize to \( 4N \)-dimensional manifolds, when \( N = 1 \) another generalization is possible. Underlying the integrability of the structures studied in this paper is the existence of a suitable twistor space, and hypercomplex manifolds automatically have such twistor spaces. However in four dimensions the existence of a twistor space follows from the Weyl tensor being anti-self-dual, and the hypercomplex condition here implies, not is implied by, this condition. Thus a possible generalization is to study metrics with anti-self-dual Weyl tensor and which are also scalar flat. Such a system has another reduction to scalar flat Kähler metrics. These different systems and their interconnections are summarized in the following diagram:

\[
\begin{array}{ccc}
\text{a.s.d. Weyl} & & \text{Kähler} \\
\{ (R = 0) \} & & \{ R = 0 \}
\end{array}
\]

\[
\begin{array}{ccc}
\text{hypercomplex} & & \text{HyperKähler} \\
\{ (R = 0) \} & & \{ \text{Ricci flat} \}
\end{array}
\]

The conditions \( (R = 0) \) in brackets indicate how the conformal factor for otherwise conformally invariant conditions have been fixed.
An analogous system of equations to (1) for the scalar-flat, anti-self-dual-Weyl systems is given by

\[
\begin{align*}
g_{tt} + (e^{-\psi} \{g, h\})_z &= e^{-\psi} \{g, h_z - g_z\}, \\
h_{tt} + (e^{-\psi} \{g, h\})_y &= e^{-\psi} \{h, h_z - g_y\}, \\
(e^\psi)_t + \{g, \psi_y\} - \{h, \psi_z\} &= 0
\end{align*}
\]

(These certainly imply the geometric conditions though whether they are implied by them is unclear). The corresponding metric is given by

\[
g = 2dy \left\{ h_t dt + h_s dx - \frac{h_t e^\psi}{\Delta} [h_t dy + g_t dz] \right\} + 2dz \left\{ g_t dt + g_s dx - \frac{g_t e^\psi}{\Delta} [h_t dy + g_t dz] \right\}.
\]

where \(\Delta = h_t g_s - g_t h_s\). The two reductions above are easy to see from this system:

- When \(\psi = 0\), this system reduces to
  \[
  \begin{align*}
  g_{tt} &= \{h, g_z\} - \{g, g_z\}, \\
  h_{tt} &= \{h, h_z\} - \{g, h_y\},
  \end{align*}
  \]
  that is, to the hypercomplex systems studied in the main body of this paper. The further reduction \(h = \theta, g = \theta\) reduces this to the hyperKähler equation (1).

- Imposing the Kähler condition on this system gives \(h = \theta, g = \theta\) and the first set of equations simplify to
  \[
  \begin{align*}
  \theta_{tt} + e^{-\psi} \{\theta_z, \theta_y\} &= 0, \\
  (e^\psi)_t + \{\theta_z, \psi_y\} - \{\theta_y, \psi_z\} &= 0.
  \end{align*}
  \]
  These are the analogues of the well known scalar-flat Kähler equations [20], written in evolutionary form. It is easy to find solutions, such as the one which gives the Burns metric. The further reduction \(\psi = 0\) reduces this to the hyperKähler equation (1).

One interesting class of solutions to all these systems comes from imposing an \(SU(2)\) symmetry on the metrics. Such metrics are often referred to as Bianchi IX metrics. This symmetry reduces the field equations from partial differential equations down to systems of coupled ordinary differential equations which may be integrated directly. These ideas may also be applied to other Bianchi metrics.

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Paper 2

Grassmann structures and integrable systems,
Grassmann structures and integrable systems

James D.E. Grant*

Department of Mathematics, University of Hull, Hull HU6 7RX, UK

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Abstract

We consider some natural connections which arise between right-flat \((p, q)\) paraconformal structures and integrable systems. We find that such systems may be formulated in Lax form with a “Lax \(p\)-tuple” of linear differential operators, depending a spectral parameter which lives in \((q-1)\)-dimensional complex projective space. Generally, the differential operators contain partial derivatives with respect to the spectral parameter. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

It has long been known that in four-dimensional Riemannian geometry there is a connection between conformal structures with anti-self-dual Weyl tensor, and three-dimensional complex manifolds:

**Theorem 1.1** (Atiyah, Hitchin and Singer [1]). *If a manifold \(M\) admits a conformal structure with \(W^+ = 0\), then the projective spin-bundle \(\mathbf{P}(\mathcal{V}^+)\) is a complex 3-manifold. Conversely, given a complex 3-manifold \(Z\) with a real structure (i.e. anti-holomorphic involution) \(\sigma\) and a 4 parameter family of embedded rational curves with normal bundle \(N \cong H \oplus H\), on which \(\sigma\) acts as the anti-podal map, then the space of real rational curves admits an anti-self-dual conformal structure. All anti-self-dual conformal structures arise in this way.*

Moreover, if one considers anti-self-dual Yang–Mills fields on an anti-self-dual background, then solutions may be constructed in terms of holomorphic vector bundles on this
complex 3-manifold. The existence of such a complex 3-manifold is looked on as being central to the notion of integrability of the anti-self-duality equations [9,16,17].

If one tries to generalise notions of anti-self-duality to higher dimensional Riemannian manifolds, there are several (inequivalent) paths one may choose to follow.

A common choice is to investigate Riemannian manifolds in higher dimension with reduced holonomy group [15], and gauge fields related to these structures [4]. In the case of (irreducible, non-symmetric) Riemannian manifolds, one may then invoke Berger’s classification of the possible holonomy groups of the Levi–Civita connection. Apart from the generic case of SO(n) holonomy group, the holonomy groups allowed by Berger’s classification correspond to Kähler, quaternionic-Kähler, Ricci-flat Kähler and hyper-Kähler manifolds, along with two exceptional possibilities of Ricci-flat metrics with holonomy group $G_2$ and Spin$_7$ in dimensions 7 and 8. This may not be the most natural approach if ones interest is in integrable systems, however, since, of these possibilities, the only systems which appear to be integrable are those which govern Kähler, quaternionic-Kähler and hyper-Kähler structures. The equations for Ricci-flat Kähler metrics are not integrable in dimensions greater than 4. Little is known concerning the integrability of the $G_2$ and Spin$_7$ holonomy equations, although they contain as special cases the (non-integrable) equations for six-dimensional and eight-dimensional Ricci-flat Kähler structures, respectively. Therefore, it is extremely unlikely that these two systems are integrable.

An alternative path is not to start with the geometrical condition of anti-self-duality of the four-dimensional metric, but to simply consider systems in higher dimensions where there is a suitable generalisation of the complex 3-manifold which appears in four dimensions. Generically, we will denote such a complex manifold by $Z$, and the idea is that one reconstructs the geometrical manifold, denoted $M$, as a parameter space of particular sub-manifolds of $Z$. Natural geometrical structures then arise on the manifold $M$ as a result of the integrability of the complex structure on $Z$. Any additional holomorphic structures that exist on $Z$ then lead to more specialised geometrical structures on $M$. The complex manifold $Z$ will be referred to as a twistor space, and the essence of the work of Ward and others [9,16,17] is that it is the existence of a twistor construction for a problem which should be interpreted as a sign of its integrability. Substantial evidence for this claim comes from the fact that many standard integrable systems in two and three dimensions may be constructed as symmetry reductions of the equations for anti-self-dual Yang–Mills fields and anti-self-dual conformal structures in four dimensions [9,17]. In this approach it is the existence of the complex manifold $Z$ that is central, the equations on the space–time then being simply a manifestation of the complex structure on $Z$.

We aim here to study (local) properties of structures for which there is a complex manifold construction, the right-flat Grassmann structures, and to see what new features of integrable systems these structures suggest. In particular, we begin by reviewing, in explicitly local terms, the construction implicit in Theorem 1.1. What we find is that even in this simplest situation, there are features of the equations which arise which are unusual from the point of view of integrable systems. In the generic case of an anti-self-dual manifold, the spin-bundle does not fibre over $\mathbb{P}_1$, the complex projective line. In integrable systems terms this means that the operators in our Lax pair contain partial derivatives with respect to the spectral parameter. Therefore the spectral parameter itself is very much part of the geometrical problem, a property which is unusual (but not unknown) in conventional integrable systems theory.
We then study, in a similar fashion \((p, q)\) right-flat Grassmann structures in dimension \(n = pq\). In this case, we find that as opposed to a Lax pair of operators depending on a spectral parameter \(\lambda \in \mathbb{P}_1\), the right-flat condition on a Grassmann structure is determined by a Lax \(p\)-tuple of differential operators depending on a spectral parameter taking values in the higher-dimensional projective space \(\mathbb{P}_{q-1}\). As in the case of anti-self-dual structures in four dimensions, these differential operators generally contain derivatives in the spectral parameter, corresponding to the fact that the complex manifold \(Z\) does not fibre holomorphically over \(\mathbb{P}_{q-1}\).

The moral of our story is that if one takes the ideas of Ward and others seriously that it is the connection with complex manifold theory which is central to integrable system theory, then one must substantially generalise what one considers to be an integrable system.

2. Anti-self-dual conformal structures

We begin by reviewing, in local terms, the construction implicit in Theorem 1.1. Consider an oriented Riemannian four-manifold \(M\). We may then define a canonical almost complex structure on the projective spin-bundle. First, we use the Levi–Civita connection to split the tangent bundle of \(\mathbb{P}(\mathbb{V}^+)\) as the direct sum of a vertical part along the fibres, \(\mathbb{V}(\mathbb{P}(\mathbb{V}^+))\), and the horizontal part, \(\mathbb{H}(\mathbb{P}(\mathbb{V}^+))\), which is the pull-back of the tangent bundle of \(M\), \(p^*TM\). The vertical fibres are complex projective lines, and so inherit a natural almost complex structure. In the horizontal direction, a non-zero spinor \(\pi \in (\mathbb{V}^+)_{x}\) identifies \(T_xM\) by Clifford multiplication with the two-dimensional complex vector space \((\mathbb{V}^-)_{x}\). At the points of \(\mathbb{P}(\mathbb{V}^+)\) corresponding to \(\pi\), we put this almost complex structure on \(\mathbb{H}_{x}(\mathbb{P}(\mathbb{V}^+))\).

It follows that this almost complex structure is integrable if and only if the Weyl tensor of the conformal structure is anti-self-dual \([1]\).

To cast this in more explicit terms, fix a Riemannian metric, \(g\), in the conformal structure. If we complexify the tangent space, and extend the metric by complex linearity to a complex metric (again denoted \(g\)) on \(TM \otimes \mathbb{C}\), then locally, we may introduce a null basis \(\{\epsilon^i| i = 1, \ldots, 4\}\) for \(T^*M \otimes \mathbb{C}\) in which the metric may be written as

\[
g = \epsilon^1 \otimes \epsilon^2 + \epsilon^2 \otimes \epsilon^1 + \epsilon^3 \otimes \epsilon^4 + \epsilon^4 \otimes \epsilon^3. \tag{2.1}\]

We can then define the Levi–Civita connection, \(\Gamma\), of the tetrad by the equation

\[
de^i + \sum_{j=1}^{4} \Gamma^i_{\ j} \wedge \epsilon^j = 0, \quad i = 1, \ldots, 4.
\]

If we adopt an affine complex coordinate, \(\lambda\), on the fibre \((\mathbb{P}(\mathbb{V}^+))_{x} \cong \mathbb{P}_1\), then we define an almost complex structure on \(\mathbb{P}(\mathbb{V}^+)\) by defining the distribution \(\Lambda \subset T^*(\mathbb{P}(\mathbb{V}^+))\) spanned by the 1-forms

\[
\sigma_1 = \epsilon^3 + \lambda \epsilon^1, \quad \sigma_2 = \epsilon^2 - \lambda \epsilon^4, \quad \sigma_3 = d\lambda + \Gamma_{14} + \lambda(\Gamma_{12} - \Gamma_{34}) + \lambda^2 \Gamma_{23}.
\]

This almost complex structure on \(\mathbb{P}(\mathbb{V}^+)\) is integrable if and only if the distribution \(\Lambda\) is involutive, i.e. \(d\Lambda \subset \Lambda^1 \wedge \Lambda\). It is straightforward to show that this is the case if and
only if the Weyl tensor of the metric $g$ defined above is anti-self-dual. It also follows straightforwardly that this construction is unaffected by conformal changes of metric, and so depends only on the conformal equivalence class of the metric [1].

The connection with integrable systems comes from taking a dual formulation of this result [9]. The anti-holomorphic tangent space of $P(V^+)$ is spanned by the vector fields

$$v_1 = \frac{1}{1 + \bar{\lambda} \lambda} \left[ e_4 + A_4 \frac{\partial}{\partial \lambda} + \bar{A}_4 \frac{\partial}{\partial \bar{\lambda}} + \lambda \left( e_2 + A_2 \frac{\partial}{\partial \lambda} + \bar{A}_2 \frac{\partial}{\partial \bar{\lambda}} \right) \right],$$

$$v_2 = \frac{1}{1 + \bar{\lambda} \lambda} \left[ e_1 + A_1 \frac{\partial}{\partial \lambda} + \bar{A}_1 \frac{\partial}{\partial \bar{\lambda}} - \lambda \left( e_3 + A_3 \frac{\partial}{\partial \lambda} + \bar{A}_3 \frac{\partial}{\partial \bar{\lambda}} \right) \right],$$

$$v_3 = \frac{\partial}{\partial \bar{\lambda}},$$

where $e_i$ are vector fields on $M$ dual to the 1-forms $e^i$

$$\langle e^i, e_j \rangle = \delta^i_j,$$

and

$$A = -\Gamma_{14} - \lambda (\Gamma_{12} - \Gamma_{34}) - \lambda^2 \Gamma_{23}, \quad \bar{A} = -\Gamma_{23} + \lambda (\Gamma_{12} - \Gamma_{34}) - \lambda^2 \Gamma_{14}. $$

The complex structure defined by these vectors is integrable if they are closed under Lie brackets. We now note that the complex structure defined by these vector fields is the same as defined by the following basis:

$$L_1 = D_4 + \lambda D_2, \quad L_2 = D_1 - \lambda D_3, \quad v = \frac{\partial}{\partial \bar{\lambda}},$$

where we have defined the vector fields

$$D_i = e_i + A_i \frac{\partial}{\partial \lambda}. \quad \tag{2.2}$$

The only non-trivial part of the integrability of the complex structure we have defined is that the Lie bracket of $L_1$ and $L_2$ must lie in $T^{(0,1)}$. Therefore for integrability we require the existence of functions $\alpha(x: \lambda), \beta(x: \lambda)$ with the property that

$$[D_4 + \lambda D_2, D_1 - \lambda D_3] = \alpha(D_4 + \lambda D_2) + \beta(D_1 - \lambda D_3). \quad \tag{2.3}$$

A power counting argument implies that the functions $\alpha$ and $\beta$ are quadratic polynomials in the variable $\lambda$. If this condition is satisfied, then the projective spin-bundle is a complex 3-manifold, and so the conformal structure must be anti-self-dual. Conversely, if the conformal structure is anti-self-dual, then the projective spin-bundle is a complex 3-manifold and so, locally, we may choose bases where the above equations are satisfied. We therefore have the following theorem.

**Theorem 2.1.** Given an anti-self-dual conformal structure and any representative metric in the conformal class written in the form (2.1), then there exists a 1-form $A$, which is a quadratic function of an arbitrary $P_1$-valued parameter $\lambda$, and two quadratic functions of $\lambda, \alpha$ and $\beta$ which obey Eq. (2.3), where the differential operators $D_i$ are as in Eq. (2.2).
It is possible to decompose Eq. (2.3) into components in the tangent space of the manifold \( M \) and components in the vertical direction \( \partial/\partial \lambda \). The components in \( TM \) tell us that the functions \( \alpha, \beta \) and the components of the form \( A \) correspond to parts of the Levi–Civita connection. The parts of the Levi–Civita connection they define are precisely the parts required to construct the self-dual part of the Weyl tensor, \( +W \). The vertical component of Eq. (2.3) then tell us that the five individual components of \( +W \) vanish identically, so the Weyl tensor is anti-self-dual.

Eq. (2.3) tells us that the operators \( L_1 \) and \( L_2 \) constitute a Lax pair for the problem, and therefore that the system is integrable. However, these operators contain derivatives with respect to the spectral parameter \( \lambda \), a feature which does not usually occur in standard integrable systems theory. The origin of these derivative terms lies in the nature of the complex manifold \( P(V^+) \). Eq. (2.3) are the integrability condition which ensures the existence of three linearly independent solutions of the over-determined set of equations for a function \( f(x: \lambda) \)

\[
L_1 f = L_2 f = 0.
\]

Solutions of these equations correspond to meromorphic functions on \( P(V^+) \). The fact that \( \lambda \) itself is not a solution of these equation is a consequence of the fact that generally \( P(V^+) \) does not fibre over \( P_1 \) (equivalently \( \lambda \) is not a meromorphic function on \( P(V^+) \)). In the case of hyper-Kähler or hyper-complex structures, where the spin-bundle does fibre over \( P_1 \), the \( \lambda \) derivatives are not present in the Lax pair [5,8]. In these cases, one can reconstruct the transition functions of the bundle from the solutions of the above equations [11].

Although, Eq. (2.3) describes the most general anti-self-dual conformal structures locally, there are various special cases of these equations:

- Letting \( A_i = \lambda \phi_i \), we recover the class of Hermitian anti-self-dual spaces, which are conformal to scalar-flat \( \bar{\partial}-\)Kähler metrics [14].
- Letting \( A_4 + \lambda A_2 = A_1 - \lambda A_3 = 0 \) defines hyper-complex structures in four dimensions [5].
- Letting \( A_i = \lambda \phi_i \), and assuming the vector fields \( e_i \) are divergence free with respect to some volume element defines a scalar-flat Kähler metric up to a known conformal factor (this is an extension of a result of Park [13]).
- Letting \( A_4 + \lambda A_2 = A_1 - \lambda A_3 = 0 \) and assuming the vector fields \( e_i \) are divergence free with respect to some volume element defines a hyper-Kähler metric up to a known conformal factor [8].

Similar results hold for complex anti-self-dual conformal structures and real conformal structures of signature \((-,-,+,+)) with suitable generalisations and modifications of the reality conditions.

### 3. Grassmann structures

Anti-self-dual conformal structures in four dimensions are a special case of a more general type of structure, a right-flat Grassmann structure. Recall that for integers \( p, q \geq 2 \), a \((p,q)\) Grassmann structure (or paraconformal structure in the terminology of Bailey and
Eastwood [2]) consists of a complex manifold \( M \) of complex dimension \( n = pq \) and an isomorphism \( \alpha \) between the (holomorphic) tangent bundle of \( M \) and the tensor product of a rank \( p \) complex vector bundle \( U \) with a rank \( q \) vector bundle \( V \)

\[
\alpha : TM \rightarrow U \otimes V.
\]  

(3.1)

Given such an isomorphism, we may introduce an isomorphism

\[
\Lambda^p U \cong \Lambda^q V
\]

(3.2)

between the highest exterior powers of these bundles.

Given connections, both denoted \( \nabla \), on the bundles \( U \) and \( V \), we may define a unique induced connection on \( TM \), again denoted \( \nabla \), by demanding that covariant differentiation commutes with the isomorphism \( \alpha \). This affine connection naturally has torsion \( T \) defined by

\[
\nabla_X Y - \nabla_Y X - [X, Y] = T(X, Y) \quad \forall X, Y \in \Gamma(TM),
\]

and curvature tensor \( R \) given by

\[
([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})V = R(X, Y)V \quad \forall X, Y, V \in \Gamma(TM).
\]

A scale for a Grassmann structure consists of a non-vanishing section, \( \epsilon \), of the bundle \( \Lambda^p U \). The isomorphism (3.2) then implies the existence of a non-vanishing section, \( \epsilon' \), of the bundle \( \Lambda^q V \). It can be shown [2] that given the sections \( \epsilon \) and \( \epsilon' \), there exist unique connections on \( U \) and \( V \) (and therefore on \( TM \)) with the property that the torsion is trace-free, and which annihilate the forms \( \epsilon \) and \( \epsilon' \):

\[
\nabla \epsilon = \nabla \epsilon' = 0.
\]

We shall generally assume the existence of a scale, and work with the unique connections which preserve it.

### 3.1. Algebraic decomposition of the torsion and curvature

If we consider the space of 2-forms on \( M \), we have

\[
\wedge^2 (T^*M) \cong \Lambda^2 (U^* \otimes V^*) \cong (\Lambda^2(U^*) \otimes S^2(V^*)) \oplus (S^2(U^*) \otimes \Lambda^2(V^*))
\]

where \( \wedge \) and \( S \) denote skew-symmetric and symmetric powers of the relevant bundles, respectively. Viewing the torsion as a \( TM \) valued 2-form on \( M \), it decomposes into two parts

\[
T = T^+ \oplus T^-,
\]

where

\[
T^+ \in TM \otimes (\Lambda^2(U^*) \otimes S^2(V^*)), \quad T^- \in TM \otimes (S^2(U^*) \otimes \Lambda^2(V^*)).
\]

One can show that the trace-free parts of these parts of the torsion are independent of the connection chosen on the vector bundles \( U \) and \( V \) [2].
There is a similar decomposition of the curvature tensor $R$. Given the direct product nature of $TM$, the curvature decomposes as a direct sum

$$R = R^- \otimes \text{Id}_V + \text{Id}_U \otimes R^+,$$

where $R^-$ and $R^+$ denote the curvatures of the connections $\nabla$ on $U$ and $V$, respectively. $R^+$ may be viewed as a section of $\Lambda^2(M) \otimes \text{End}(V)$, and

$$\Lambda^2(M) \otimes \text{End}(V) \cong (\Lambda^2U^* \otimes S^2V^*) \otimes (V \otimes V^*) \cong (\Lambda^2U^* \otimes (V \otimes V^* \otimes S^2V^*)) \otimes (S^2U^* \otimes (V \otimes V^* \otimes \Lambda^2V^*)�)(V \otimes V^* \otimes S^2V^*)) \otimes (S^2U^* \otimes (V \otimes V^* \otimes \Lambda^2V^*)�).$$

We now wish to consider the component of $R^+$ which is a section of $\Lambda^2U^* \otimes (V \otimes V^* \otimes S^2V^*)$. If we completely symmetrise in the $V$ components, then the trace-free part of the remaining object will be referred to as the positive part of the Weyl tensor:

$$W^+ \in \Gamma(\text{trace-free part of } \Lambda^2U^* \otimes (V \otimes V^* \otimes S^2V^*)�).$$

**Definition 3.1.** A $(p, q)$ Grassmann structure is right-flat if $T^+ = 0$, $p > 2$, $W^+ = 0$, $p = 2$.

In the case $p > 2$, the vanishing of the torsion implies automatically that $W^+ = 0$, whereas if $p = 2$, the torsion $T^+$ automatically vanishes, so the condition $W^+ = 0$ is non-trivial [2]. A complex four-dimensional spin-manifold with a metric is a particular case of a Grassmann manifold with $p = q = 2$ since, due to the structure of the complexified rotation group, the complexified tangent bundle decomposes as a product of spin-bundles. In this case, $W^+$ may be identified with the self-dual part of the Weyl tensor of the conformal structure [1]. In higher dimensions, with $p = 2k$ and $q = 2$, special cases of right-flat Grassmann structures include quaternion-Kähler and hyper-Kähler structures.

### 3.2. Twistor spaces

In the case of right-flat Grassmann structures, there is an associated complex manifold $Z$ of dimension $(p + 1)(q - 1)$ which defines the structure. This manifold is constructed as follows.

Consider a $(p, q)$ Grassmann structure on a complex manifold $M$ as above, and assume we have a local basis $\{\epsilon^a | a = 1, \ldots, n\}$ for $T^*M$. The isomorphism (3.1) implies we may write this as $\{\epsilon^{AA'} | A = 1, \ldots, p, A' = 1, \ldots, q\}$. Given any $\pi_x \in V_x$, we define the annihilator

$$\pi_x \perp := \{\phi_x \in V_x^* | \langle \phi_x, \pi_x \rangle = 0\} \subset V_x^*.$$

Let $\Lambda \subset \Omega(V)$ be the distribution on the total space of the bundle $V$ generated by the 1-forms

$$\sigma^A := \phi_{A'} \epsilon^{AA'}, \quad \phi \in \Gamma(\pi \perp), \quad (3.3)$$

$$\sigma^{A'} := d\pi^{A'} + \gamma^{A' B'} \pi^{B'}, \quad (3.4)$$
where $\gamma_{A'B'}$ denote the components of the connection on $V$. Complex conjugation gives the complex conjugate distribution $\bar{A}$. The sub-bundle of $T(V)$ annihilated by $\Lambda$ and $\bar{A}$ is spanned by the distributions $T$ and $\bar{T}$, where $T \subset T(V)$ is spanned by the vector fields

$$v_A := \pi^{A'} D_{AA'}, \quad A = 1, \ldots, p.$$  \hfill (3.5)

The distribution $\Lambda$ is closed ($d\Lambda \subset \Lambda^1 \wedge \Lambda$) if and only if the distribution $T$ is closed under Lie brackets $[T, T] \subset T$. It is straightforward to show from Eq. (3.5) that given $\lambda, \chi \in \Gamma(U)$

$$[v_\lambda \otimes \pi, v_\chi \otimes \pi] \equiv -T(\lambda \otimes \pi, \chi \otimes \pi) + (R(\lambda \otimes \pi, \chi \otimes \pi)\pi)^{A'} \frac{\partial}{\partial \pi^A} \pmod{T}.$$  \hfill (3.6)

Therefore, if we wish the space $T$ to be closed under Lie bracket, we require

$$\langle \phi, T(\lambda \otimes \pi, \chi \otimes \pi) \rangle = 0 \quad \forall \phi \in \Gamma(\pi^\perp) \quad \forall \lambda, \chi \in \Gamma(U),$$  \hfill (3.7)

$$R(\lambda \otimes \pi, \chi \otimes \pi)\pi = 0 \quad \forall \lambda, \chi \in \Gamma(U).$$  \hfill (3.8)

It is straightforward to show that if we fix the connections $\nabla$ on $U$ and $V$ so as to preserve the scale, as mentioned in Section 3, then Eq. (3.8) implies that the Grassmann structure is right-flat. Therefore, the distribution $T$ is integrable (equivalently $\Lambda$ is closed under exterior differentiation) if and only if the Grassmann structure is right-flat.

We wish to consider the projective version of this construction. Treating the section $\pi$ as homogeneous coordinates on the projective space $P_{q-1} \cong \mathbb{P}(V)p$ for each $p \in M$, we may introduce complex coordinates on the region $U_1 = \{ \pi \in C^q | \pi^1 \neq 0 \}$

$$\lambda^i = \frac{\pi^i}{\pi^1}, \quad i = 2, \ldots, q.$$  \hfill (3.9)

The projections of the 1-forms above are

$$\sigma^A := \epsilon^A + \lambda^2 e^{A2} + \ldots + \lambda^q e^{Aq}, \quad A = 1, \ldots, p,$$

$$\sigma^i := d\lambda^i - A^i, \quad i = 2, \ldots, q.$$  \hfill (3.10)

where $A$ is the projective version of the connection. We again denote the distribution in $\mathbb{P}(V)$ defined by these 1-forms by $\Lambda$. Similarly, a distribution, again denoted $T \subset T(\mathbb{P}(V))$, is spanned by the projection of the vector fields (3.5)

$$v_A := D_1 + \lambda^2 D_2 + \ldots + \lambda^q D_q, \quad A = 1, \ldots, p.$$  \hfill (3.11)

The distribution spanned by these vector fields is integrable if and only if the Grassmann structure on $M$ is right-flat. The integrability of this distribution implies we have a set of integrable $p$-dimensional planes in $\mathbb{P}(V)$. Quotienting out $\mathbb{P}(V^+)$ by this distribution\(^1\) therefore defines a quotient manifold $Z$ of dimension $(p + 1)(q - 1)$, which we denote by

\(^1\) We are assuming that there is nothing globally pathological about the fibration, and that such a quotient operation is justified.
Z. We therefore have a map $p : Z \to M$, where the image of a point in $Z$ is a $p$-dimensional plane in $\mathbf{P}(V)$ (in twistorial terminology, an $\alpha$-plane). We can then define the distribution $p^* \Lambda \subset \Lambda(Z)$, which is involutive on $Z$. Since the dimension of $p^* \Lambda$ equals the dimension of $Z$, this distribution therefore determines an almost-complex structure on $Z$. Moreover, since $p^* \Lambda$ is involutive, this almost-complex structure is integrable. Therefore $Z$ is a complex manifold of dimension $(p + 1)(q - 1)$.

We now wish to invert this process and construct the manifold $M$ from a generic complex manifold $Z$. Given a point $p \in M$, its image in the manifold $Z$ constructed above is a copy of $\mathbf{P}^{q-1} \subset Z$, corresponding to the fibre $\mathbf{P}(V)_p$. We therefore wish to reconstruct the manifold $M$ as the parameter space of embedded $\mathbf{P}^{q-1}$'s in $Z$. In order to carry out this construction, we need to determine the normal bundle, $N$, of such an embedded sub-manifold.

In the notation of Eq. (3.4), the co-normal bundle, $N^*$, is spanned by the forms $\{ \phi_A \in \Lambda^A | \phi \in \pi^\perp \subset V^*, A = 1, \ldots, p\}$. The co-normal bundle is therefore isomorphic to $p$ copies of the bundle $\pi^\perp \subset V^*$, which annihilates the element $\pi \in V$. Given $x \in P_{q-1}$, $\pi_x$ is an element of the complex line in $C^q$ corresponding to the point $x$, i.e. an element of the $L_x$, where $L$ denotes the tautological bundle $L := H^{-1}$. We define the Universal Quotient bundle, $Q$, so that the short sequence of vector bundles

$$0 \to L \to C^q \to Q \to 0$$

is exact, where $C^q$ denotes the trivial rank $q$ vector bundle over $P_{q-1}$. The bundle $\pi^\perp$ is therefore isomorphic to $Q^*$, the dual of the quotient bundle. From the fact that $Q \cong H \otimes TP_{q-1}$ [6], we deduce that

$$N^* \cong \bigoplus_1^p \Omega^1(1), \tag{3.12}$$

where for a general manifold $X$, $\Omega^r$ denotes the bundle of $r$-forms on $X$, and in the particular case of $X = P_{q-1}$, we define

$$\Omega^r(k) := \Omega^r(P_n) \otimes H^k.$$

The dual of Eq. (3.12) provides the normal bundle of $P_{q-1} \subset Z$

$$N \cong (\bigoplus_1^p H^{-1}) \otimes T(P_{q-1}). \tag{3.13}$$

The Grassmann manifold $M$ is reconstructed as the set of embedded $P_{q-1}$'s in $Z$. Given that we know the form of the normal bundle of an embedded $P_{q-1}$ corresponding to a point $x \in M$, the number of deformations of the projective space follows from Kodaira’s theorem: if $H^1(P_{q-1}, N) = 0$, then the space of embedded $P_{q-1}$’s is a complex analytic manifold $M$, and the tangent space, $T_x M$, is isomorphic to $H^0(P_{q-1}, N)$. To calculate these cohomology groups, we need some results concerning vector bundles over complex projective spaces [12]. Serre duality states that for a holomorphic vector bundle $E$ over a (projective algebraic) complex $n$ manifold $X$, we have the isomorphism

$$H^q(X, E) \cong (H^{n-q}(X, K_X \otimes E^*))^*,$$
where $K_X$ denotes the canonical bundle of $X$. On such a manifold we also have the identification
\[(\Omega^r)^* \cong (\Omega^n)^* \otimes \Omega^{n-r}.\]
For a complex projective space
\[K_{P_n} \cong H^{-(n+1)},\]
so in this case, we have
\[H^q(P_n, \Omega^p(k)) \cong (H^{n-q}(P_n, \Omega^n(-k))^*). \tag{3.14}\]
Results of Bott [3] then tell us that
\[
\dim_C H^q(P_n, \Omega^p(k)) = \begin{cases} 
(n + k - p) \binom{k - 1}{p}, & q = 0, \ 0 \leq p \leq n, \ k > p, \\
1, & k = 0, \ 0 \leq p = q \leq n, \\
\binom{-k + p}{-k} \binom{-k - 1}{n - p}, & q = n, \ 0 \leq p \leq n, \ k < p - n, \\
0, & \text{otherwise}.
\end{cases} \tag{3.15}
\]
Applying these results, we first show that $H^1(P_{q-1}, N) = 0$. From Eqs. (3.12) and (3.14), we find that
\[H^1(P_{q-1}, N) \cong (H^{q-2}(P_{q-1}, K \otimes N^*))^* \cong (H^{q-2}(P_{q-1}, H^{-q} \oplus \bigoplus_1 \Omega^1))^* \cong \bigoplus_1 (H^{q-2}(P_{q-1}, \Omega^1(1-q)))^* \cong \bigoplus_1 H^1(P_{q-1}, \Omega^{q-2}(q - 1)) \cong 0,
\]
where the last equality follows from Eq. (3.15). Therefore $T_x M$ is isomorphic to $H^0(P_{q-1}, N)$, where
\[H^0(P_{q-1}, N) \cong \bigoplus_1 H^0(P_{q-1}, \Omega^{q-2}(q - 1)) \cong \mathbb{C}^{pq}.
\]
by a similar argument to that given above. Therefore, given an embedded $P_{q-1}$ in a complex manifold $Z$ of complex dimension $(p + 1)(q - 1)$ with normal bundle as in Eq. (3.13), there will exist an $n = pq$ parameter family of such spaces. In the usual fashion, the integrability of the complex structure on $Z$ then implies that $M$ carries a right-flat Grassmann structure.

4. Integrable systems interpretation

The integrability of the distribution $T$ defined by the vector fields (3.11) implies the existence of functions $C^C_{AB}$ with the property that
\[
[v_A, v_B] = \sum_{C=1}^{p} C^C_{AB} v_C, \quad A, B = 1, \ldots, p, \tag{4.1}
\]
where, we recall, the vector fields $v_A$ are defined by

$$v_A = D_1 + \lambda^2 D_2 + \cdots + \lambda^q D_q$$

with

$$D_i = e_i - A_i^j \frac{\partial}{\partial \lambda^j}$$

with the $A_{ij}$ quadratic polynomials in the spectral parameters $\lambda^i$. As such, the $p$ vector fields $v_A$ are sections of the tangent bundle of $P(V)$, and correspond to differential operators which depend on a set of $(q-1)$ spectral parameters $(\lambda^2, \ldots, \lambda^p)$. More properly, these parameters correspond to a section of the line bundle $H$ over $P_{q-1}$, so our “spectral parameter” now lives in $P_{q-1}$, unlike the usual case where we have a single spectral parameter in $P_1$. A power counting argument implies that the functions $C_{AB}^C$ are quadratic polynomials in the complex coordinates $\lambda^i$, corresponding to sections of the bundle $H^2$.

As in the description of anti-self-dual conformal structures in four dimensions, the differential operators $v_A$ contain partial derivatives with respect to these spectral parameters, corresponding to the fact that the complex manifold $Z$ generally does not fibre holomorphically over $P_{q-1}$.

The integrability of the distribution $T$ is equivalent to the fact that the distribution $A$ is involutive. Integrability of $T$ implies the integrability of a distribution of $p$-dimensional planes in $P(V)$, and the existence of $(p+1)(q-1)$ functions $f^\alpha$ such that the planes are level sets of these functions. Equivalently, the distribution $A$ is spanned by the differentials $\{df^\alpha\}$. If we then quotient out by the $p$-dimensional distribution to construct the manifold $Z$, then the functions $f^\alpha$ descend to holomorphic functions on the manifold $Z$, and $\{df^\alpha\}$ generate $A^{(1,0)}(Z)$.

In terms of the Grassmann manifold $M$, the $\frac{1}{2} p(p-1)$ equations (4.1) are the integrability condition for over-determined set of equations for a function $f(x: \lambda)$:

$$v_A f = 0, \quad A = 1, \ldots, p.$$  \hspace{1cm} (4.2)

When Eq. (4.1) are satisfied, there exist $(p+1)(q-1)$ linearly independent solutions of these equations $\{f_\alpha\}$. The sub-space $\{f_\alpha = \text{constant}\} \subset P(V)$ are then the $\alpha$ planes of our right-flat Grassmann structure. The functions $\{f_\alpha\}$ then descend to holomorphic functions on the quotient manifold $Z$.

In integrable systems terminology, Eq. (4.2) is the associated linear problem for the right-flat Grassmann structure. The compatibility condition Eq. (4.1) then ensures the integrability of the system. There are several non-standard elements of this construction, however. Firstly, the analogue of the spectral parameter of standard integrable systems theory in these equations is the set of affine coordinates $\{\lambda^i\}$ on the $(q-1)$-dimensional complex projective space $P_{q-1}$.

Secondly, as opposed to the usual “Lax pair” formulation of integrable systems, we are here forced to consider a “Lax $p$-tuple” of operators, i.e. the vector fields $v_A$, which must define an integrable distribution for the complex structure on the manifold $Z$ to be integrable.

As in the simpler case of anti-self-dual conformal structures in dimension 4 (and similarly three-dimensional Einstein–Weyl structures), the differential operators we consider...
generally contain derivatives in the spectral projective space, corresponding to the fact that the complex manifold $Z$ generally does not fibre over the complex projective space $P_{q-1}$.

4.1. Relations with Ward’s systems

Eq. (4.1) are, in some sense, an analogue of a construction due to Ward [16] for gauge fields. Ward considered principal $G$-bundles with a connection $A \in \Gamma(\Lambda^1 \otimes \mathfrak{g})$. We consider a field $\psi$ in a representation of $G$, and consider the over-determined set of linear equations

$$D\psi \alpha = 0, \quad \alpha = 1, \ldots, p,$$

(4.3)

where $D\psi$ denotes the covariant derivative of the field $\psi$ with respect to the connection $A$, and the $V_\alpha$ are vector fields. Moreover, the vector fields $V_\alpha$ are taken to depend on a set of complex parameters $\{\lambda^A | A = 1, \ldots, q\}$, being a homogeneous polynomial of degree $N$ in these parameters. (The vector fields may therefore be identified with a section of $TM \otimes H^N$, where $H$ denotes the Hopf bundle over the complex projective space $P_{q-1}$.) For fixed $\lambda^A$, Eq. (4.3) are actually $p \dim \mathfrak{g}$ differential equations for $\dim \mathfrak{g}$ unknowns, and so are over-determined if $p > 1$. Since the system is over-determined, the existence of a maximal family of solutions places a set of algebraic constraints on the curvature $F$ of the connection

$$F(V_\alpha, V_\beta) = 0, \quad \alpha, \beta = 1, \ldots, p.$$  

(4.4)

In the cases where the set of polynomial vector fields $V_\alpha$ are suitably non-degenerate, Eq. (4.4) can be completely solved by twistorial techniques [16].

The connection with Grassmann structures arises if we consider the case of linear polynomials corresponding to $N = 1$. In this case, the non-degeneracy condition mentioned above is analogous to the defining isomorphism (3.1). If we assume the underlying manifold of the theory is $\mathbb{R}^{pq}$ with coordinates $\{x^a : a = 1, \ldots, pq\}$, and that the connection is constant (i.e. independent of the $x^a$), then the integrability conditions above become a set of algebraic equations on the connection

$$[A(V_\alpha), A(V_\beta)] = 0, \quad \alpha, \beta = 1, \ldots, p.$$

If we now take the connection $A$ to have values in the tangent bundle of an auxiliary manifold $\tilde{M}$, then we may write

$$A(V_\alpha) = \sum_{a=1}^n \sum_{A=1}^q \sigma^a_{\alpha A} \lambda^A e_a,$$

(4.5)

where $\{e_a | a = 1, \ldots, n\}$ denotes a basis of vector fields on the manifold $\tilde{M}$. The integrability conditions (4.4) then reduce to a set of relations on the commutators of the vector fields $\{e_a\}$ on $\tilde{M}$

$$\sum_{A,B=0}^q \sum_{a,b=1}^n \lambda^A \lambda^B \sigma^a_{\alpha A} \sigma^b_{\beta B} [e_a, e_b] = 0.$$  

(4.6)
where \([\ ,\]\] denotes the Lie bracket of vector fields. Imposing Eq. (4.6) for all values of the parameters \(\lambda^A\), we recover Eq. (4.1) with \(C_{AB}^C = 0\) in the case when the derivatives with respect to the spectral parameter are not present.

If we allow derivatives with respect to the spectral parameter, then, in the terminology of Park [13], our equations for general right-flat Grassmann structures may therefore be considered as a \(P_{q-1}\)-extension of Ward’s equations for a constant connection on flat space with values in the tangent bundle of an auxiliary \(n\)-manifold \(\bar{M}\).

A special case of these equations without derivatives with respect to the spectral parameters is Joyce’s interpretation of the equations for hyper-complex conformal structures in four dimensions [7], which in turn is a generalisation of the description of anti-self-dual Ricci-flat structures due to Mason and Newman [8].

In terms of Ward’s classification of systems in dimensions up to 11, Grassmann structures of type \((p, q) = (k, 2)\) correspond to Ward’s systems \(A_k\), \(p = 2, q = m + 1\) correspond to his \(C_m\), and \(p = q = 3\) correspond to his system \(D\). The geometrical analogue of Ward’s systems with higher order homogeneous polynomials correspond to twistor spaces \(Z\) containing embedded \(P_{q-1}\)’s with more complicated normal bundle, sections of which can be identified with a collection of sections of the bundle \(H^n\). Unfortunately, there does not seem to be any simple geometrical interpretation of these systems in general.

5. Remarks and conclusions

If we wish to take seriously the idea that at the heart of classical integrable systems is a connection with complex geometry, then implicit in the formulation of Grassmann structures given in Eq. (4.1) are several generalisations of standard notions of integrability.

Firstly, the analogue of the spectral parameter of standard integrable systems theory in these equations is the set of affine coordinates \(\{\lambda^i\}\) on the \((q-1)\)-dimensional complex projective space \(P_{q-1}\). In other words, the spectral parameter lives in a general complex projective space.

Secondly, as opposed to the usual “Lax pair” formulation of integrable systems, we are here forced to consider a “Lax \(p\)-tuple” of operators, i.e. the vector fields \(v_A\), which must define an integrable distribution for the complex structure on the manifold \(Z\) to be integrable.

Finally, as in the simpler case of anti-self-dual conformal structures in dimension 4 (and similarly three-dimensional Einstein–Weyl structures), the differential operators we consider generally contain derivatives in the spectral projective space, corresponding to the fact that the complex manifold \(Z\) generally does not fibre over the complex projective space \(P_{q-1}\).

The only case in which we recover a standard Lax pair construction with spectral parameter in \(P_1\) is the case \((p, q) = (2, 2)\), when the twistor space fibres over \(P_1\). Geometrically, this corresponds to the description of (complexified) hyper-complex structures in four dimensions.

The second observation above is consistent with the complex-manifold approach to hyper-complex and quaternionic-Kähler manifolds of real dimension \(4k\), where the points of the manifold correspond to rational curves \((q = 2)\) with normal bundle \(\bigoplus_1^{2k} H\) in a complex manifold \(Z\) of dimension \(2k + 1\). One could further generalise this picture by considering
more general embedded complex sub-manifolds than $P_{q-1}$, with more complicated normal bundles (see, for example [10]). The geometrical structures induced on the space of such sub-manifolds is, however, rather unclear, and does not appear to have any straightforward interpretation in terms of integrable systems. Even if we restrict ourselves to embedded rational curves, Grothendieck’s theorem implies that the most general normal bundle is of the form $N \cong \bigoplus_{i=1}^{n} H^{m_i}$, for integers $m_i$, but the geometrical interpretation of the induced structure on the space of rational curves for general $m_i$ is far from apparent.

Finally, we should note that we have considered only complex Grassmann structures. If we consider an analytic real Grassmann manifold, where the complexified tangent bundle splits as a tensor product, then we may complexify the manifold and use the complex construction of the twistor space given above. However, there does not seem to be any straightforward definition of the twistor space in the case of non-analytic real Grassmann manifolds. The hope would be that analyticity follows from existence of a right-flat Grassmann structure, in the same way that in four dimensions the existence of an anti-self-dual conformal structure implies the existence of a real analytic structure [1]. From the twistorial point of view, we require that the complex manifold $Z$ admit a real structure (i.e. an anti-holomorphic involution), $\sigma$, and that there be a $n$-parameter family of real embedded $P_{q-1}$’s which are invariant under this map. These invariant $P_{q-1}$’s then correspond to points of the manifold $M$. Since the manifold $M$ is then a real sub-manifold of a complex-analytic manifold, it then necessarily admits a real-analytic structure. The existence (or not) of fixed points of $\sigma$ then allows us to attribute a signature to the induced Grassmann structure on $M$, with the fixed point set generically defining a real projective space, which determines the set of null planes at a given point in $M$. A real structure on $Z$ with no fixed points would define the analogue of a Riemannian structure of Theorem 1.1.

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References

Paper 3

Coisotropic variational problems,
with E. Musso,
Coisotropic variational problems

Emilio Musso*, James D.E. Grant

Dipartimento di Matematica Pura ed Applicata, Università di L’Aquila,
Via Vetoio, 67100 L’Aquila, Italy

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Abstract

In this paper we study constrained variational problems in one independent variable defined on the space of integral curves of a Frenet system in a homogeneous space $G/H$. We prove that if the Lagrangian is $G$-invariant and coisotropic then the extremal curves can be found by quadratures. Our proof is constructive and relies on the reduction theory for coisotropic optimal control problems. This gives a unified explanation of the integrability of several classical variational problems such as the total squared curvature functional, the projective, conformal and pseudo-conformal arc-length functionals, the Delaunay and the Poincaré variational problems.

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1. Introduction

The present paper is an outcome of our attempt to understand the general mechanisms underlying the integrability of constrained variational problems for curves of constant type in homogeneous spaces [10,11,17,18]. The Pfaffian differential systems arising from curves of constant type lead to the notion of generalized Frenet system for curves of a homogeneous space $G/H$. Roughly speaking, a generalized Frenet system of order $k$ on $G/H$ is a $G$-invariant submanifold $S \subset J^k(\mathbb{R}, G/H)$ of the jet space $J^k(\mathbb{R}, G/H)$, which may be linearized by a left-invariant affine sub-bundle of $T(G)$. From the geometrical viewpoint the integral curves of such systems are canonical lifts of curves of constant type on $G/H$.

* Corresponding author. Tel.: +39-0862-433128; fax: +39-0862-433180.
E-mail addresses: musso@univaq.it (E. Musso), grant@dm.univaq.it (J.D.E. Grant).
The most elementary example is the classical Frenet–Serret differential system for generic curves in Euclidean space. We then consider a $G$-invariant Lagrangian and we investigate the corresponding Euler–Lagrange system. The general construction of the momentum space and of the Euler–Lagrange system of a constrained variational problem in one independent variable is due to Griffiths (we refer to [5,12,16] as the standard references on the subject and to [6] as the original source of inspiration of the approach developed by Griffiths). We adhere to the terminology introduced in [5,12] and say that a Lagrangian $L$ is non-degenerate if the momentum space $Y$ is odd-dimensional and if the canonical 2-form on $Y$ has maximal rank. We prove that if the Lagrangian $L$ is $G$-invariant and coisotropic (see Definition 4.3) then the extremal curves of the variational problem can be found by quadratures. The proof relies on the reduction theory of Hamiltonian systems with symmetries (see [2,9,14,15] for the standard theory in the symplectic category and [1,5,24,27] for generalizations to contact geometry, time-dependent Hamiltonian systems and Poisson manifolds). One of the ingredients of the proof is a concrete geometric description of the Marsden–Weinstein reduced spaces in terms of the phase portraits of the system. This procedure is constructive and applies to several concrete examples (see Refs. [5,7,12,16,20–23]).

The paper is organized as follows. In the next section we recall the basic definitions and properties of linear control systems on Lie groups and Frenet systems of curves in homogeneous spaces. In Section 3, we examine variational problems defined by invariant Lagrangians for linear control systems on Lie groups. From a geometrical viewpoint we deal with $k$th order variational problems for curves of constant type in a homogeneous space that depend on the generalized curvatures. Since all the derived systems have constant rank, the extremal curves of the variational problem are the projections of the integral curves of the Euler–Lagrange system. Therefore, we focus our attention on the momentum space and on the Euler–Lagrange system. First we investigate the geometry of the momentum space $Y$ of a regular invariant Lagrangian of a linear control system of a Lie group $G$. We show that $Y$ is of the form $G \times \mathcal{F}$, where $\mathcal{F}$ is an immersed submanifold of $g \times g^*$ (we call $\mathcal{F}$ the phase space of the variational problem). Next we study non-degenerate Lagrangians. We prove that if $L$ is non-degenerate, then the phase space $\mathcal{F}$ can be realized as a submanifold of $g^*$. We define the linearized phase portraits and the Legendre transform and analyze the structure of the characteristic vector field of a non-degenerate Lagrangian. In Section 4 we study coisotropic Lagrangians. We prove that the integral curves of the characteristic vector field passing through a point of the bifurcation set are orbits of one-parameter subgroups of the symmetry group $G$. Therefore, from this point on, we focus our attention on the regular part $Y_t$ of the momentum space. We show that $Y_t$ is of the form $G \times \mathcal{F}_t$, where $\mathcal{F}_t$ is an open subset of the phase space. We prove that $\mathcal{F}_t$ intersects the coadjoint orbits, $O(\mu)$, of $G$ transversally and that $\mathcal{P}_t(\mu) = \mathcal{F}_t \cap O(\mu)$ are smooth curves (referred to as the phase portraits). Subsequently we introduce the moment map $J : Y_t \to g^*$ and prove that the Marsden–Weinstein reduction $J^{-1}(\mu)/G_\mu$ can be naturally identified with the phase portrait $\mathcal{P}_t(\mu)$. We also show that every $\mu \in J(Y_t)$ is a regular element of $g^*$ which implies that the isotropy subgroups $G_\mu$ are Abelian, for every $\mu \in J(Y_t)$. We then examine more closely the phase flow $\phi$ and the characteristic vector field $\xi$. We prove that if the Lie algebra $g$ possesses a non-degenerate Ad-invariant inner product then the differential equation fulfilled by the phase flow can be written in Lax form. From the Noether conservation theorem we know that the characteristic vector field $\xi$ is tangent to
the fibers $J^{-1}(\mu)$ of the moment map. We define a canonical connection form $\theta^\mu$ on the Marsden–Weinstein fibrations $J^{-1}_r(\mu) \to \mathcal{P}(\mu)$ whose horizontal curves are the integral curves of the characteristic vector field. Since the base is one-dimensional and the structure group $G_\mu$ is Abelian, the horizontal curves can be found by a single quadrature. This shows that the extremal curves of an invariant coisotropic Lagrangian are integrable by quadratures. As a byproduct, we prove that if the canonical connection $\theta^\mu$ is complete, which is generically the case when $\mu$ is a regular value of the moment map, then the connected components of $J^{-1}_r(\mu)$ are Euclidean cylinders and the characteristic vector field $\xi$ can be linearized on $J^{-1}_r(\mu)$. We would like to stress that the connection form $\theta^\mu$ can be constructed explicitly from the data of the problem, so that the integration process can be performed in a completely explicit way.

Finally, in two appendices, we summarize the background material that we use from the theory of Pfaffian differential systems and constrained variational problems in one independent variable.

Throughout the paper, we demonstrate how our general results apply to the specific example of isotropic curves in $\mathbb{R}^{(2,1)}$. We show how to derive the Frenet system for such curves, and show that the variational problem is coisotropic if we take the Lagrangian to be a linear function of the curvature. We prove that the phase portraits may be parameterized in terms of elliptic functions, and construct the sections of the Marsden–Weinstein fibration required to reduce the integration to quadratures. Other concrete geometrical examples where the general scheme described in this paper are implemented may be found in [5,6,20–23]. In all of these cases the generic phase portrait is an elliptic curve, so that the extremal curves can be integrated in terms of elliptic functions and elliptic integrals.

2. Linear control systems on Lie groups and Frenet systems in homogeneous spaces

2.1. Linear control systems on Lie groups

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ will be denoted by $(\eta, V) \in \mathfrak{g}^* \times \mathfrak{g} \to \langle \eta; V \rangle \in \mathbb{R}$. We let $\Theta \in \mathfrak{g}^* \otimes \mathfrak{g}$ be the Maurer–Cartan form of $G$. If we fix a basis $(e_0, \ldots, e_n)$ of $\mathfrak{g}$, then $\Theta = \theta^J \otimes e_J$, where $(\theta^0, \ldots, \theta^n)$ is the basis of $\mathfrak{g}^*$ dual to $(e_0, \ldots, e_n)$.

**Definition 2.1.** Let $\mathbb{A} \subset \mathfrak{g}$ denote an affine subspace of $\mathfrak{g}$ of the form $P + A = \{ P + A : A \in \mathfrak{a} \}$, where $P \in \mathfrak{g}$ and $\mathfrak{a} \in \text{Gr}_h(\mathfrak{g})$ with $P \notin \mathfrak{a}$. The set of such affine subspaces of $\mathfrak{g}$ will be denoted as $P^h(\mathfrak{g})$. We call $M := G \times \mathbb{A}$ the configuration space of the affine subspace $\mathbb{A}$, and denote by $\pi_G : M \to G$ and $\pi_{\mathbb{A}} : M \to \mathbb{A}$ the natural projections onto the two factors.

We now fix a left-invariant form $\omega \in \mathfrak{g}^*$ such that $\langle \omega; P \rangle = 1$ and $\omega \in \mathfrak{a}^\perp$ (i.e. $\langle \omega; A \rangle = 0$, for all $A \in \mathfrak{a}$). We may fix a basis $(e_0, \ldots, e_n)$ of $\mathfrak{g}$ such that

$$P = e_0, \quad \mathfrak{a} = \text{span}(e_1, \ldots, e_h),$$
and we let \( \theta^0, \ldots, \theta^n \) be the components of the Maurer–Cartan form with respect to \((e_0, \ldots, e_n)\). Such a basis may be chosen so that \( \theta^0 = \omega \). Using the projection \( \pi_G \), we may pull-back the differential 1-forms \( \omega, \theta^1, \ldots, \theta^n \) to \( M \), to define a set of 1-forms on \( M \) which, by the standard abuse of notation, we again denote by \( \omega, \theta^1, \ldots, \theta^n \in \Omega^1(M) \). Let \( k^1, \ldots, k^h \) denote the affine coordinates on \( \mathbb{A} \) defined by the affine frame \((P, e_1, \ldots, e_h)\). We then define the 1-forms

\[
\eta_j := \begin{cases}
\theta^j - k^j \omega, & j = 1, \ldots, h, \\
\theta^j, & j = h + 1, \ldots, n.
\end{cases}
\]

We then define the Pfaffian differential system \((\mathcal{A}, \omega)\) on \( M \) to be the Pfaffian differential ideal generated by the 1-forms \( \{\eta^j : j = 1, \ldots, n\} \) with the independence condition given by \( \omega \).

**Definition 2.2.** \((\mathcal{A}, \omega)\) is the linear control system associated to \( \mathbb{A} \in P^h(\mathfrak{g}) \).

Note that the ideal \( \mathcal{A} \) has constant rank, being generated by a rank \( n \) sub-bundle, \( Z \subset T^*(M) \). The sub-bundle \( Z \) is of the form \( G \times \mathcal{Z} \), where

\[
\mathcal{Z} = \{(Q, \eta) \in \mathbb{A} \times \mathfrak{g}^* : \langle \eta; Q \rangle = 0\} \subset \mathbb{A} \times \mathfrak{g}^*,
\]

and where the embedding of \( Z \) as a sub-bundle of \( T^*(M) \) is given by

\[
(g, Q, \eta) \in G \times \mathcal{Z} \rightarrow \pi_G^*(\eta)_{(g, Q)} \in T^*_G(g, Q), M.
\]

We may use the left-invariant trivialization to identify \( T(G) \) and \( G \times \mathfrak{g} \). The tangent space to \( M \) at \((g, Q)\) is then identified with \( \mathfrak{g} \oplus \mathfrak{a} \). With this identification at hand, the integral elements of \((\mathcal{A}, \omega)\) at \((g, Q)\) are the one-dimensional subspaces of \( \mathfrak{g} \oplus \mathfrak{a} \) of the form \((Q, v)\), where \( v \in \mathfrak{a} \).

A smooth curve \( \gamma = (\alpha, \beta) : (a, b) \rightarrow M \), where \((a, b) \subseteq \mathbb{R} \) is a parameterized integral curve of the control system \((\mathcal{A}, \omega)\) if and only if \( \alpha : (a, b) \rightarrow G \) is a solution of the linear system \( \alpha(t)^{-1} \alpha'(t) = \beta(t) \). Thus, as a control system, the points of the affine space \( \mathbb{A} \) play the role of the inputs. Note that if we assign a smooth map \( \beta : (-\epsilon, \epsilon) \rightarrow \mathbb{A} \) and a point \( g_0 \in G \), then there exists a unique integral curve of the control system, \( \gamma = (\alpha, \beta) \), satisfying the initial condition \( \alpha(0) = g_0 \).

Consider the linear subspaces \( a_k \subset \mathfrak{g} \) defined recursively by

\[
a_1 = \mathfrak{a} + \text{span}(P), \ a_2 = a_1 + [a_1, a_1], \ldots, a_k = a_{k-1} + [a_{k-1}, a_{k-1}].
\]

The smallest integer \( N \) such that \( a_N = a_{N+1} \) is called the *derived length* of \( \mathbb{A} \). Note that \( Z_s := \mathbb{A} \times a_s^\perp \) is contained in \( Z_s \), for \( s = 1, \ldots, N \). We set \( Z_s = M \times Z_s, s = 1, \ldots, N \)

---

1. We will generally follow the usual practice in the method of moving frames and omit the pull-back signs to simplify notation. This should cause no confusion as we will clearly specify the manifolds that we are working on.

2. More invariantly, if we fix \( \omega \in \mathfrak{g}^* \) with \( \omega \in a_1^\perp \) and \( \langle \omega; P \rangle = 1 \), then we define the \( \mathfrak{g} \)-valued 1-form \( \hat{\theta} \in \Omega^1(M, \mathfrak{g}) \) by the formula \( \hat{\theta}|_{(g, Q)} := \pi^*_G(\theta - Q\omega)|_{(g, Q)} \). \( \mathcal{A} \) is then the differential ideal generated by \( \{\mu; \hat{\theta} : \mu \in \mathfrak{g}^*\} \).
and we consider the sequence of sub-bundles
\[ Z_N \subset Z_{N-1} \subset \cdots \subset Z_1 \subset Z. \]
If we denote by \( \mathcal{A}_s \) the Pfaffian differential ideal generated by \( Z_s \), then
\[ \mathcal{A}_N \subset \mathcal{A}_{N-1} \subset \cdots \subset \mathcal{A}_1 \subset \mathcal{A} \]
is the derived flag of the control system (see Refs. [4,12] for more details about derived flags). We have thus proved the following proposition.

**Proposition 2.3.** All the derived systems of a linear control system on a Lie group \( G \) have constant rank.

### 2.2. Frenet systems in homogeneous spaces

Let \( H \subset G \) be a closed Lie subgroup and consider the homogeneous space \( G/H \). The left-action of \( G \) on \( G/H \) induces an action of \( G \) on the jet space \( J^k(\mathbb{R}, G/H) \), called the \( k \)th prolongation of the action of \( G \) on \( G/H \).

**Definition 2.4.** A differential relation (in one independent variable) of order \( k \) on \( G/H \) is a submanifold \( S \) of \( J^k(\mathbb{R}, G/H) \) such that \( dt\big|_S \) is nowhere vanishing. We define the Pfaffian differential system \( (\mathcal{I}, dt) \) on \( S \) given by restriction to \( S \) of the canonical contact system on \( J^k(\mathbb{R}, G/H) \). A smooth curve \( \gamma : (a, b) \to G/H \) is said to be of type \( S \) if \( j^k(\gamma)|_t \in S \), for all \( t \in (a, b) \).

Note that the integral curves of the Pfaffian differential system \( (\mathcal{I}, dt) \) are the \( k \)-order jets \( j^k(\gamma) \) of curves \( \gamma : (a, b) \to G/H \) that satisfy the differential relation \( j^k(\gamma)|_t \in S \), for all \( t \in (a, b) \).

**Definition 2.5.** A Frenet system of order \( k \) on \( G/H \) is a triple \( (S, \mathcal{A}, \Phi) \), where:

1. \( S \subset J^k(\mathbb{R}, G/H) \) is a \( G \)-invariant differential relation of order \( k \) endowed with the induced contact system \( (\mathcal{I}, dt) \),
2. \( \mathcal{A} \in P^h(g) \),
3. \( \Phi : S \to M \) is a smooth equivariant map from \( S \) onto an open subset \( \Phi(S) \) of \( M = G \times \mathbb{A} \), the configuration space,

with the properties that:

- If \( \gamma : (a, b) \to G/H \) is a smooth curve of type \( S \) then \( \Gamma = \Phi \circ j^k(\gamma) \) is an integral curve of the control system \( (\mathcal{A}, \omega) \).
- If \( \Gamma : (a, b) \to M \) is an integral curve of \( (\mathcal{A}, \omega) \) such that \( \text{Im}(\Gamma) \subset \Phi(S) \), then \( \gamma = \pi_{G/H} \circ \Gamma : (a, b) \to G/H \) is a curve of type \( S \) and \( \Gamma = \Phi \circ j^k(\gamma) \).

The method of moving frames [11,12] gives an algorithmic procedure for the construction of the Frenet systems for curves of constant type in homogeneous spaces (see [10,17,18]). We refer the reader to [12] for the explicit construction of the Frenet system of generic...
curves in the affine space \( \mathbb{R}^3 \), to [7,23] for the Frenet system of generic curves in \( \mathbb{RP}^2 \), to [20,26] for the Frenet system of generic curves in the conformal 3-sphere and to [22] for the Frenet systems of generic Legendrian curves in the strongly pseudo-convex real hyperquadric \( Q^3 \) of \( \mathbb{CP}^2 \).

**Definition 2.6.** Let \( F := \pi_S \circ \Phi : S \to G \) and \( K := \pi_{\mathbb{A}} \circ \Phi : S \to \mathbb{A} \) denote the two components of the map \( \Phi \). We call \( F \) the Frenet map and \( K \) the curvature map.

Let \( \gamma : (a, b) \to G/H \) be a curve of type \( S \), then \( \Gamma = \Phi \circ J^k(\gamma) : (a, b) \to M \) is called the canonical lift of \( \gamma \). The maps

\[
F_\gamma := F \circ J^k(\gamma) : (a, b) \to G, \quad K_\gamma := K \circ J^k(\gamma) : (a, b) \to \mathbb{A}
\]

are called the Frenet frame field and the curvature function of \( \gamma \), respectively.

**Definition 2.7.** The generalized arc-length of a curve \( \gamma : (a, b) \to G/H \) of type \( S \) is the smooth function \( s_\gamma : (a, b) \to \mathbb{R} \), unique up to a constant, such that \( ds_\gamma = \Gamma^*(\omega) \), where \( \Gamma : (a, b) \to M \) is the canonical lift of \( \gamma \). Each curve \( \gamma \subset G/H \) of type \( S \) may be parameterized in such a way that \( ds_\gamma = dt \). In this case, we say that the curve \( \gamma \) is normalized.

**Proposition 2.8.** Let \((S, \mathbb{A}, \Phi)\) be a Frenet system. Then \( \Phi(S) = G \times U_\Phi \), where \( U_\Phi \) is an open subset of \( \mathbb{A} \).

**Proof.** We set \( U_\Phi = \pi_{\mathbb{A}}(\Phi(S)) \). Thus, \( U_\Phi \) is an open subset of \( \mathbb{A} \) such that \( \Phi(S) \subseteq G \times U_\Phi \). Take any \((g_0, Q_0) \in G \times U_\Phi \). Since \( Q_0 \in U_\Phi \) then there exists \( g_1 \in G \) such that \((g_1, Q_0) \in \Phi(S) \). Now let \( \Gamma : (-\epsilon, \epsilon) \to M \) be an integral curve of the control system \((\mathbb{A}, \omega)\) such that \( \Gamma(0) = (g_1, Q_0) \). Since \( \Phi(S) \) is an open set we may, by restricting the value of \( \epsilon \) if necessary, assume that \( \text{Im}(\Gamma) \subset \Phi(S) \). Then, the projection of \( \Gamma \) onto \( G/H \) is a curve \( \gamma : (-\epsilon, \epsilon) \to G/H \) of type \( S \) such that \( \Gamma = \Phi(J^k(\gamma)) \). Using the \( G \)-invariance of \( S \) it follows that \( g_0 g_1^{-1} \gamma \) is another curve of type \( S \). Thus, from the equivariance of \( \Phi \) it follows that \( g_0 g_1^{-1} \Gamma(0) = (g_0, Q_0) \) belongs to \( \Phi(S) \). This shows that \( G \times U_\Phi \subseteq \Phi(S) \).

The elements of the open subset \( U_\Phi \) may therefore be considered as the “geometrical inputs” of the control system \((\mathbb{A}, \omega)\). In particular, the curvature function \( K \) gives a complete set of local differential invariants for curves of type \( S \). More precisely, if \( \gamma, \tilde{\gamma} : (a, b) \to G/H \) are normalized curves of type \( S \) with \( K_\gamma = K_{\tilde{\gamma}} \), then \( \gamma \) and \( \tilde{\gamma} \) are congruent to one other, in the sense that there exists a \( g \in G \) such that \( g\gamma(t) = \tilde{\gamma}(t) \), for all \( t \in (a, b) \). Moreover, given any smooth map \( K : (a, b) \to U_\Phi \subset \mathbb{A} \) there exists a normalized curve \( \gamma : (a, b) \to G/H \) of type \( S \), unique up to congruence, such that \( K_\gamma = K \).

If we fix an affine frame \((P, e_1, \ldots, e_h)\) of \( \mathbb{A} \) and if we let \( k_1, \ldots, k_h \) be the corresponding coordinates, we may identify the configuration space \( M \) with \( G \times \mathbb{R}^h \). Thus, we may write \( K_\gamma = (k_\gamma^1, \ldots, k_\gamma^h) \), where \( k_\gamma^1, \ldots, k_\gamma^h \) are smooth functions that depend on the \( k \)-jet of \( \gamma \). These functions can be viewed as the generalized curvatures of \( \gamma \).
2.3. Isotropic curves in \( \mathbb{R}^{(2,1)} \)

An example that will illustrate our considerations concerns variational principles for isotropic curves in three-dimensional Minkowski space. Let \( \mathbb{R}^{(2,1)} \) denote Minkowski 3-space endowed with the Lorentzian inner product
\[
\langle v, w \rangle = -(v^1 w^3 + v^3 w^1) + v^2 w^2 =: g_{ij} v^i w^j.
\]
We fix the spatial orientation by requiring that the standard basis \( (e_1, e_2, e_3) \) is positively oriented, and we fix the time orientation defined by the positive light cone
\[
\mathcal{L}^+ = \{ v \in \mathbb{R}^{(2,1)} : \langle v, e_1 + e_3 \rangle < 0 \}.
\]
Let \( G \) be the restricted Poincaré group \( E(2,1) \), i.e. the group of isometries of \( \mathbb{R}^{(2,1)} \) that preserve the given orientations. The group \( G \) may conveniently be described as the space of pairs \( g = (q, A) \) where \( q \in \mathbb{R}^{(2,1)} \) and \( A = (A_1, A_2, A_3) \) is a \( 3 \times 3 \) matrix such that
\[
\det(A_1, A_2, A_3) = 1, \quad \langle A_i, A_j \rangle = g_{ij}, \quad A_1, A_3 \in \mathcal{L}^+.
\]
We let \( \mathfrak{g} \) denote the Lie algebra of \( G \), consisting of all matrices of the form
\[
X(q, v) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
q^1 v_1^1 & v_1^2 & 0 \\
q^2 v_2^1 & 0 & v_2^2 \\
q^3 & 0 & v_1^3 & -v_1^1
\end{pmatrix}.
\]
We now define the Maurer–Cartan form \( \Omega \in \Omega^1(G, \mathfrak{g}) \), which takes the form
\[
\Omega = \begin{pmatrix}
0 & 0 & \omega^1 & 0 \\
\omega^1 & \omega^1 & \omega^1 & 0 \\
\omega^2 & \omega^2 & 0 & \omega^2 \\
\omega^3 & 0 & \omega^2 & -\omega^1
\end{pmatrix},
\]
such that
\[
dq = \omega^i A_i, \quad dA_i = \omega^j A_j, \quad i = 1, 2, 3.
\]
Differentiating these relations, we obtain the structure equations
\[
d\omega^j = -\omega^j \wedge \omega^j, \quad d\omega^j_k = -\omega^j \wedge \omega^j_k, \quad i, k = 1, 2, 3.
\]
Recall that the Maurer–Cartan forms \( \omega^1, \omega^2, \omega^3, \omega^1, \omega^2, \omega^1 \) are linearly independent and generate the space, \( \mathfrak{g}^* \), of left-invariant 1-forms on \( G \).

**Definition 2.9.** A null (or isotropic) curve in \( \mathbb{R}^{(2,1)} \) is a smooth parameterized curve
\[
\alpha : (a, b) \subset \mathbb{R} \to \mathbb{R}^{(2,1)}
\]
such that $\alpha'(t) \in L^+$ for all $t \in (a, b)$. We shall assume that $\alpha$ is without flex points, in the sense that

$$\alpha'(t) \wedge \alpha''(t) \neq 0 \quad \forall t \in (a, b).$$

The linear differential form $\omega_{\alpha} := \|\alpha''(t)\|^{1/2} \, dt$ is nowhere vanishing, and is invariant under changes of parameter and the action of the group $G$. Without loss of generality we may assume that $\alpha$ is normalized, in the sense that

$$\|\alpha''(t)\|^{1/2} = 1 \quad \forall t \in (a, b).$$

(This condition fixes the parameter $t$ up to an additive constant.) The curvature of $\alpha$ is defined by

$$k(t) = -\frac{1}{2} \|\alpha'''(t)\|^2 \quad \forall t \in (a, b).$$

At each point of the curve we may define the frame $g(t) = (\alpha(t), A(t)) \in G$ given by

$$A_1(t) = \alpha'(t), \quad A_2(t) = \alpha''(t), \quad A_3(t) = \alpha'''(t) + \frac{1}{2} \|\alpha'''(t)\|^2 \alpha'(t).$$

This frame defines a canonical lift

$$g : t \in (a, b) \to g(t) = (\alpha(t), A(t)) \in G$$

of the curve $\alpha$ to the group $G$, referred to as the Frenet frame field along $\alpha$. An application of the method of moving frames shows that the Frenet frame field is the unique lift of $\alpha$ to $G$ with the property that

$$g^*(\Omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & \kappa & 0 \\ 0 & 1 & 0 & \kappa \\ 0 & 0 & 1 & 0 \end{pmatrix} \, dt.$$  

We illustrate the construction of the Frenet system for isotropic curves in $\mathbb{R}(2,1)$, viewed as a homogeneous space of the group $G$. Let

$$t, \quad X = (x^1, x^2, x^3), \quad X_1 = (x_1^1, x_1^2, x_1^3), \quad X_2 = (x_2^1, x_2^2, x_2^3), \quad X_3 = (x_3^1, x_3^2, x_3^3)$$

be the standard coordinates on the jet space $J^3(\mathbb{R}, \mathbb{R}(2,1)) \cong \mathbb{R} \times \mathbb{R}(2,1) \times \mathbb{R}(2,1) \times \mathbb{R}(2,1)$. The differential relation $S \subset J^3(\mathbb{R}, \mathbb{R}(2,1))$ is defined by

$$X_1 \in L^+, \quad \|X_2\| = 1, \quad (X_1, X_2) = (X_2, X_3) = 0, \quad X_1 \wedge X_2 \wedge X_3 \neq 0.$$  

Holonomic sections of $S$ are third-order jets $j^3(\alpha)$ of normalized isotropic curves $\alpha : (a, b) \to \mathbb{R}(2,1)$. We define $\kappa : S \to \mathbb{R}$ by

$$\kappa(t, X, X_1, X_2, X_3) = -\frac{1}{2} \|X_3\|^2.$$  

$\kappa[j^3(\alpha)]$ is then the curvature of the isotropic curve $\alpha$. 

The affine space $A = P + a \subset g$ is then the straight line

$$k \in \mathbb{R} \rightarrow Q(k) = e_0 + ke_1 \in g,$$

where

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

It is convenient to identify the configuration space $M = G \times A$ with $G \times \mathbb{R}$ by means of the map

$$(g, e_0 + ke_1) \in G \times A \rightarrow (g, k) \in G \times \mathbb{R}.$$  

With this identification at hand, the linear control system $(A, \omega)$ is generated by the linear differential forms

$$\eta^1 = \omega^2 - k\omega, \quad \eta^2 = \omega^1, \quad \eta^3 = \omega^2 - \omega, \quad \eta^4 = \omega^2, \quad \eta^5 = \omega^3,$$

along with independence condition

$$\omega = \omega^1.$$

We now consider a smooth curve $\Gamma : (a, b) \rightarrow M$ and let

$$g : t \in (a, b) \rightarrow (\alpha(t), A(t)) \in G, \quad k : t \in (a, b) \rightarrow k(t) \in A,$$

be the two components of $\Gamma$. Then $\Gamma$ is an integral curve of the control system $(A, \omega)$ if and only if $g : (a, b) \rightarrow G$ is the Frenet field along the isotropic curve $\alpha : (a, b) \rightarrow \mathbb{R}^{(2,1)}$, and $k$ is the curvature of the curve $\alpha$. The mapping $\Phi : S \rightarrow M = G \times \mathbb{R}$ linearizing the differential relation $S$ is defined by

$$S : (t, X, X_1, X_2, X_3) \in S \rightarrow ((X; X_1, X_2, X_3 + \frac{1}{2}\|X_3\|^2X_1), -\frac{1}{2}\|X_3\|^2) \in M.$$  

**Remark 2.10.** Using the structure equations for $\mathbb{E}(2, 1)$ we find that

$$d\omega = (\kappa \eta^4 - \eta^2) \wedge \omega - \eta^1 \wedge \eta^4,$$  

(1a)

$$d\eta^1 = -\pi \wedge \omega + \eta^1 \wedge \eta^2 + \kappa \eta^1 \wedge \eta^4,$$  

(1b)

$$d\eta^2 = (\kappa \eta^3 - \eta^1) \wedge \omega - \eta^1 \wedge \eta^3,$$  

(1c)

$$d\eta^3 = (2\eta^2 - \kappa \eta^4) \wedge \omega + \eta^1 \wedge \eta^4 + \eta^2 \wedge \eta^3,$$  

(1d)

$$d\eta^4 = (\kappa \eta^5 - \eta^4) \wedge \omega - \eta^1 \wedge \eta^5,$$  

(1e)

$$d\eta^5 = \eta^1 \wedge \omega + \eta^2 \wedge \eta^5 - \eta^3 \wedge \eta^4,$$  

(1f)

where

$$\pi = d\kappa + \kappa^2 \eta^4.$$
2.4. Coadjoint action of $\mathbb{E}(2, 1)$

For later convenience, we now discuss the coadjoint action of $\mathbb{E}(2, 1)$ on $\mathfrak{e}(2, 1)^*$, the dual of its Lie algebra. Our discussion follows the discussion of the coadjoint representation of $\mathbb{E}(3)$ given in Guillemin and Sternberg [14].

Using the Maurer–Cartan forms, we identify $\mathfrak{g}^*$ with $\mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)}$ by means of the map

$$(p, v) \in \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)} \rightarrow p_i \omega^i - v^1 \omega^2_1 + v^2 \omega^1_1 + v^3 \omega^2_2 \in \mathfrak{g}^*.$$

The coadjoint action of $G$ on $\mathfrak{g}^*$ then takes the form

$$g \cdot (p, v) = (Ap, Av - (Ap) \times q), \quad \tag{2}$$

for all $g = \begin{pmatrix} 1 & 0 \\ q & A \end{pmatrix} \in G = \mathbb{E}(2, 1)$, where $\times$ denotes the vector cross product

$$\langle v \times w, u \rangle = \det(v, w, u) \quad \forall v, w, u \in \mathbb{R}^{(2,1)}.$$

We now define the map

$$C : (p, v) \in \mathfrak{g}^* \rightarrow (\|p\|^2, \langle p, v \rangle) \in \mathbb{R}^2,$$

the components which, $C_1$ and $C_2$, generate the space of Casimir functions. We recall the following standard material:

- Let $G$ be a Lie group, and $\mathfrak{g}^*$ the dual of the Lie algebra of $G$. Let $\mu \in \mathfrak{g}^*$. The isotropy group of $G$ at $\mu$ is the closed subgroup of $G$ defined by

$$G_\mu := \{ g \in G : \text{Ad}^*(g)\mu = \mu \} = \{ g \in G : \langle \mu; \text{Ad}(g^{-1})A \rangle = \langle \mu; A \rangle, \ \forall A \in \mathfrak{g} \}.$$

- The Lie algebra of $G_\mu$ is

$$\mathfrak{g}_\mu = \{ A \in \mathfrak{g} : \text{ad}^*(A)\mu = 0 \} = \{ A \in \mathfrak{g} : \langle \mu; [A, B] \rangle = 0, \ \forall B \in \mathfrak{g} \}.$$

- The rank of the group $G$ is defined as

$$\text{rank}(G) = \inf\{\dim(\mathfrak{g}_\mu) : \mu \in \mathfrak{g}^* \}.$$

- An element $\mu \in \mathfrak{g}^*$ is regular if $\dim(\mathfrak{g}_\mu) = \text{rank}(G)$, otherwise $\mu$ is a singular element of $\mathfrak{g}^*$. The set of regular elements of $\mathfrak{g}^*$ will be denoted by $\mathfrak{g}_r^*$, while $\mathfrak{g}_s^*$ will denote the set of singular elements.

- By a theorem of Dixmier (cf. [8,9]), the isotropy group $G_\mu$ and the isotropy Lie algebra $\mathfrak{g}_\mu$ of a regular element $\mu \in \mathfrak{g}_r^*$ are Abelian.

In the case of $\mathbb{E}(2, 1)$, $\mathfrak{g}_r^*$ is the open subset of $\mathfrak{g}^*$ consisting of elements

$$\mathfrak{g}_r^* = \{ (p, v) \in \mathfrak{g} : p \neq 0 \}.$$
The co-adjoint orbit, \( O(p_0, v_0) \), through a regular element \((p_0, v_0) \in g^*_r\) is therefore the four-dimensional sub-manifold

\[
O(p_0, v_0) = \{(p, v) \in \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)} : \|p\|^2 = \|p_0\|^2, \langle p_0, v_0 \rangle = \langle p, v \rangle \}.
\]

There are three types of regular orbit:

- orbits of positive type: \( O(p_0, v_0) \) with \( C_1 = \|p_0\|^2 > 0 \);
- orbits of negative type: \( O(p_0, v_0) \) with \( C_1 = \|p_0\|^2 < 0 \);
- orbits of null type: \( O(p_0, v_0) \) with \( C_1 = \|p_0\|^2 = 0 \);

The orbits of negative and null type also break into sub-classes according to whether \( p_0 \) is future-directed, with \( \langle p_0, e_1 + e_3 \rangle < 0 \), or past-directed, with \( \langle p_0, e_1 + e_3 \rangle > 0 \).

3. Variational problems

3.1. Non-degenerate invariant variational problems

**Definition 3.1.** Given an affine subspace \( A \in P^h(g) \), an invariant Lagrangian of type \( A \) is a smooth real-valued function \( L : A \rightarrow \mathbb{R} \).

An invariant Lagrangian \( L \) gives rise to a variational problem defined on the integral curves of the linear control system \((A, \omega)\). From this point of view the Lagrangian \( L \) is considered as a cost function. It is then an optimal control problem to minimize the cost

\[
\mathcal{L} : \Gamma \rightarrow \int_{\gamma} \Gamma^*(L\omega)
\]

among the integral curves of \((A, \omega)\). If \((A, \omega)\) comes from a Frenet system \((S, A, \Phi)\) on the homogeneous space \( G/H \) then the Lagrangian \( L \) defines a geometric action functional \( \tilde{L} : S \rightarrow \mathbb{R} \) acting on the space of the normalized curves of type \( S \):

\[
\tilde{L} : \gamma \in S \rightarrow \int_{\gamma} L(K[j^k(\gamma)(t)]) \, dt.
\]

Note that the geometric action functional \( \tilde{L} \) depends only on the generalized curvatures of \( \gamma \).

**Example 3.2.** The simplest invariant variational problem for a Frenet systems is the arc-length functional, which is defined by a constant Lagrangian (see [7,12,20,22,23] for more details about the arc-length functionals for generic curves in the conformal and pseudo-conformal three-dimensional sphere, in the real projective plane and in the affine plane). Another typical example of an invariant Lagrangian is the Kirchhoff variational problem for the Frenet system of generic curves in \( \mathbb{R}^3 \), defined by the action functional

\[
\mathcal{L} : \gamma \subset \mathbb{R}^3 \rightarrow \int_{\gamma} (\kappa^2(u) + a\tau(u)) \, du.
\]
The extremal curves are the canonical lifts of the Kirchhoff elastic rods of \( \mathbb{R}^3 \). When \( a = 0 \), we get the total squared curvature functional. Other examples of invariant variational problems for curves in \( \mathbb{R}^3 \) have been considered in Ref. [19].

Given an invariant Lagrangian \( L : \mathbb{A} \to \mathbb{R} \), we construct the corresponding affine sub-bundle \( \tilde{Z} \subset T^*(M) \) over the configuration space \( M = G \times \mathbb{A} \). The fiber of \( \tilde{Z} \) over the point \( (g, Q) \in M \) is given by the affine space
\[
\tilde{Z} \mid_{(g, Q)} = \{ \eta \in g^* : \langle \eta; Q \rangle = L(Q) \}.
\]
Note that \( \tilde{Z} \) is of the form \( G \times \tilde{Z} \), where
\[
\tilde{Z} = \{ (Q, \eta) \in \mathbb{A} \times g^* : \langle \eta; Q \rangle = L(Q) \}.
\]
The Liouville 1-form \( \psi \) is given by
\[
\psi \mid_{(g, Q, \eta)} = \pi^*(\eta) \mid_{(g, Q, \eta)} \quad \forall (g, Q, \eta) \in \tilde{Z},
\]
where \( \pi : G \times \tilde{Z} \to G \) denotes the projection onto the first factor.

**Remark 3.3.** Pick a basis \((e_0, e_1, \ldots, e_h, e_{h+1}, \ldots, e_n)\) of \( g \) such that
\[
P = e_0, \quad a = \text{span}(e_1, \ldots, e_h), \quad \omega = \theta^0,
\]
where \((\theta^0, \ldots, \theta^n)\) is the dual basis of \( g^* \). We use the following index range: \( i, j = 1, \ldots, h \), \( a, b = h + 1, \ldots, n \). The map
\[
(g, k, \lambda) \in G \times \mathbb{R}^h \times \mathbb{R}^n \to (g, e_0 + k^i e_j, L(e_0 + k^i e_j)\omega + \lambda_j (\theta^j - k^j \omega) + \lambda_a \theta^a) \in \tilde{Z}
\]
gives an explicit identification between \( G \times \mathbb{R}^h \times \mathbb{R}^n \) and \( \tilde{Z} \). With this identification at hand, the tautological 1-form can be written as
\[
\psi = (L(k^1, \ldots, k^h) - k^j \lambda_j)\omega + \lambda_j \theta^j + \lambda_a \theta^a.
\]

**Definition 3.4.** An invariant Lagrangian \( L : \mathbb{A} \to \mathbb{R} \) is said to be regular if the corresponding variational problem \((\mathbb{A}, \omega, L)\) is regular, i.e. if the Cartan system of \( \Psi = d\psi \), with the independence condition \( \omega \), is reducible (see Definition A.6). For a regular Lagrangian we denote by \( Y \subset \tilde{Z} \) the momentum space of the variational problem \((\mathbb{A}, \omega, L)\).

**Remark 3.5.** We have seen that all the derived systems of \((\mathbb{A}, \omega)\) have constant rank. This implies that the extremal curves of a regular invariant variational problem are the projections of the integral curves of the Euler–Lagrange system on \( Y \) (cf. [3]).

**Proposition 3.6.** Let \( L : \mathbb{A} \to \mathbb{R} \) be a regular Lagrangian with momentum space \( Y \). Then \( Y = G \times F \), where \( F \) is an immersed submanifold of \( \mathbb{A} \times g^* \).

**Proof.** First we claim that the momentum space, \( Y \), is \( G \)-invariant. To show this, for any \( g \in G \), we consider the submanifold \( g \cdot Y \subset \tilde{Z} \). The \( G \)-invariance of the exterior differential
forms $\psi$, $\Psi$ and $\omega$ implies that left translation $L_g : \tilde{Z} \to \tilde{Z}$ sends integral elements of $(C(\Psi), \omega)$ into integral elements of $(C(\Psi), \omega)$. Hence, for every point $p \in g \cdot Y$, there exists an integral element of $(C(\Psi), \omega)$ tangent to $g \cdot Y$. Since the momentum space $Y$ is maximal with respect to this property, it follows that $g \cdot Y \subseteq Y$. Thus the group $G$ acts on $Y$. Since this action is free and proper, the quotient space $\mathcal{F} = Y/G$ exists as a manifold. The natural projection $\pi : Y \to \mathcal{F}$ is constant along the fibers of the map $(g, Q, \eta) \in Y \to (Q, \eta) \in \mathbb{A} \times g^*$. Thus it induces a smooth one-to-one immersion $j : \mathcal{F} \to \mathbb{A} \times g^*$. We conclude the proof by observing that the map $(\text{id}, j) : G \times \mathcal{F} \to Y$ is a smooth diffeomorphism.

\textbf{Definition 3.7.} We call $\mathcal{F}$ the \textit{phase space} of the system. Note that a point $p \in \mathcal{F}$ is of the form $p = (Q, \eta)$, where $Q \in \mathbb{A}$, $\eta \in g^*$. We define the maps

$$\Lambda : (Q, \eta) \in \mathcal{F} \to \eta \in g^*, \quad \mathcal{H} : (Q, \eta) \in \mathcal{F} \to Q \in \mathbb{A} \subset g.$$ 

We refer to $\Lambda$ as the \textit{Legendre transform} and $\mathcal{H}$ as the \textit{Hamiltonian}. Let $F(p) := T_p(\mathcal{F}) \subset \mathbb{A} \oplus g^*$ be the tangent space of $\mathcal{F}$ at $p$. We then define

$$R(p) := d\Lambda|_p[F(p)] \subset g^*, \quad S(p) := d\mathcal{H}|_p[F(p)] \subset \mathbb{A} \quad \forall p \in \mathcal{F}.$$ 

\textbf{Definition 3.8.} A regular invariant Lagrangian $L : \mathbb{A} \to \mathbb{R}$ is said to be \textit{non-degenerate} if the momentum space $Y$ is odd-dimensional, of dimension $2m + 1$, and if the restriction of the canonical 2-form $\Psi$ to $Y$, $\Psi_Y$, has the property that $\omega \wedge (\Psi_Y)^m$ is non-vanishing.

Examples of invariant non-degenerate variational problems include the total squared curvature functional in two- and three-dimensional space forms [5,12], the Kirchhoff variational problem in $\mathbb{R}^3$, the Poincaré and the Delaunay functionals [12,16,21], the projective, the conformal and the pseudo-conformal arc-length functionals (cf. [7,20,22]).

Given a non-degenerate variational problem, it follows that $\omega \wedge (\Psi_Y)^m$ defines a volume form on $Y$, and that $\Psi_Y$ is of maximal rank on $Y$. Therefore there exists a unique vector field $\xi \in \mathfrak{X}(Y)$ such that $i_\xi(\Psi_Y) = 0$ and $\omega(\xi) = 1$.

\textbf{Definition 3.9.} $\xi$ is the \textit{characteristic vector field} of the non-degenerate variational problem $(\mathbb{A}, \omega, L)$.

If $(\mathbb{A}, \omega, L)$ is non-degenerate then the Euler–Lagrange system is simply the Cartan system of the canonical 2-form restricted to the momentum space: $\mathcal{E} = C(\Psi_Y)$. Therefore, for such variational problems, the integral curves of the Euler–Lagrange system are the integral curves of the characteristic vector field $\xi$ (see Ref. [12]). We therefore have the following theorem.

\textbf{Theorem 3.10.} Let $\Gamma : (a, b) \to Y$ be an integral curve of the characteristic vector field $\xi$ of a non-degenerate variational problem $(\mathbb{A}, \omega, L)$. Then $\gamma = \pi_M \circ \Gamma : (a, b) \to M$ is a critical point of the action functional $L$.

\textbf{Proposition 3.11.} If $L$ is non-degenerate then the Legendre transform $\Lambda : \mathcal{F} \to g^*$ is an immersion.
Proof. Let $\xi$ be the characteristic vector field of the momentum space. The Liouville form $\psi$, the canonical 2-form $\Psi$ and the independence condition $\omega$ are $G$-invariants, therefore the characteristic vector field is also $G$-invariant. Since $\xi|_{(g,p)} \in T_{(g,p)}Y \cong \mathfrak{g} \oplus F(p) \subset \mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{g}^*$, this implies that there exist smooth maps $A_\xi : \mathcal{F} \to \mathfrak{g}$ and $\Phi_\xi : \mathcal{F} \to \mathfrak{a} \oplus \mathfrak{g}^*$ with the property that

$$
\xi|_{(g,p)} = A_\xi(p)|_g + \Phi_\xi(p) \quad \forall (g,p) \in Y,
$$

where $\Phi_\xi(p) \in F(p)$ for all $p \in \mathcal{F}$. Since $\xi$ satisfies the transversality condition $1 = \omega(\xi) = \langle \omega; A_\xi \rangle$, then $A_\xi : \mathcal{F} \to \mathfrak{g}$ is a nowhere vanishing function. If we now consider $\{0\} \oplus \ker[d\Lambda|_p] \subset \mathfrak{g} \oplus F(p)$ then it is simple to check that every such vector lies in the kernel of the canonical 2-form $\Psi$. Since this null-distribution is generated by $\xi$, we therefore have

$$
\{0\} \oplus \ker[d\Lambda|_p] \subset \text{span}[A_\xi(p) + \Phi_\xi(g)].
$$

Since $A_\xi$ is non-vanishing, however, this holds if and only if $\ker[d\Lambda|_p] = \{0\}$. 

From now on we will assume that the Legendre transform $\Lambda$ is a one-to-one immersion, so that the phase space, $\mathcal{F}$, can be considered as a submanifold (not necessarily embedded) of $\mathfrak{g}^*$. Consequently, we will think of the momentum space as an immersed submanifold of $G \times \mathfrak{g}^*$. The notation introduced in the preceding paragraphs can then be simplified as follows:

- the Legendre map $\Lambda$ is the inclusion of $\mathcal{F}$ into $\mathfrak{g}^*$;
- the tangent space $F(\eta)$ of $\mathcal{F}$ at $\eta \in \mathcal{F}$ is a linear subspace of $\mathfrak{g}^*$ and $R(\eta) = F(\eta)$;
- the tangent space $T_{(g,\eta)}(Y)$ is identified with $\mathfrak{g} \oplus F(\eta) \subset \mathfrak{g} \oplus \mathfrak{g}^*$;
- the Liouville form and the canonical 2-form on $Y$ are the restrictions to $Y$ of the Liouville form and the standard symplectic form on $T^*(G)$;
- the characteristic vector field $\xi$ can be written as

$$
\xi|_{(g,\eta)} = A_\xi(\eta)|_g + \Phi_\xi(\eta) \quad \forall (g,\eta) \in Y,
$$

where $A_\xi : \mathcal{F} \to \mathfrak{g}$ and $\Phi_\xi : \mathcal{F} \to \mathfrak{g}^*$ are smooth functions such that $\Phi_\xi(\eta) \in F(\eta)$, for all $\eta \in \mathcal{F}$.

From now on, we will adhere to these simplifications.

With this notation at hand, we may use the left-invariant trivialization of $T(G)$ to identify the tangent space

$$
T_{(g,\eta)}(Y) \cong T_g G \oplus T_\eta \mathcal{F} \cong \mathfrak{g} \oplus F(\eta) \subset \mathfrak{g} \oplus \mathfrak{g}^*.
$$

We then have the explicit isomorphism

$$
A + v \in \mathfrak{g} \oplus F(\eta) \to A|_g + v \in T_{(g,\eta)}(Y),
$$

where $A \in \mathfrak{g} = T_{id}(G)$ and $A|_g = (L_g)^*A \in T_g(G)$. With this identification, the Liouville form $\psi$ becomes the cross-section of $T^*(Y)$ defined by

$$
\psi|_{(g,\eta)}(A + v) = \langle \eta, A \rangle \quad \forall (g,\eta) \in Y \forall A + v \in \mathfrak{g} \oplus F(\eta).
$$

(3)
Then, from the standard formula
\[ d\psi(X, Y) = \frac{1}{2} \{ X[\psi(Y)] - Y[\psi(X)] - \psi([X, Y]) \}, \]
it follows that the canonical 2-form \( \Psi = d\psi \in \Omega^2(Y) \) takes the form
\[ \Psi|_{(g, \eta)}(A + v; B + w) = -\frac{1}{2} \langle w; A \rangle + \frac{1}{2} \langle \text{ad}^*(A)\eta + v; B \rangle, \]
for all \( \eta \in \mathcal{F} \) and for all \( A + v, B + w \in g \oplus F(\eta) \).

**Definition 3.12.** Given a left-invariant 1-form \( \mu \in g^* \), let \( O(\mu) \subset g^* \) be the coadjoint orbit passing through \( \mu \), and let \( O(\mu) := \text{ad}^*(g)\mu \subset g^* \) denote the tangent space to the orbit \( O(\mu) \) at \( \mu \). The linearized phase portrait of the point \( \eta \in \mathcal{F} \) is the linear subspace \( \Pi(\eta) := F(\eta) \cap O(\eta) \) of \( g^* \). The subset \( \mathcal{P}(\mu) = F(\eta) \cap O(\mu) \) is referred to as the phase portrait of \( \mu \in g^* \).

The following result shows that the characteristic vector field \( \xi \) may be written in terms of the Hamiltonian \( \mathcal{H} \):

**Theorem 3.13.** The characteristic vector field \( \xi \) is given by
\[ \xi|_{(g, \eta)} = \mathcal{H}(\eta)|_g - \text{ad}^*[\mathcal{H}(\eta)]\eta \quad \forall (g, \eta) \in Y. \tag{5} \]

**Proof.** Given a point \( \eta \in \mathcal{F} \), we set
\[ \text{Ann}(F(\eta)) = \{ A \in g : \langle v; A \rangle = 0, \forall v \in F(\eta) \} \]
and let
\[ \rho(\eta) : \text{Ann}(F(\eta)) \to O(\eta) \]
be the linear map
\[ \rho(\eta) : A \in \text{Ann}(F(\eta)) \to \text{ad}^*(A)\eta \in O(\eta). \tag{6} \]
It then follows from Eqs. (3) and (4) that a tangent vector \( A + v \in g \oplus F(\eta) \) to the momentum space \( Y \) at the point \((g, \eta)\) belongs to the kernel of \( \Psi \) if and only if
\[ A \in \rho(\eta)^{-1}(\Pi(\eta)), \quad v = -\rho(\eta)A. \tag{7} \]

We now let \((g_0, \eta_0) \in Y \) and let \( \Gamma : (\epsilon, \epsilon) \to Y \) be the integral curve of the characteristic vector field \( \xi \) with initial condition \( \Gamma(0) = (g_0, \eta_0) \). We write \( \Gamma(t) = (g(t), \eta(t)) \), where \( g : (\epsilon, \epsilon) \to G \) and \( \eta : (\epsilon, \epsilon) \to \mathcal{F} \) are smooth maps such that
\[ g^{-1}(t)g'(t) \, dt = g^*(\Theta)|_t, \quad g^{-1}(t)g'(t) = A_{\xi}[\eta(t)], \quad g(0) = g_0, \quad \eta(0) = \eta_0. \]

On the other hand\(^3\)
\[ t \in (\epsilon, \epsilon) \to (g(t), \mathcal{H}[\eta(t)]) \in G \times \mathbb{A} = M \]

\(^3\) It is a general fact that if \( \pi : Y \to M \) is the momentum space of a regular variational problem \((\mathcal{I}, \omega, L)\) on the configuration space \( M \) and if \( \Gamma : (a, b) \to Y \) is an integral curve of the Euler–Lagrange system, then \( \gamma = \pi \circ \Gamma \) is an integral curve of the Pfaffian differential system \((\mathcal{I}, \omega)\) on \( M \) (see [12]).
is an integral curve of the linear control system \((A, \omega)\). We then have
\[
g^*(\Theta)_{\mid t} = \mathcal{H}[\eta(t)]g^*(\omega)_{\mid t} = \mathcal{H}[\eta(t)]dt_{\mid t}.
\]
Therefore, we conclude that
\[
A_{\xi}[\eta(t)] = \mathcal{H}[\eta(t)] \quad \forall t \in (-\epsilon, \epsilon).
\]
Since \(\xi\) belongs to the kernel of \(\Psi\), we conclude from Eq. (7) that
\[
\Phi_{\xi}[\eta(t)] = -\text{ad}^*[\mathcal{H}(\eta(t))]\eta(t) \quad \forall t \in (-\epsilon, \epsilon).
\]
This yields the required result.

**Definition 3.14.** The phase flow is the flow of the vector field \(\Phi_{\xi}: \eta \in \mathcal{F} \rightarrow -\text{ad}^*[\mathcal{H}(\eta)]\eta \in \mathfrak{g}^*\).

**Remark 3.15.** We use the notation \(\phi_{\xi}: D \subset \mathbb{R} \times \mathcal{F} \rightarrow \mathcal{F}\) to indicate the phase flow. We observe the following facts:

- The domain of definition \(D\) of the phase flow is of the form
  \[
  D = \{(t, \eta) \in \mathbb{R} \times \mathcal{F}: t \in (\epsilon^-(\eta), \epsilon^+(\eta))\},
  \]
  where \(\epsilon^-: \mathcal{F} \rightarrow \mathbb{R}^- \cup \{-\infty\}\) and \(\epsilon^+: \mathcal{F} \rightarrow \mathbb{R}^+ \cup \infty\).
- For every \(\eta \in \mathcal{F}\), the curve \(\phi_{\eta}: (\epsilon^-(\eta), \epsilon^+(\eta)) \rightarrow \phi_{\xi}(t, \eta) \in \mathcal{F}\)
  is the maximal integral curve of \(\Phi_{\xi}\) with the initial condition \(\phi_{\eta}(0) = \eta\).
- \(\Phi_{\xi}(\eta) \in \Pi(\eta)\) and \(\phi_{\eta}(t) \in \mathcal{P}(\eta)\), for every \(\eta \in \mathcal{F}\) and every \(t \in (\epsilon^- (\eta), \epsilon^+ (\eta))\).
- If we fix a point \((g_0, \eta_0) \in Y = G \times \mathcal{F}\), then the maximal integral curve of the characteristic vector field \(\xi\) with the initial condition \((g_0, \eta_0)\) is given by
  \[
  \Gamma_{(g_0, \eta_0)}: t \in (\epsilon^- (\eta_0), \epsilon^+ (\eta_0)) \rightarrow (h_{(g_0, \eta_0)}(t), \phi_{\eta_0}(t)) \in G \times \mathcal{F},
  \]
  where \(h_{(g_0, \eta_0)}\) is the (unique) solution of the equation
  \[
  h^{-1} h' = \mathcal{H}[\phi_{\eta_0}(t)], \quad h(0) = g_0.
  \]
- We set \(\tilde{D} = \{(t; (g, \eta)) \in \mathbb{R} \times Y: t \in (\epsilon^- (\eta), \epsilon^+ (\eta))\}\). The flow \(\Gamma\) of the characteristic vector field \(\xi\) is the local one-parameter group of transformations \(\Gamma: \tilde{D} \subset \mathbb{R} \times Y \rightarrow Y\) given by
  \[
  \Gamma(t, g, \eta) = (h_{(g, \eta)}(t), \phi(t, \eta)) \quad \forall (t; (g, \eta)) \in \tilde{D}.
  \]

**Remark 3.16.** The phase flow \(\phi_{\xi}: D \rightarrow \mathcal{F}\) satisfies the Euler equation
\[
\left. \frac{\partial \phi_{\xi}}{\partial t} \right|_{(t, \eta)} = -\text{ad}^*[\mathcal{H}[\phi_{\xi}(t, \eta)]\phi_{\xi}(t, \eta), \quad \phi_{\xi}(0, \eta) = \eta, \quad \forall (t, \eta) \in D. \quad (8)
\]
If there exists a $G$-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, then we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$. (For example, if $G$ is semisimple, then take the pairing defined by the Killing form of $\mathfrak{g}$.) Using this identification, the Euler equation can be written in the Lax form

$$
\frac{\partial \phi_\xi}{\partial t}(t, \eta) \bigg|_{(t, \eta)} = -[H(\phi_\xi(t, \eta)), \phi_\xi(t, \eta)], \quad \phi_\xi(0, \eta) = \eta, \; \forall (t, \eta) \in \mathcal{D}.
$$

**Definition 3.17.** We denote by $\mathcal{F}_s = \{ \eta \in \mathcal{F} : \Phi_\xi(\eta) = 0 \}$ the set of all fixed points of the phase flow and by $\mathcal{F}_r$ the complement of $\mathcal{F}_s$. We call $\mathcal{F}_s$ and $\mathcal{F}_r$ the singular and the regular parts of the phase space, respectively. We call $\Sigma = G \times \mathcal{F}_s \subset Y$ the bifurcation set and refer to $Y_r = Y \setminus \Sigma$ as the regular part of the momentum space. The intersection $\mathcal{P}_r(\mu) = \mathcal{F}_r \cap \mathcal{O}(\mu) \subset \mathcal{P}(\mu)$ is called the regular part of the phase portrait $\mathcal{P}(\mu)$. The connected component $\tilde{\mathcal{P}}(\mu)$ of $\mathcal{P}_r(\mu)$ containing $\mu$ is referred to as the reduced phase portrait of $\mu$.

The following result, which may be verified by applying the uniqueness theorems for ordinary differential equations, characterizes integral curves of the characteristic vector field that intersect the bifurcation set.

**Proposition 3.18.** Let $p = (g, \eta) \in \Sigma$ be a point of the bifurcation set. The integral curve $\Gamma_\xi(-, p) : \mathbb{R} \to Y$ of the characteristic vector field $\xi$ passing through $p$ is the orbit of the one-parameter subgroup generated by $H(\eta)$:

$$
\Gamma_\xi(t, p) = (\exp(H(\eta)t)g, \eta) \; \forall t \in \mathbb{R}.
$$

This result implies that if $(\mathcal{A}, \omega)$ comes from a Frenet system of curves in $G/H$, then the curve $\gamma \subset G/H$ of type $S$ that corresponds to $\Gamma_\xi(-, p)$, where $p \in \Sigma$, has constant curvature (i.e. $K_\gamma$ = constant).

Since this result completely characterizes the behavior of integral curves that intersect the bifurcation set $\Sigma$, we shall henceforth restrict our attention to the regular parts of the phase space and momentum space. Therefore, to simplify the notation, $\mathcal{F}, Y$ and $\mathcal{P}(\mu)$ will be used to denote the regular parts of the phase space, the momentum space and the phase portraits, respectively.

### 3.2. The Poincaré variational problem for isotropic curves in $\mathbb{R}^{(2,1)}$

We now return to our example of isotropic curves in $\mathbb{R}^{(2,1)}$ considered in Section 2.3. Let $m$ be a non-zero constant and consider the variational problem on the space $\mathcal{V}$ of parameterized integral curves $\Gamma : t \in (a, b) \to (g(t), k(t)) \in G \times \mathbb{R}$ of the Pfaffian system $(\mathcal{A}, \omega)$ defined by the action functional

$$
\mathcal{L}_m : \Gamma \in \mathcal{V} \to \int_{\Gamma}(1 + mk)\omega.
$$

Geometrically, this amounts to an analogue of the Poincaré variational problem where we minimize the arc-length functional (defined by the integral of the canonical line-element
of the null curve) amongst normalized null curves \( \alpha \subset \mathbb{R}^{(2,1)} \) subject to the additional constraint that the integral of the curvature \( k \) along the curve be held constant.

The affine sub-bundle \( \tilde{Z} \subset T^*(M) \) is given by \( M \times \tilde{Z} \), where \( \tilde{Z} \subset \mathfrak{g} \oplus \mathfrak{g}^* \) is the submanifold consisting of all \( (Q(k), \eta) \in \mathbb{A} \oplus \mathfrak{g}^* \) such that \( \langle \eta; Q(k) \rangle = 1 + mk \). (See Section 2.3 for the definition of the map \( Q : \mathbb{R} \rightarrow \mathfrak{g} \).) Therefore \( (Q(k), \eta) \) belongs to \( \tilde{Z} \) if and only if

\[
\eta = \eta(k, \lambda_1, \ldots, \lambda_5) := (1 + mk)\omega + \lambda_1 \eta^1 + \lambda_2 \eta^2 + \lambda_3 \eta^3 + \lambda_4 \eta^4 + \lambda_5 \eta^5,
\]

where \( \lambda_1, \ldots, \lambda_5 \in \mathbb{R} \). For simplicity, we identify \( \tilde{Z} \) with \( G \times \mathbb{R}^6 \) by means of the map

\[
(g : k, \lambda_1, \ldots, \lambda_5) \in G \times \mathbb{R}^6 \rightarrow (g, Q(k), \eta(k, \lambda_1, \ldots, \lambda_5)) \in \tilde{Z}.
\]

Thus the Liouville form on \( \tilde{Z} \) is given by

\[
\psi = (1 + mk)\omega + \lambda_1 \eta^1 + \lambda_2 \eta^2 + \lambda_3 \eta^3 + \lambda_4 \eta^4 + \lambda_5 \eta^5.
\]

From the structure (1), we find that

\[
\Psi \equiv m\pi \wedge \omega - (1 + mk)\eta^2 \wedge \omega + \sum \alpha \wedge \eta^\alpha - \lambda_1 \pi \wedge \omega + \lambda_2 (\kappa \eta^3 - \eta^1) \wedge \omega + \lambda_3 (2\eta^2 - \kappa \eta^4) \wedge \omega + \lambda_4 (\kappa \eta^5 - \eta^3) \wedge \omega + \lambda_5 \eta^4 \wedge \omega,
\]

where \( \Psi := \text{d}\psi \) and where \( \equiv \) denotes equality modulo \( \text{span}(\{\eta^\alpha \wedge \eta^\beta\})_{\alpha, \beta = 1, \ldots, 5} \). Let

\[
(\partial_\omega, \partial_{\eta^1}, \ldots, \partial_{\eta^5}, \partial_{\lambda_1}, \ldots, \partial_{\lambda_5}, \partial_\pi)
\]

denote the parallelization of \( \tilde{Z} \) dual to the coframing

\[
(\omega, \eta^1, \ldots, \eta^5, d\lambda_1, \ldots, d\lambda_5, \pi).
\]

We then have

\[
i_{\partial_\omega} \Psi = \eta^i, \quad i = 1, \ldots, 5,
\]

along with

\[
i_{\partial_\eta^i} \Psi = -\alpha, \quad i_{\partial_\pi} \Psi = -\beta, \quad i_{\partial_{\eta^i}} \Psi = -\beta_i, \quad i = 1, \ldots, 5,
\]

where

\[
\alpha = (m - \lambda_1)\,dk, \quad (9a)
\]
\[
\beta = (m - \lambda_1)\omega, \quad (9b)
\]
\[
\beta_1 = d\lambda_1 + \lambda_2 \omega, \quad (9c)
\]
\[
\beta_2 = d\lambda_2 + (1 + mk - 2\lambda_3)\omega, \quad (9d)
\]
\[
\beta_3 = d\lambda_3 + (\lambda_4 - k\lambda_2)\omega, \quad (9e)
\]
\[
\beta_4 = d\lambda_4 + (k\lambda_3 - \lambda_5 - k)\omega, \quad (9f)
\]
\[
\beta_5 = d\lambda_5 - k\lambda_4 \omega, \quad (9g)
\]
and where \( \equiv \) denotes equality modulo \( \text{span}(\eta^1, \ldots, \eta^5) \). From these equations, we deduce that the Cartan system \((C(\Psi), \omega)\) is generated by the differential 1-forms \((\eta^1, \ldots, \eta^5, \alpha, \beta, \beta_1, \ldots, \beta_5)\).

**Theorem 3.19.** The momentum space, \( Y \), is the nine-dimensional sub-manifold of \( \tilde{Z} \) defined by the equations

\[
\lambda_2 = \lambda_1 - m = \lambda_3 - \frac{1}{2}(1 + mk) = 0,
\]

and the Euler–Lagrange system \((E, \omega)\) is the Pfaffian differential system on \( Y \) with independence condition \( \omega \) generated by the linear differential forms \((\eta^1, \ldots, \eta^5, \sigma_1, \sigma_2, \sigma_3)\), where

\[
\sigma_1 = \frac{1}{2} mk \partial_k + \lambda_4 \omega, \quad \sigma_2 = d\lambda_4 - (\lambda_5 + \frac{1}{2}k(1 - mk))\omega, \quad \sigma_3 = d\lambda_5 - k\lambda_4 \omega.
\]

**Proof.** We let \( V_1 \subset T(\tilde{Z}) \) be the sub-variety of one-dimensional integral elements of the Cartan system \((C(\Psi), \omega)\) and denote by \( \tilde{Z}_1 \subset \tilde{Z} \) the projection of \( V_1 \) under the bundle map \( T(\tilde{Z}) \to \tilde{Z} \). From Eqs. (9a) and (9b) we then deduce that \( \tilde{Z}_1 \) is the sub-manifold defined by \( \lambda_1 = m \). Denote by \( C(\Psi)_1 \) the restriction to \( \tilde{Z}_1 \) of the Cartan system. Then \( C(\Psi)_1 \) is generated by the linear differential forms \((\eta^1, \ldots, \eta^5, \lambda_2 \omega, \beta_2, \ldots, \beta_5)\). We then consider the sub-variety \( V_2 \subset T(\tilde{Z}_1) \) consisting of integral elements of \((C(\Psi)_1, \omega)\) and let \( \tilde{Z}_2 \subset \tilde{Z}_1 \) denote the projection of \( V_2 \). We therefore have that \( \tilde{Z}_2 \) is the sub-manifold of \( \tilde{Z}_1 \) defined by \( \lambda_2 = 0 \). Denote by \( C(\Psi)_2 \) the restriction to \( \tilde{Z}_2 \) of \( C(\Psi)_1 \). Then \( C(\Psi)_2 \) is generated by the linear differential forms \((\eta^1, \ldots, \eta^5, (1 + mk - 2\lambda_3)\omega, \beta_3, \beta_4, \beta_5)\). We proceed as above and let \( V_3 \subset T(\tilde{Z}_2) \) be the sub-variety of integral elements of \((C(\Psi)_2, \omega)\) and define \( \tilde{Z}_3 \subset \tilde{Z}_2 \) to be the image of \( V_3 \) under the projection \( T(\tilde{Z}_2) \to \tilde{Z}_2 \). It follows that \( \tilde{Z}_3 \) is the sub-manifold of \( \tilde{Z}_2 \) defined by the equation \( \lambda_3 = (1/2)(1 + mk) \) and that the restriction \( C(\Psi)_3 \) of \( C(\Psi)_2 \) to \( \tilde{Z}_3 \) is the Pfaffian differential system generated by \((\eta^1, \ldots, \eta^5, \sigma_1, \sigma_2, \sigma_3)\). If we let \( V_4 \subset T(\tilde{Z}_3) \) be the set of integral elements of \((C(\Psi)_3, \omega)\) then the bundle map \( V_4 \to \tilde{Z}_3 \) is surjective. Hence \( Y = \tilde{Z}_3 \) and \((C(\Psi)_3, \omega)\) is the reduced space of \((C(\Psi), \omega)\).

**Corollary 3.20.** The momentum space \( Y \) associated with the Poincaré variational problem for isotropic curves in \( \mathbb{R}^{(2,1)} \) is rank 3, affine sub-bundle \( Y = G \times \mathcal{F} \subset T^*(G) \cong G \times \mathfrak{g}^* \), where \( \mathcal{F} \subset \mathfrak{g}^* \) is defined by

\[
\mathcal{F} = \frac{1}{2}(\omega^1 + \omega_1^2) + m\omega_2^1 + \text{span}(\omega_1^2 - \omega^1, \omega^2, \omega^3).
\]

The variational problem is non-degenerate, and the characteristic vector field takes the form

\[
\xi = \partial_\omega - \frac{2\lambda_4}{m} \partial_k - \lambda_4 \partial_{\lambda_3} + \left(\lambda_5 + \frac{1}{2}k(1 - mk)\right) \partial_{\lambda_4} + k\lambda_4 \partial_{\lambda_5}.
\]

**Proof.** It follows from the preceding theorem that the restriction of the Liouville to the momentum space takes the form

\[
\psi_Y = (1 + mk)\omega + m\eta^1 + \frac{1}{2}(1 + mk)\eta^3 + \lambda_4 \eta^4 + \lambda_5 \eta^5
\]

\[
= \frac{1}{2}(\omega^1 + \omega_1^2) + m\omega_2^1 + \frac{1}{2}mk(\omega_1^2 - \omega^1) + \lambda_4 \omega^2 + \lambda_5 \omega^3.
\]
The form of $Y$ and $\mathcal{F}$ follow directly from this equation. The dimension of $Y$ is equal to 9, and a straightforward calculation shows that

$$\omega \wedge (\Psi Y)^4 = -12m^2 \omega \wedge dk \wedge d\lambda_4 \wedge d\lambda_5 \wedge \eta^4 \wedge \eta^5,$$

which is nowhere vanishing. Hence the variational problem is non-degenerate. The form of the characteristic vector field follows from a direct calculation.

**Remark 3.21.** Since the variational problem is non-degenerate, the Euler–Lagrange system $\mathcal{E}$ coincides with the Cartan system of $\Psi$. The characteristic line-distribution $\Xi \subset T(Y)$ of $\Psi$ is transverse to the independence condition $\omega$, and is generated by the characteristic vector field $\xi$.

**Remark 3.22.** Using the explicit form of the Liouville form, we may identify $Y = G \times \mathbb{R}^3$, where $(k, \lambda_4, \lambda_5)$ serve as coordinates on $\mathbb{R}^3$. The explicit form for the characteristic vector field and the 1-forms $\eta_i$ and $\omega$ then imply that the map $\mathcal{H} : Y \to \mathfrak{g}$ is given by

$$\mathcal{H}[\eta(k, \lambda_4, \lambda_5)] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & k & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathfrak{g}.$$

A smooth map $\Gamma : (a, b) \to Y$ is an integral curve of the Euler–Lagrange system if and only if it satisfies

$$\Gamma^*(\eta^i) = 0, \quad i = 1, \ldots, 5, \quad \Gamma^*(\sigma_i) = 0, \quad i = 1, 2, 3$$

with the independence condition

$$\Gamma^*(\omega) \neq 0.$$

Without loss of generality, we may choose a parameterization of our integral curve such that

$$\Gamma^*(\omega) = dt.$$

In this case, we may write $\Gamma : t \in (a, b) \to (g(t), k(t), \lambda_4(t), \lambda_5(t)) \in Y = G \times \mathbb{R}^3$. From these relations, and the explicit form of the differential forms $\eta^i$ and $\sigma_i$, we deduce the following result.

**Proposition 3.23.** The smooth map $\Gamma : (a, b) \to Y$, parameterized such that $\Gamma^*(\omega) = dt$, is an integral curve of the Euler–Lagrange system if and only if the real-valued functions $k(t), \lambda_4(t), \lambda_5(t)$ satisfy the relations

$$\frac{dk}{dt} = -\frac{2\lambda_4}{m}, \quad \frac{d\lambda_4}{dt} = \left(\lambda_5 + \frac{1}{2}k(1 - mk)\right), \quad \frac{d\lambda_5}{dt} = k\lambda_4,$$

(12)
and \( g(t) \in G \) is a solution of

\[
    g(t)^{-1} \frac{dg(t)}{dt} = \mathcal{H}(\eta(t)) = \begin{pmatrix}
        0 & 0 & 0 & 0 \\
        1 & 0 & k(t) & 0 \\
        0 & 1 & 0 & k(t) \\
        0 & 0 & 1 & 0
    \end{pmatrix}.
\]

(13)

**Remark 3.24.** Although, in the present case, the Lie group \( G \) is not semisimple, it is naturally embedded in \( SL(4, \mathbb{R}) \). Using the Killing form on \( \mathfrak{sl}(4, \mathbb{R}) \) we deduce that the Euler equation (12) may be written in Lax form

\[
    L' = [L, \mathcal{H}],
\]

where

\[
    L(k, \lambda_4, \lambda_5) = \begin{pmatrix}
        \frac{1}{2}(1 + mk) & 0 & -\lambda_4 & -\lambda_5 \\
        0 & \frac{1}{2}(1 - mk) & 0 & -\lambda_5 \\
        -m & 0 & \frac{1}{2}(1 - mk) & \lambda_4 \\
    \end{pmatrix}.
\]

**Proposition 3.25.** If \( \Gamma : (a, b) \to Y \) is an integral curve of the Euler–Lagrange system, with \( \Gamma^*(\omega) = dt \), then the curvature \( k(t) \) satisfies the third-order ordinary differential equation

\[
    m \frac{d^3k}{dt^3} - 3mk \frac{dk}{dt} + \frac{dk}{dt} = 0.
\]

(14)

Conversely, any non-constant solution \( k : (a, b) \to \mathbb{R} \) of this equation determines a parameterized integral curve of the Euler–Lagrange system, unique up to the action of \( E(2, 1) \).

**Proof.** Eq. (14) for \( k(t) \) follows directly from (12). Conversely, given a solution of (14), we can uniquely reconstruct \( \lambda_4(t), \lambda_5(t) \) from (12), and \( g(t) \) is determined, up to initial conditions, by (13). \( \square \)

**Remark 3.26.** Under the identification \( g^* \cong \mathbb{R}^{(2, 1)} \oplus \mathbb{R}^{(2, 1)} \) introduced in Section 2.4, the Liouville form (11) maps to \( (p, v) \in \mathbb{R}^{(2, 1)} \oplus \mathbb{R}^{(2, 1)} \), where

\[
    p = -\lambda_5 e_1 + \lambda_4 e_2 - \frac{1}{2}(1 - mk)e_3, \quad v = -\frac{1}{2}(1 + mk)e_1 + me_3,
\]

and \( \{e_i\} \) is the standard basis of \( \mathbb{R}^{(2, 1)} \). The Casimir operators therefore take the form

\[
    C_1 = \|p\|^2 = \lambda_4^2 - \lambda_5(1 - mk), \quad C_2 = \langle p, v \rangle = m\lambda_5 - \frac{1}{4}(1 - m^2k^2),
\]

(15)

and are constant along integral curves of the Euler–Lagrange system.

The explicit form of the Casimir operators implies the following result.
Proposition 3.27. If \( \Gamma : (a, b) \to Y \), parameterized such that \( \Gamma^*(\omega) = dt \), is an integral curve of the Euler–Lagrange system then the curvature \( k(t) \) satisfies the first-order ordinary differential equation
\[
\left( \frac{dk}{dt} \right)^2 = k^3 - \frac{1}{m} k^2 - \frac{1}{m^2} (4C_2 + 1) k + \frac{1}{m^2} (4mC_1 + 4C_2 + 1).
\]

Remark 3.28. Letting \( h(t) := (1/4)(k - (1/3)m) \), we deduce that \( h \) satisfies
\[
\left( \frac{dh}{dt} \right)^2 = 4h^3 - g_2 h - g_3,
\]
where
\[
g_2 = \frac{1}{m^2} \left( C_2 + \frac{1}{3} \right), \quad g_3 = \frac{1}{m^3} \left( \frac{mC_1}{4} + \frac{C_2}{6} + \frac{1}{27} \right).
\]
Hence the curvature \( k \) and the functions \( \lambda_4, \lambda_5 \) corresponding to any solution of the Euler–Lagrange system may be expressed in terms of Weierstrass elliptic functions with invariants \( g_2, g_3 \).

Remark 3.29. A short calculation using the explicit form of \( \eta = \psi_Y \) given in (11) and the coadjoint action of \( g \) on \( g^* \), which can be derived from (2), shows that in the present case the linearized phase portrait \( \Pi(\eta) = F(\eta) \cap O(\eta) \) is one-dimensional, and is spanned by the vector \( \text{ad}^* [H(\eta)] \eta = \lambda_4 (\omega_1^2 - \omega_1^1) - (\lambda_5 + (1/2) k(1-mk)) \omega^1 - k\lambda_4 \omega^3 \). Hence the regular parts of the phase portraits are one-dimensional in the current problem. We now introduce a more general class of variational problems for which this is the case.

4. Coisotropic variational problems

Consider a smooth manifold \( M \) equipped with an exterior differential 2-form \( \Psi \). The kernel of \( \Psi_x \) will be denoted by \( N(\Psi)_x \). Suppose that a Lie group \( G \) acts on \( M \). Denote by \( A^\sharp \) the fundamental vector field on \( M \) corresponding to \( A \in g \) and, for each \( x \in M \), let \( g^\sharp(M)_x \subset T_x(M) \) be the vector subspace \( \{ A^\sharp_x : A \in g \} \). We denote by \( g^\sharp(M)_x^\perp \) the polar space of \( g^\sharp(M)_x \) with respect to \( \Psi_x \):
\[
g^\sharp(M)_x^\perp := \{ v \in T_x(M) : \Psi_x(v, A^\sharp) = 0, \forall A^\sharp \in g^\sharp(M)_x \}.
\]

Definition 4.1. The action of \( G \) on \( M \) is coisotropic with respect to \( \Psi \) if
\[
g^\sharp(M)_x^\perp \subset g^\sharp(M)_x + N(\Psi)_x \quad \forall x \in M.
\]

Remark 4.2. The notion of a coisotropic action arises naturally when studying collective complete integrability of Hamiltonian systems (see [13,14,25]).

Definition 4.3. An invariant Lagrangian \( L : A \to \mathbb{R} \) is said to be coisotropic if it is non-degenerate and if the action of \( G \) on the regular part of the momentum space \( Y \) is coisotropic with respect to \( \psi_Y \), the restriction of the canonical 2-form \( \Psi \) to \( Y \).
(Recall that we are now using the notation \(Y_F\) and \(P(\mu)\) to denote the regular parts of the momentum space, phase space and phase portraits, respectively.)

**Proposition 4.4.** A non-degenerate invariant Lagrangian \(L : A \to \mathbb{R}\) is coisotropic if and only if the linearized phase portrait \(\Pi(\eta)\) is spanned by \(\text{ad}^*[\mathcal{H}(\eta)]\eta\), for every \(\eta \in \mathcal{F}\).

**Proof.** Using the left-invariant trivialization, we find that the polar space of \(g^\#(Y)(g,\eta)\) is given by
\[
g(\eta) \perp := g^\#(Y)_{(g,\eta)} \perp = \{A + V \in g \oplus F(\eta) : V = -\text{ad}^*[A]\eta\}.
\]

First, assume that \(L\) is coisotropic, i.e.
\[
g(\eta) \perp \subset g + \text{span}(\mathcal{H}(\eta) - \text{ad}^*[\mathcal{H}(\eta)]\eta).
\]
Let \(V \in \Pi(\eta)\). Then \(V \in F(\eta)\) and there exists an \(A \in g\) such that \(V = -\text{ad}^*(A)\eta\). Then \(A + V\) belongs to \(g(\eta) \perp\) and hence \(V\) must be a real multiple of \(\text{ad}^*[\mathcal{H}(\eta)]\eta\).

Conversely, assume that \(\Pi(\eta)\) is spanned by \(\text{ad}^*[\mathcal{H}(\eta)]\eta\). Given any element, \(A + V\), of the polar space \(g(\eta) \perp\), then \(V \in \Pi(\eta)\) so there exists \(s \in \mathbb{R}\) such that \(V = s \cdot \text{ad}^*[\mathcal{H}(\eta)]\eta\). Therefore, we can write
\[
A + V = -s(\mathcal{H}(\eta) - \text{ad}^*[\mathcal{H}(\eta)]\eta) + (A + s\mathcal{H}(\eta)).
\]
Since \(A + s\mathcal{H}(\eta)\) is an element of \(g\), it follows that \(A + V \in g + \text{span}(\mathcal{H}(\eta) - \text{ad}^*[\mathcal{H}(\eta)]\eta)\). Therefore \(g(\eta) \perp \subset g + \text{span}(\mathcal{H}(\eta) - \text{ad}^*[\mathcal{H}(\eta)]\eta)\), as required. \(\square\)

**Remark 4.5.** Note that if \(\eta \in \mathcal{F}\) then \(\Pi(\eta)\) is one-dimensional and the map \(\rho(\eta) : \text{Ann}(F(\eta)) \to O(\eta)\) defined in (6) is injective.

**Proposition 4.6.** Let \(L : A \to \mathbb{R}\) be an invariant coisotropic Lagrangian and let \(Y = G \times \mathcal{F}\) be the corresponding momentum space. Suppose that \(\mathcal{F}\) is non-empty, we then have:

- \(\dim(Y) = \dim(G) + \text{rank}(G) + 1\);
- the regular part of the phase space, \(\mathcal{F}\), intersects the coadjoint orbits transversally;
- the regular parts \(P(\mu)\) of the phase portraits are smooth and one-dimensional;
- every \(\eta \in \mathcal{F}\) is a regular element of \(g^*\). In particular, the isotropy group \(G_\eta\) and the isotropy algebra \(g_\eta\) are Abelian.

**Proof.** For each \(\eta \in \mathcal{F}\), let \(k(\eta)\) be the dimension of the isotropy Lie algebra \(g_\eta\). Note that
\[
\dim(F(\eta) \cap O(\eta)) = 1, \quad \dim(O(\eta)) = \dim(G) - k(\eta),
\]
\[
\dim(\text{Ann}(F(\eta))) = \dim(G) - \dim(F(\eta)).
\]
We then have
\[
\dim(F(\eta)) + \dim(O(\eta)) - 1 \leq \dim(G),
\]
which in turn implies that $\dim(\mathcal{F}) \leq k(\eta) + 1$. On the other hand, from the injectivity of the map $\rho(\eta) : \text{Ann}(F(\eta)) \to \mathcal{O}(\eta)$, it follows that $k(\eta) \leq \dim(\mathcal{F})$. Therefore we have

$$k(\eta) \leq \dim(\mathcal{F}) \leq k(\eta) + 1.$$  

Notice that $\dim(G) + k(\eta)$ is even and that $\dim(Y) = \dim(G) + \dim(\mathcal{F})$ is odd. Thus, we must have $k(\eta) + 1 = \dim(\mathcal{F})$. In particular, $k(\eta) = k$ is constant and

$$\dim(Y) = \dim(G) + k + 1.$$  

This implies that

$$\dim(g) = \dim(F(\eta)) + \dim(\mathcal{O}(\eta)) - 1.$$  

Thus $\mathcal{F}$ intersects the coadjoint orbits transversally. Since $\dim(F(\eta) \cap \mathcal{O}(\eta)) = 1$, it follows that $\mathcal{P}(\eta) = \mathcal{F} \cap \mathcal{O}(\eta)$ is a smooth curve such that $T_\eta[\mathcal{P}(\eta)] = \mathcal{I}(\eta)$. Moreover, from the transversality condition, it follows that $\mathcal{F}$ cannot be contained in the set $g^*_s$ of the singular element of $g^*$. Thus, $\mathcal{F} \cap g^*_s$ is non-empty. Therefore, there exists an $\eta \in \mathcal{F}$ such that $k(\eta) = \text{rank}(G)$. This gives the required result.

**Remark 4.7.** The regular part of the phase space, $\mathcal{F}$, is foliated by the nowhere vanishing vector field $\Phi_\xi$ and the leaves are the phase portraits. Furthermore, if $X \subset \mathcal{F}$ is a local section of such a foliation then $X$ is also a local section of the coadjoint representation.

**Definition 4.8.** Let $L : A \to \mathbb{R}$ be a coisotropic Lagrangian. The **moment map** $J : Y \to g^*$ of the Hamiltonian action of $G$ on $Y$ is defined by

$$J(g, \eta) = \text{Ad}^*(g)\eta \quad \forall (g, \eta) \in Y.$$  

**Proposition 4.9.** Let $L : A \to \mathbb{R}$ be a coisotropic Lagrangian. Then:

- $J(Y) \subset g^*_r$;
- $J : Y \to g^*$ is a submersion;
- $J^{-1}(\mu)$ is a $(k + 1)$-dimensional submanifold of $Y$ such that

$$T_{(g,\eta)}[J^{-1}(\mu)] = \ker[dJ|_{(g,\eta)}] = \text{span}[\xi|_{(g,\eta)}] + g_\eta.$$  

so that the characteristic vector field $\xi|_{J^{-1}(\mu)}$ is tangent of $J^{-1}(\mu)$, and $G_\mu$ acts freely and properly on $J^{-1}(\mu)$.

- $Y_\mu := J^{-1}(\mu)/G_\mu$ is a one-dimensional manifold and $J^{-1}(\mu) \to Y_\mu$ is a principal $G_\mu$ bundle.

- $Y_\mu \cong \mathcal{P}(\mu)$, the phase portrait.

**Proof.** From **Proposition 4.6** we know that each $\eta \in \mathcal{F}$ is an element of $g^*_r$, and hence $J(g, \eta) \in g^*_r$ for all $(g, \eta) \in Y$. The differential of the moment map is given by the formula

$$\langle dJ|_{(g,\eta)}(A + V); B \rangle = \langle \text{ad}^*(A)\eta + V; B \rangle \quad \forall A + V \in g \oplus F(\eta), \; \forall B \in g.$$  

(16)
This implies that
$$\text{Im}[dJ|_{(g,\eta)}] = F(\eta) + O(\eta)$$
for all $\eta \in F$. Since $F$ and $O(\eta)$ intersect transversally, this implies that $J$ is a submersion. Therefore $J^{-1}(\mu)$ is a sub-manifold of $Y$ and the tangent space $T_{(g,\eta)}[J^{-1}(\mu)]$ is naturally isomorphic to $\ker[dJ|_{(g,\eta)}]$. From the formula (16) and the fact that the Lagrangian is coisotropic, we deduce that
$$\ker[dJ|_{(g,\eta)}] = \text{span}[\xi|_{(g,\eta)}] + g\eta$$
for all $\eta \in F$, as required. Note that this relation implies that the characteristic vector field $\xi$ belongs to $\ker[dJ]$, and therefore that $\xi|_{(g,\eta)}$ is tangent to $J^{-1}(\mu)$. We shall denote the restriction of $\xi$ to the fiber $J^{-1}(\mu)$ by $\xi^\mu$.

The isotropy group $G_\mu$ acts on $J^{-1}(\mu)$ by $(g, \eta) \mapsto (hg, \eta)$ for each $h \in G_\mu$. This action is clearly free and proper, so the quotient space $Y_\mu := J^{-1}(\mu)/G_\mu$ exists as a one-dimensional manifold. The map $\pi_\mu : (g, \eta) \in J^{-1}(\mu) \to [(g, \eta)] \in Y_\mu$
gives $J^{-1}(\mu)$ the structure of a principal fiber-bundle with structure group $G_\mu$. Moreover, the vector field $\xi^\mu$ is horizontal with respect to the fibration $J^{-1}(\mu) \to Y_\mu$.

We also consider the fibration of $J^{-1}(\mu)$ over $P(\mu)$ defined by
$$\tilde{\pi}_\mu : (g, \eta) \in J^{-1}(\mu) \to \eta \in P(\mu).$$

The structure group is again the isotropy subgroup $G_\mu$. Furthermore, $\tilde{\pi}_\mu$ is constant along the fibers of the fibration $\pi_\mu$, and therefore descends to a diffeomorphism of $Y_\mu$ onto $P(\mu)$. \hfill \Box

**Definition 4.10.** We adopt standard terminology, referring to $Y_\mu$ as the Marsden–Weinstein reduction of $Y$ at $\mu$, and to $\pi_\mu : J^{-1}(\mu) \to Y_\mu$ as the Marsden–Weinstein fibration at $\mu$. We may consider $\pi_\mu : J^{-1}(\mu) \to Y_\mu$ as a principal $G_\mu$ bundle over $Y_\mu$, where the right-action of $G_\mu$ on $J^{-1}(\mu)$ is given by $R_h(g, \eta) = (hg, \eta)$, for all $h \in G_\mu$, for $(g, \eta) \in J^{-1}(\mu)$.

**Definition 4.11.** Let $\mu \in J(Y)$. The restriction of the Marsden–Weinstein fibration $J^{-1}(\mu)$ to the reduced phase portrait $\tilde{P}(\mu)$ is said to be the reduced Marsden–Weinstein fibration. We shall denote this fibration by $\tilde{\pi}_\mu : P^\mu \to \tilde{P}(\mu)$.

**Remark 4.12.** The vector field $\Phi_\xi : \eta \mapsto -\text{ad}^*[\mathcal{H}(\eta)]\eta$ is tangent to the phase portraits. We denote by $\Phi_\xi^\mu$ the restriction of $\Phi_\xi$ to $\tilde{P}(\mu)$. Note that the vector fields $\xi^\mu$ and $\Phi_\xi^\mu$ are related by the fibration $\tilde{\pi}_\mu : P^\mu \to \tilde{P}(\mu)$.

**Remark 4.13.** On the reduced phase portrait $\tilde{P}(\mu)$ there exists a unique nowhere vanishing 1-form $\sigma^\mu$ such that $\sigma^\mu(\Phi_\xi^\mu) = 1$. Take $\eta \in \tilde{P}(\mu)$, then the integral curve $\phi_\eta : (\epsilon^-(\eta), \epsilon^+(\eta)) \to g^\mu$ is a maximal parameterization of $\tilde{P}(\mu)$ such that $\phi_\eta^*(\sigma^\mu) = dt$. 

Definition 4.14. On \( P^\mu \) we consider the \( g^\mu \)-valued 1-form \( \theta^\mu \) defined by

\[
\theta^\mu|_{(g, \eta)} := \text{Ad}(g)(\Theta - \mathcal{H}\sigma^\mu).
\]

This defines a connection on the reduced Marsden–Weinstein fibration \( P^\mu \to \tilde{\mathcal{P}}(\mu) \). We call \( \theta^\mu \) the canonical connection of the reduced Marsden–Weinstein fibration \( P^\mu \to \tilde{\mathcal{P}}(\mu) \).

4.1. Isotropic curves in \( \mathbb{R}^{(2, 1)} \)

In the case of our problem for isotropic curves in \( \mathbb{R}^{(2, 1)} \), we have defined a map

\[
\mathbb{R}^3 \hookrightarrow g* \cong \mathbb{R}^{(2, 1)} \oplus \mathbb{R}^{(2, 1)}, \quad y = (k, \lambda_4, \lambda_5) \mapsto (p(y), v(y)),
\]

where

\[
p = \begin{pmatrix}
-\lambda_5 \\
\lambda_4 \\
-\frac{1}{2}(1 - mk)
\end{pmatrix}, \quad
v = \begin{pmatrix}
-\frac{1}{2}(1 + mk) \\
0 \\
m
\end{pmatrix}.
\]

Given the form (10) of the characteristic vector field \( \xi \), we see that the regular part of the phase space \( \mathcal{F} \) is given by the complement of the set of points with

\[
\lambda_4 = 0, \quad \lambda_5 + \frac{1}{2}k(1 - mk) = 0.
\]

To show that the action of \( \mathbb{E}(2, 1) \) on the regular part of the momentum space, \( Y \), is coisotropic, we consider a general vector field on \( Y \):

\[
Z^1 \frac{\partial}{\partial k} + Z^2 \frac{\partial}{\partial \lambda_4} + Z^4 \frac{\partial}{\partial \lambda_5} + X^i \frac{\partial}{\partial \omega^i} + Y^1 \frac{\partial}{\partial \omega_1^1} + Y^2 \frac{\partial}{\partial \omega_1^2} + Y^3 \frac{\partial}{\partial \omega_2^3}.
\]

This vector field lies in \( g(Y)^\perp|_{(g, k, \lambda_4, \lambda_5)} \) if and only if

\[
\begin{align*}
\frac{1}{2}mZ^1 &= -\frac{1}{2}(1 - mk)Y^1 - \lambda_4 Y^2, \\
Z^2 &= \frac{1}{2}(1 - mk)Y^3 + \lambda_5 Y^2, \\
Z^3 &= \lambda_4 Y^3 - \lambda_5 Y^1,
\end{align*}
\]

and

\[
\begin{align*}
-\frac{1}{2}(1 - mk)X^1 + \lambda_5 X^3 - mY^3 + \frac{1}{2}(1 + mk)Y^2 &= 0, \\
-\lambda_4 X^1 - \lambda_5 X^2 - \frac{1}{2}mZ^1 - \frac{1}{2}(1 + mk)Y^1 &= 0, \\
-\frac{1}{2}(1 - mk)X^2 - \lambda_1 X^3 + mY^1 &= 0.
\end{align*}
\]

If \( (k, \lambda_4, \lambda_5) \) lies in the regular part of \( \mathcal{F} \) then conditions (19) and (20) are linearly independent. Therefore in this case

\[
\dim g(Y)^\perp = 3.
\]
It is easily checked that the characteristic vector field $\xi$ belongs to $g(Y)_{\perp}$, as do the following vector fields:

$$
S_1 = -\lambda_4 \frac{\partial}{\partial \omega_1} + \frac{1}{2}(1 - mk) \frac{\partial^2}{\partial \omega_1^2} - \lambda_5 \frac{\partial}{\partial \omega_2} - m \frac{\partial}{\partial \omega^3},
$$

$$
S_2 = \lambda_5 \frac{\partial}{\partial \omega_1} - \lambda_4 \frac{\partial}{\partial \omega_2} + \frac{1}{2}(1 - mk) \frac{\partial}{\partial \omega^3}.
$$

Hence we have that

$$
g(Y)_{\perp} = \text{span}(\xi, S_1, S_2) \subset \text{span}(\xi) \oplus g.
$$

Hence the action is coisotropic.

**The momentum map and the basic invariants.** From the form of the coadjoint action of $E(2, 1)$ given earlier, we deduce that the moment map takes the form

$$
J(g; y) = (Ap(y), Av(y) - (Ap(y)) \times Q)
$$

for all $g = (Q, A) \in E(2, 1)$. Note that $J(Y) \subseteq g^*_r$. The basic invariants, which correspond to constants of motion of the system, are the Casimir operators

$$
C_1(g, y) := \|p(y)\|^2 = \lambda_4^2 - \lambda_5(1 - mk),
$$

$$
C_2(g, y) := \langle p(y), v(y) \rangle = m\lambda_5 - \frac{1}{2}(1 - m^2 k^2).
$$

If we choose $\mu = (m_1, m_2) \in g^*_r$, where $m_1, m_1 \in \mathbb{R}^{(2, 1)}$, then $J^{-1}(\mu)$ is the set

$$
\lambda_4^2 = \frac{m^2}{4} k^3 - \frac{m}{4} k^2 - \left(\frac{1}{4} + \langle m_1, m_2 \rangle\right) k + \left(\|m_1\|^2 + \frac{1}{4m} + \frac{\langle m_1, m_2 \rangle}{4}\right),
$$

$$
\lambda_5 = \frac{1}{m} \left(\frac{1}{4}(1 - m^2 k^2) + \langle m_1, m_2 \rangle\right).
$$

If we perform the substitution

$$
k = \left(\frac{4}{m}\right)^{2/3} \chi + \frac{1}{3m},
$$

then the first equation becomes the cubic relation

$$
\lambda_4^2 = 4\chi^3 - g_2\chi - g_3,
$$

where we have defined the “modified Casimirs”

$$
g_2(m_1, m_2) = \left(\frac{4}{m}\right)^{2/3} \left(\frac{1}{3} + \langle m_1, m_2 \rangle\right),
$$

$$
g_3(m_1, m_2) = -\left(\|m_1\|^2 + \frac{2}{3m} \langle m_1, m_2 \rangle + \frac{4}{27m}\right).
$$
Parameterization of the phase portraits. We define the discriminant of the cubic polynomial appearing in Eq. (23)

\[ D(m_1, m_2) = 27g_3^2 - g_2^3. \]

There are two non-degenerate cases that we must consider.

Case I: \( D(m_1, m_2) > 0 \). In this case the cubic polynomial has one real root and two complex-conjugate roots. We may parameterize the curve by taking \( \chi(t) = \wp(t; g_2, g_3) \) with \( t \in (0, 2\omega_1) \), where \( \omega_1, \omega_2 \) and \( \omega_3 = (1/2)(\omega_1 + \omega_2) \) are the half-periods of the \( \wp \) function. From (22), we then find that \( k \) may be written in terms of elliptic functions and, solving (21) then gives \( \lambda_4 \) and \( \lambda_5 \) in terms of elliptic functions.

Case II: \( D(m_1, m_2) < 0 \). In this case the cubic polynomial has three distinct real roots, and the curve (23) has two disjoint components. The “compact” component may be parameterized by \( \chi(t) = \wp_3(t; g_2, g_3) := \wp(t + \omega_3; g_2, g_3) \) with \( t \in \mathbb{R} \). This solution is periodic, with period \( 2\omega_1 \). The “unbounded” component of the curve is parameterized by \( \chi(t) = \wp(t; g_2, g_3) \), \( t \in (0, 2\omega_1) \).

In the degenerate cases where the discriminant vanishes at \( D = 0 \), the cubic is singular, and our curve is rational. In this case, the \( \wp \) and \( \wp_3 \) functions degenerate into elementary functions.

5. Integrability by quadratures

Proposition 5.1. The integral curves of the characteristic vector field \( \xi \) with momentum \( \mu \in J(Y) \) are the horizontal curves of the canonical connection \( \theta^\mu \) on \( P^\mu \).

Proof. Consider a horizontal curve \( \Gamma : (a, b) \to P^\mu \) of the canonical connection. We write \( \Gamma(t) = (h(t), \eta(t)) \), where \( h : (a, b) \to G \) and \( \eta : (a, b) \to \mathcal{P}(\mu) \) are smooth curves. Without loss of generality we can assume that \( \eta^* (\sigma^\mu) = dt \), so that

\[ \eta'(t) = \Phi^\mu_\xi |_{\eta(t)} = -\text{ad}^*[\mathcal{H}(\eta(t))] \eta(t) \quad \forall t \in (a, b). \]

Since \( \Gamma \) is horizontal, we then have

\[ 0 = \Gamma^* (\theta^\mu) = g(t)(h^{-1}(t)h'(t) - \mathcal{H}[\eta(t)])g(t)^{-1} \, dt. \]

This implies that \( \Gamma'(t) = \xi |_{\Gamma(t)} \), for all \( t \in (a, b) \).

Conversely, if \( \Gamma : (a, b) \to Y \) is an integral curve of \( \xi \) with momentum \( \mu \) then \( \Gamma(t) \in P^\mu \), for all \( t \in (a, b) \). Furthermore, we know that

\[ \Gamma'(t) = \mathcal{H}(\eta(t)) - \text{ad}^*[\mathcal{H}(\eta(t))] \eta(t). \]

Thus \( h^{-1}(t)h'(t) = \mathcal{H}[\eta(t)] \) and hence \( \Gamma^* [\theta^\mu] = 0. \)

Since the structure group of the Marsden–Weinstein fibrations is Abelian and the base manifolds are one-dimensional, the horizontal curves of the canonical connection can be found by a single quadrature. The explicit integration of the horizontal curves requires four...
steps:

- Step 1: take a smooth parameterization \( \eta : (a, b) \rightarrow \tilde{P}(\mu) \) of the phase portrait.
- Step 2: compute \( v^\mu : (a, b) \rightarrow \mathbb{R} \) such that \( \eta^*(\sigma^\mu) = v^\mu \, dt \).
- Step 3: take any map \( g : (a, b) \rightarrow G \) such that \((g^{-1}, \eta) : (a, b) \rightarrow Y\) is a cross-section of the reduced Marsden–Weinstein fibration \( \tilde{\pi}_\mu : P^\mu \rightarrow \tilde{P}(\mu) \). This involves solving the equation

\[
\text{Ad}^*(g(t)^{-1})\eta(t) = \mu
\]

for \( g(t) \).
- Step 4: compute the gauge transformation

\[
h(t) = \exp \left[ \int_{t_0}^t (g^{-1}(u)H[\eta(u)]g(u)v^\mu(u) + g^{-1}(u)g'(u)) \, du \right] \quad \forall t \in (a, b).
\]

(24)

Note that the fact that \((g(t)^{-1}, \eta(t))\) is a section of \( \tilde{\pi}_\mu : P^\mu \rightarrow \tilde{P}(\mu) \), along with the definitions of \( H[\eta] \) and \( v^\mu \) imply that \( g^{-1}(t)H[\eta(t)]g(t)v^\mu(t) + g^{-1}(t)g'(t) \in \mathfrak{g}_\mu \) for all \( t \in (a, b) \). Hence \( h(t) \in G_\mu \), for all \( t \in (a, b) \).

Conclusion. The image of the curve \( \Gamma : (a, b) \rightarrow Y \) defined by

\[
\Gamma(t) = (h(t)g(t)^{-1}, \eta(t)) \quad \forall t \in (a, b)
\]

(25)

is contained in \( P^\mu \) and \( \Gamma \) is horizontal for the canonical connection \( \theta^\mu \). Any horizontal curve of \( P^\mu \) arises in this way.

Remark 5.2. In the case where the symmetry group \( G \) is a classical matrix group, with \( G \subset \text{Aut}(V) \) for \( V \) a finite-dimensional vector space, and the Lagrangian is a polynomial function, then we have:

- The phase space \( \mathcal{F} \) is an algebraic subset of \( \mathfrak{g}^* \).
- The generic coadjoint orbit is defined by polynomial equations \( F_j(\eta) = 0, j = 1, \ldots, k \), where \( k \) is the rank of \( \mathfrak{g} \) and where \( F_1, \ldots, F_k \) is a basis of the \( \text{Ad}^* \)-invariant polynomial functions \( \mathfrak{g}^* \rightarrow \mathbb{R} \).
- The phase portraits are real-algebraic curves.

In the simplest cases the phase portraits are rational or elliptic curves, so they can be easily parameterized by means of elementary or elliptic functions (see the examples considered in Refs. [5,12,16,20–23]). The third step in the construction above can be treated in a rather easy way if \( \mathfrak{g} \) is semisimple and if the momentum \( \mu \) is a regular semisimple element of \( \mathfrak{g}^* \). In this case the construction of a cross-section of the Marsden–Weinstein fibration is a linear-algebra problem involving the structure of the Cartan subalgebras of \( \mathfrak{g} \).

Definition 5.3. Consider \( \mu \in J(Y) \). We say that \( \mu \) is a complete momentum if \( \epsilon^-(\eta) = -\infty \) and \( \epsilon^+(\eta) = \infty \) for some (and hence for all) \( \eta \in \tilde{\mathcal{P}}(\mu) \).
Proposition 5.4. If $\mu \in J(Y)$ is a complete momentum, then the connected components of the reduced Marsden–Weinstein fibration $P^\mu \to \tilde{P}(\mu)$ are Euclidean cylinders and $\xi^\mu$ is a linear vector field.

Proof. Let $Q(\mu)$ be a connected component of $P^\mu$ and let $\eta : \mathbb{R} \to \tilde{P}(\mu)$ be an integral curve of the phase flow that parameterizes the reduced phase portrait. Since $\mathbb{R}$ is contractible, $Q(\mu) \to \tilde{P}(\mu)$ is a trivial fiber bundle. This implies that there exists a smooth map $g : \mathbb{R} \to G$ such that $(g^{-1}, \eta) : \mathbb{R} \to G \times \tilde{P}(\mu)$ is a cross-section of $Q(\mu) \to \tilde{P}(\mu)$. Fix $(g_0, \eta_0) \in Q(\mu)$ and let $t_0 \in \mathbb{R}$ such that $\eta(t_0) = \eta_0$ and consider the curve $\Gamma_{(g_0, \eta_0)} : \mathbb{R} \to Q(\mu)$ defined by

$$
\Gamma_{(g_0, \eta_0)}(t) = (g_0 g(t_0) k(t), \eta(t)) \quad \forall t \in \mathbb{R},
$$

where

$$
k(t) = \exp \left[ \int_{t_0}^{t} (g(u)^{-1} \mathcal{H}[\eta(u)] g(u) + g(u)^{-1} g'(u)) \, du \right] g(t)^{-1} \quad \forall t \in \mathbb{R}.
$$

Then, $\Gamma_{(g_0, \eta_0)}$ is the integral curve of $\xi^\mu$ with initial condition $\Gamma_{(g_0, \eta_0)}(t_0) = (g_0, \eta_0)$. This shows that the restriction of the vector field $\xi^\mu$ to $Q(\mu)$ is complete. Now fix a basis $(e_1, \ldots, e_k)$ of $g_\mu$ and let $e_1^*, \ldots, e_k^*$ denote the corresponding fundamental vector fields on $Q(\mu)$. Then $\{\xi^\mu, e_1^*, \ldots, e_k^*\}$ is a set of complete, linearly independent and commuting vector fields on $Q^\mu$. It is then a standard fact that $Q(\mu)$ is a $(k+1)$-dimensional cylinder, that is $Q(\mu) = \mathbb{R}^{k+1}/K$, where $K \subset \mathbb{R}^{k+1}$ is a subgroup of $\mathbb{R}^{k+1}$ generated over $\mathbb{Z}$ by $m \leq k+1$ linearly independent vectors $a_1, \ldots, a_m$.

$$
K = \left\{ \sum_{j=1}^{m} n_j a_j : n_j \in \mathbb{Z} \right\}.
$$

The vector fields $\{\xi^\mu, e_1^*, \ldots, e_k^*\}$ are then the push-forward of linear vector fields $b_0, b_1, \ldots, b_k$ on $\mathbb{R}^{k+1}$. This yields the required result. \qed

Remark 5.5. If $\tilde{P}(\mu)$ is compact and the isotropy subgroup $G_\mu$ is compact, then the connected components of the reduced Marsden–Weinstein fibration $P^\mu$ are $(k+1)$-dimensional tori.

5.1. Cross-sections of the Marsden–Weinstein fibration for isotropic curves in $\mathbb{R}^{(2,1)}$

Finally, we show how the above integration procedure may be carried out in our example. Given $y = (k, \lambda_4, \lambda_5) \in \mathbb{R}^3$, we have defined the vectors $(p, v) \in \mathbb{R}^{(2,1)} \oplus \mathbb{R}^{(2,1)} \cong g^*$. With respect to the standard basis $(e_1, e_2, e_3)$ for $\mathbb{R}^{(2,1)}$ these take the form

$$
p = \sum_{i=1}^{3} p_i^j e_i := -\lambda_5 e_1 + \lambda_4 e_2 - \frac{1}{2} (1 - mk) e_3,
$$

$$
v = \sum_{i=1}^{3} v_i^j e_i := -\frac{1}{2} (1 + mk) e_1 + me_3.
$$
Letting $\mu = (m_1, m_2) \in \text{Im } J \subseteq g^*$, we wish to construct the map $g : (a, b) \subset \mathbb{R} \rightarrow G$ with the property that $(g^{-1}, \eta)$ is a section of the reduced Marsden–Weinstein fibration $\tilde{\pi}_\mu : P^\mu \rightarrow \tilde{P}(\mu)$. We must consider separately the cases where the coadjoint orbit is of positive, negative or null type.

**Positive type.** For an orbit of positive type, where $C_1 = \|p\|^2 > 0$, we may assume, up to the action of $G$ on $O(\mu)$, that $\mu = (m_1, m_2)$ is in the standard form:

$$m_1 = \sqrt{C_1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad m_2 = \frac{C_2}{\sqrt{C_1}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where $C_2 := \langle p, v \rangle$.

We now wish to construct $g = (Q, A) \in \mathbb{R}(2, 1)$ with the property that $\eta = (p, v) = \text{Ad}^*(g)\mu$.

Since $\|p\|^2 > 0$, we may define the $\mathbb{R}(2, 1)$-valued map

$$A_2 := \frac{p}{\sqrt{C_1}} = \frac{p}{\|p\|},$$

with the property that $\|A_2\|^2 = 1$. We can now complete $A_2$ to a frame field $(A_1, A_2, A_3)$ by adding any $\mathbb{R}(2, 1)$-valued functions $A_1, A_3$ with the property that

$$\langle A_i, A_j \rangle = g_{ij}, \quad i, j = 1, 2, 3,$$

and we fix the orientation of this basis by the requirements that

$$A_2 \times A_1 = A_1, \quad A_2 \times A_3 = -A_3, \quad A_3 \times A_1 = A_2.$$

More explicitly, we can define the vector $S = \lambda_4 e_1 + (\lambda_5 - (1/2)(1 - mk))e_2 - \lambda_4 e_3$, with the property that $(p, S) = 0$. We then define $A = (A_1, A_2, A_3) : P^\mu \rightarrow SO(2, 1)$ by

$$A_1 = \frac{1}{\sqrt{2}} \left( \frac{p}{\|p\|} \times \frac{S}{\|S\|} + \frac{S}{\|S\|} \right), \quad A_2 = \frac{p}{\|p\|},$$

$$A_3 = \frac{1}{\sqrt{2}} \left( \frac{p}{\|p\|} \times \frac{S}{\|S\|} - \frac{S}{\|S\|} \right).$$

Defining the map $Q : P^\mu \rightarrow \mathbb{R}(2, 1)$ by

$$Q = -\frac{1}{\sqrt{C_1}} A_2 \times v = \frac{\langle v, A_3 \rangle}{\|p\|^2} A_1 - \frac{\langle v, A_1 \rangle}{\|p\|^2} A_3,$$

we let

$$g := (Q, A) : P^\mu \rightarrow \tilde{P}(\mu).$$

It then follows that $(p, v) = \text{Ad}^*(g)\mu$, as required. Therefore the map $(p, v) \in \tilde{P}(\mu) \rightarrow (g(p, v)^{-1}, (p, v)) \in P^\mu$ is a cross-section of the Marsden–Weinstein fibration.
Negative type. For orbits of negative type we have $C_1 := \|p\|^2 < 0$. We treat the case where the vector $p$ is future-directed, although the past-directed case may be treated similarly. The standard form of elements in this case is $\mu = (m_1, m_2)$, where

$$m_1 = \frac{1}{\sqrt{|C_1|/2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad m_2 = -\frac{C_2}{\sqrt{2|C_1|}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

To define a suitable basis, we fix a vector $S \in \mathbb{R}^{(2,1)}$ with $\|S\|^2 = 1$ and $\langle p, S \rangle = 0$. (For example, the vector $S = e_2 + p^2/p^3 e_1$.) We then define a basis

$$A_1 = \frac{1}{\sqrt{2|C_1|}}(p - p \times S), \quad A_2 = S, \quad A_3 = \frac{1}{\sqrt{2|C_1|}}(p + p \times S).$$

Letting $A := (A_1, A_2, A_3) \in SO(2, 1)$, and

$$Q := \frac{1}{|C_1|} p \times v = \frac{1}{\sqrt{2|C_1|}}((v, A_1 - A_3)A_2 - (v, A_2)(A_1 - A_3)),$$

we then define $g := (Q, A)$. It follows that $\eta = \text{Ad}^*(g)\mu$, and hence that the map $(p, v) \mapsto (g(p, v)^{-1}, (p, v))$ is a cross-section of the Marsden–Weinstein fibration.

Null type. Finally, orbits of negative type have $C_1 := \|p\|^2 = 0$ with $p \neq 0$. Again we treat the case where the vector $p$ is future-directed, the past-directed case being similar. In this case, we may use the action of $\mathbb{E}(2, 1)$ to reduce $\mu = (m_1, m_2)$ to the standard form

$$m_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad m_2 = -C_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We now define the null basis vector $A_1 := p$, and extend to a basis $(A_1, A_2, A_3)$ by defining, for example,

$$A_2 = e_2 + \frac{p^2}{p^3} e_1, \quad A_3 = -\frac{1}{p^3} e_1.$$

We let

$$Q := A_3 \times v = \langle v, A_2 \rangle A_3 - \langle v, A_3 \rangle A_2,$$

and then define $g = (Q, A)$. Again it follows that $\eta = \text{Ad}^*(g)\mu$ and therefore that the map $(p, v) \mapsto (g(p, v)^{-1}, (p, v))$ is a cross-section of the Marsden–Weinstein fibration.

The explicit parameterizations of the orbits given in Section 4.1 have the property that $\eta^*(\sigma^\mu) = dt$, and hence $v^\mu = 1$. From the explicit forms of the cross-sections, $g$, it is straightforward to check for each type of orbit that $g(t)^{-1} g(t)' + g(t)^{-1} \mathcal{H}[\eta(t)]g(t)$ lies in $g_\mu$ for all $t$ in the relevant range, as required. We may then, by direct integration, compute the gauge transformation (24). The integral curves of the Euler–Lagrange system are then given by (25).
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Appendix A. Pfaffian differential systems with one independent variable

Definition A.1. Let $M$ be a smooth manifold. A Pfaffian differential system $(\mathcal{I}, \omega)$ with one independent variable on $M$ consists of a Pfaffian differential ideal $\mathcal{I} \subset \Omega^*(M)$ and a non-vanishing 1-form $\omega \in \Omega^1(M)$ such that $\omega \not\equiv 0 \pmod{\mathcal{I}}$.

Definition A.2. An integral element of $(\mathcal{I}, \omega)$ is a pair $(x, E)$ consisting of a point $x \in M$ and a one-dimensional linear subspace $E \subset T_x(M)$ such that $\eta|_E = 0$, $\forall \eta \in \mathcal{I}$ and $\omega|_E \neq 0$. We denote by $V(\mathcal{I}, \omega)$ the set of integral elements of $(\mathcal{I}, \omega)$. We say that $\mathcal{I}$ has constant rank if it is generated by the cross-sections of a sub-bundle $Z$ of $T^*(M)$.

Definition A.3. A (parameterized) integral curve of $(\mathcal{I}, \omega)$ is a smooth curve $\alpha : (a, b) \subseteq \mathbb{R} \to M$ such that

$$\alpha^*(\eta) = 0 \quad \forall \eta \in \mathcal{I}, \quad \gamma^*(\omega) = dt.$$  

We denote the set of integral curves of the system by $V(\mathcal{I}, \omega)$.

Definition A.4. We say that the Pfaffian system in one independent variable $(\mathcal{I}, \omega)$ is reducible if there exists a non-empty submanifold $M^* \subseteq M$ such that:

- for each point $x \in M^*$ there exists an integral element $(x, E) \in V(\mathcal{I}, \omega)$ tangent to $M^*$;
- if $N \subseteq M$ is any other submanifold with the same property then $N \subseteq M^*$.

We call $M^*$ the reduced space. We define on $M^*$ the reduced Pfaffian system, denoted by $(\mathcal{I}^*, \omega)$, which is obtained by restricting the original system $(\mathcal{I}, \omega)$ to $M^*$.

A basic result is the following, a proof of which may be found in [12].

Proposition A.5. The Pfaffian systems $(\mathcal{I}, \omega)$ and $(\mathcal{I}^*, \omega)$ have the same integral curves.

There is an algorithmic procedure for constructing the reduction of a Pfaffian system [12]. To construct the reduced space $M^*$, we consider the projection $M_1 \subseteq M$ of $V(\mathcal{I}, \omega)$ to $M$. If $M_1$ is a non-empty submanifold of $M$, we then define $(\mathcal{I}_1, \omega_1)$ to be the restriction of $(\mathcal{I}, \omega)$ to $M_1$. We then construct $V(\mathcal{I}_1, \omega_1)$, the set of integral elements of $(\mathcal{I}_1, \omega_1)$. Repeating this construction, we inductively define

$$M_k = (M_{k-1})_1, \quad \mathcal{I}_k = (\mathcal{I}_{k-1})_1, \quad \omega_k = (\omega_{k-1})_1.$$
This process defines a sequence \( M \supseteq M_1 \supseteq \cdots \supseteq M_k \supseteq \cdots \) of submanifolds of \( M \). If \( M^* := \bigcap_{k \in \mathbb{N}} M_k \neq \emptyset \) then \( M^* \) is the reduced space of the system. Notice that this procedure requires that, at each stage, the subset \( M_k \subseteq M_{k-1} \) is a non-empty submanifold.

A.1. Cartan systems

Let \( \Psi \in \Omega^2(M) \) be an exterior differential 2-form on \( M \). We define the Cartan ideal to be the Pfaffian differential ideal \( \mathcal{C}(\Psi) \subseteq \Omega^*(M) \) generated by the set of 1-forms \( \eta_V := i_V \Psi \) obtained by contracting \( \Psi \) with vector fields on \( M \). If \( \theta^1, \ldots, \theta^n \) is a local coframing on \( M \) and if \( \Psi = a_i \theta^i \wedge \theta^j \), then \( \mathcal{C}(\Psi) \) is locally generated by the 1-forms \( a_i \theta^j \). A Cartan system is a pair \((\mathcal{C}(\Psi), \omega)\) consisting of a Cartan ideal \( \mathcal{C}(\Psi) \) and a 1-form \( \omega \in \Omega^1(M) \) such that \( \omega \vert_p \notin \mathcal{C}(\Psi) \vert_p \), for all \( p \in M \).

**Definition A.6.** A Cartan system \((\mathcal{C}(\Psi), \omega)\) is regular if:

- it is reducible and the reduced phase space \( M^* \) is odd-dimensional;
- the 2-form \( \Psi^* := \Psi \vert_{M^*} \in \Omega^2(M^*) \) is of maximal rank on \( M^* \).

An important fact is that if \((\mathcal{C}(\Psi), \omega)\) is regular then the Cartan ideal \( \mathcal{C}(\Psi^* \vert) \) on \( M^* \) is the restriction to \( M^* \) of the Cartan ideal \( \mathcal{C}(\Psi) \) on \( M \):

\[ \mathcal{I}^* := \mathcal{C}(\Psi) \vert_{M^*} = \mathcal{C}(\Psi^* \vert). \]

Again, a proof of this result may be found in [12].

If \((\mathcal{C}(\Psi), \omega)\) is regular, then there exists a unique vector field \( \xi \) on \( M^* \) such that \( i_\xi \Psi^* = 0 \) and \( \omega(\xi) = 1 \). We call \( \xi \) the characteristic vector field of the Cartan system \((\mathcal{C}(\Psi), \omega)\). The integral curves of the characteristic vector field coincide with the parameterized integral curves of \((\mathcal{C}(\Psi^* \vert), \omega)\), and hence with those of \((\mathcal{C}(\Psi), \omega)\).

A.2. Contact systems on jet spaces

Given a manifold \( M \), we denote by \( J^k(\mathbb{R}, M) \) the bundle of the \( k \)-order jets of maps \( \gamma : \mathbb{R} \to M \). The \( k \)-jet of \( \gamma \) at \( t \) will be denoted by \( j^k(\gamma) \vert_t \). Local coordinates \((x^1, \ldots, x^n)\) on \( M \) give standard local coordinates \((t, x^1, \ldots, x^n, x^1_1, \ldots, x^n_1, x^1_2, \ldots, x^n_k)\) on the jet space \( J^k(\mathbb{R}, M) \). With respect to such a coordinate system a \( k \)-jet with coordinates \((t, x^1, \ldots, x^n, x^1_1, \ldots, x^n_1, x^1_2, \ldots, x^n_k)\) is represented by \( j^k(\gamma) \vert_t \), where \( \gamma \) is the curve defined by

\[ \gamma : s \mapsto (x^1_0, \ldots, x^n_0) + (x^1_1, \ldots, x^n_1)(t - s) + \cdots + \frac{1}{k!}(x^1_k, \ldots, x^n_k)(t - s)^k. \]

The canonical contact system \( \mathcal{I} \) on \( J^k(\mathbb{R}, M) \) is defined to be the Pfaffian differential ideal generated by the forms \( \eta^i_0 := dx^i_0 - x^i_{a+1} \, dt, i = 0, \ldots, n, a = 0, \ldots, k - 1 \) (where \( x^i_0 = x^i \)). The independence condition of the system is given by the 1-form \( dt \). The integral curves \( I^* : (a, b) \to J^k(\mathbb{R}, M) \) of \( \mathcal{I} \) such that \( I^*(\omega) = dt \) are the canonical lifts \( j^k(\gamma) \) of maps \( \gamma : (a, b) \to M \).
Appendix B. Constrained variational problems in one independent variable

Definition B.1. Let \((I, \omega)\) be a Pfaffian differential system on a smooth manifold \(M\) and let \(L : M \to \mathbb{R}\) be a smooth function. The triple \((I, \omega, L)\) is said to be a constrained variational problem in one independent variable. The function \(L\) is referred to as the Lagrangian of the variational problem.

The Lagrangian \(L\) gives rise to the action functional \(\mathcal{L} : \mathcal{V}(I, \omega) \to \mathbb{R}\) defined (perhaps not everywhere) on the space of the integral curves of \((I, \omega)\) by

\[
\mathcal{L}(\gamma) = \int_{\gamma} \gamma^*(L\omega).
\]

Definition B.2. By an extremal curve of \((I, \omega, L)\) we mean an integral curve \(\gamma\) that is a critical point of the functional \(\mathcal{L}\) when one considers compactly supported variations of \(\gamma\) through integral curves of the system.

Let us suppose that the Pfaffian ideal \(I\) is generated by a sub-bundle \(Z \subset T^*(M)\). We then let \(\tilde{Z} \subset T^*(M)\) be the affine sub-bundle \(L\omega + Z\). We denote by \(\psi \in \Omega^1(\tilde{Z})\) the restriction to \(\tilde{Z}\) of the tautological 1-form on \(T^*(M)\), and call \(\psi\) the Liouville form of the variational problem. We let \(\Psi\) be the 2-form \(d\psi\) and we consider on \(\tilde{Z}\) the Cartan system \(\mathcal{C}(\Psi)\) together with the independence condition \(\omega\).

Definition B.3. We say that \((I, \omega, L)\) is a regular variational problem if the Cartan system \((\mathcal{C}(\Psi), \omega)\) is reducible. The reduced space \(Y \subset \tilde{Z}\) of \((\mathcal{C}(\Psi), \omega)\) is called the momentum space of the variational problem. The restriction of the Cartan system \((\mathcal{C}(\Psi), \omega)\) to \(Y\) is called the Euler–Lagrange system of the variational problem, and denoted \((\mathcal{E}, \omega)\).

The importance of the Euler–Lagrange system comes from the following theorem (cf. [3,12]).

Theorem B.4. Let \(\Gamma : (a, b) \to Y\) be an integral curve of the Euler–Lagrange system. Then \(\gamma = \pi_M \circ \Gamma : (a, b) \to M\) is a critical point of the action functional \(\mathcal{L}\), where \(\pi_M : Y \to M\) denotes the restriction to \(Y\) of the projection \(T^*(M) \to M\).

This theorem allows us to find critical points of the variational problem from the integral curves the Euler–Lagrange system. However, not all the extremals arise this way for a general variational problem. It is known that if all the derived systems of \(Z\) have constant rank, then all the extremals are projections of the integral curves of the Euler–Lagrange system [3]. Other results in this direction have been proved by Hsu [16].

Definition B.5. A variational problem \((I, \omega, L)\) is said to be non-degenerate if the Cartan system \((\mathcal{C}(\Psi), \omega)\) is regular, in the sense of Definition A.6.
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Paper 4

*Symmetries and moduli spaces*

*of the*

*self-dual Yang-Mills equations,*

SYMMETRIES AND MODULI SPACES OF THE SELF-DUAL
YANG-MILLS EQUATIONS

JAMES D.E. GRANT

ABSTRACT. We review the construction of infinite-dimensional symmetry algebras of the
self-dual Yang-Mills equations on \( \mathbb{R}^4 \) and the ADHM description of the moduli space of
instantons on \( S^4 \). We report on recent work describing the action of the corresponding
symmetry (pseudo)-group on the instanton moduli spaces.

1. Introduction

In Hamiltonian mechanics, one has a well defined notion of an integrable system. Such a
system is defined by a 2\( n \)-dimensional symplectic manifold \((X, \omega)\) with the dynamics being
determined by a function \( H : X \to \mathbb{R} \). The system is completely integrable in the sense of
Liouville if there exist \( n \) functions preserved by the flow generated by the Hamiltonian that
have vanishing Poisson brackets with one another and that are functionally independent on
\( X \), perhaps minus a set of measure zero, \( S \). From a more global perspective, this implies that
there is an action of an \( n \)-dimensional Abelian Lie group on \( X \setminus S \) that, in the case that this
group is compact, corresponds to a Hamiltonian torus action. Thus the space \( X \setminus S \) is foliated
by \( n \)-dimensional tori.

When one considers systems of partial differential equations, it is more difficult to define a
suitable notion of integrability. One approach has been to study systems that admit infinite-
dimensional symmetry algebras. In order to make contact with the finite-dimensional theory,
however, one would like to know whether there is a corresponding group action on the space
of solutions of the system and, if so, what the corresponding orbit structure is. As an example
for particular types of harmonic maps, the action of the corresponding symmetry group (the
dressing action) on the space of solutions is well understood (see [Gu] and references therein).

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173.
A set of equations that is of particular interest from the point of view of integrable systems theory is the self-dual Yang-Mills equations, which is a system of partial differential equations defined on an arbitrary oriented four-manifold, $X$. If the four-manifold is self-dual then the self-dual Yang-Mills equations are generally considered to be integrable in the sense that there is a twistor-theoretic method of constructing solutions which is analogous to the inverse-scattering methods ubiquitous in standard integrable systems theory [AHS, WW, MW]. Most known integrable systems can be derived as symmetry reductions of the self-dual Yang-Mills equations on a self-dual manifold for particular choices of gauge group and manifold $X$ [MW]. It is therefore important to know whether there is a corresponding group action on the space of solutions to the self-dual Yang-Mills equations corresponding to the above symmetry algebra and, if so, what the orbits of this group action are.

The purpose of this article is twofold. Firstly to review the relevant literature and results on the various aspects of the self-dual Yang-Mills equations that this problem entails, and secondly to briefly present some preliminary results on the action of non-local symmetries on instanton moduli spaces (for full details see [Gr1]). The problem of understanding the orbit structure lies at the interface between local considerations (i.e. symmetry algebras of differential equations) and global ones (moduli spaces of instanton solutions). One of our aims at this point is give a unified notation appropriate to both points of view. For simplicity we restrict ourselves to the self-dual Yang-Mills equations on $\mathbb{R}^4$ and take the gauge group to be SU(2). In this case one can construct explicitly an infinite-dimensional algebra of non-local symmetries of the self-dual Yang-Mills equations [Ch, D]. Since for standard integrable systems (e.g. the KdV equation) it is necessary to restrict oneself to a suitable class of solutions in order for inverse-scattering methods to work (e.g. periodic solutions or solutions with rapid asymptotic fall-off), we restrict ourselves to instanton solutions of the self-dual Yang-Mills equations on $\mathbb{R}^4$ (i.e. solutions with curvature that is $L^2$). By conformal invariance and a theorem of Uhlenbeck [U], this means we may equivalently consider the instanton problem on $S^4$. In this case we have a full description of the moduli space of solutions of the self-dual Yang-Mills equations given by the ADHM construction [ADHM, At1]. Our main result (see Section 4) is that the tangent space to each instanton moduli space is generated by non-local symmetries of the form given in [Ch, D]. As such, since the instanton moduli spaces are connected [Do], the corresponding symmetry pseudo-group acts transitively on them.
We end by discussing some further lines of research that we are pursuing.

2. THE SELF-DUAL YANG-MILLS EQUATIONS

Let \((X, g)\) be a connected, oriented, Riemannian four-manifold. Since \(X\) is oriented, we have a volume form \(\nu \in \Omega^4(X)\), and thus we may define a Hodge \(*\) operation

\[* : \Omega^p(X) \to \Omega^{4-p}(X), \quad p = 0, \ldots, 4,\]

by the relation

\[\alpha \wedge *\beta = \langle \alpha, \beta \rangle \nu, \quad \forall \alpha, \beta \in \Omega^p(X).\]

When restricted to \(\Omega^2(X)\) on a Riemannian four-manifold, the map \(*\) has two important properties:

- It is conformally invariant;
- \(\left( *\right|_{\Omega^2(X)} \right)^2 = \text{Id}_{\Omega^2(X)}\). We therefore have a direct sum decomposition

\[\Omega^2(X) = \Omega^2^+(X) \oplus \Omega^2^-(X)\]

of the space of two-forms into self-dual and anti-self-dual two-forms.

Let \(\pi : E \to X\) be a vector bundle over \(X\) with structure group \(G\). A connection on \(E\) may be represented by a \(g\)-valued one-form \(A \in \Omega^1(X, g)\), with curvature \(F_A \in \Omega^2(X, g)\). Using the decomposition of \(\Omega^2(X)\) we may therefore write

\[F_A = F_A^+ + F_A^-,
\]

where \(F_A^\pm \in \Omega^{2\pm}(X, g)\) are the self-dual and anti-self-dual parts of the curvature.

**Definition 2.1.** A connection on a vector bundle \(\pi : E \to X\) is a solution of the self-dual Yang-Mills equations if its curvature obeys the condition

\[*F_A = F_A.\]

(Equivalently \(F_A^- = 0\).)

If we take \(E\) to be \(TX\), the tangent bundle of \(X\), with the Levi-Civita connection then the curvature of the connection is the Riemann tensor, \(R\). Viewing this as an element of
we may decompose it in block diagonal form relative to the decomposition (*), to yield

$R = \begin{pmatrix}
W^+ + \frac{s}{12} & \text{Ric}_0 \\
\text{Ric}_0 & W^- + \frac{s}{12}
\end{pmatrix}$

where $W^\pm$ denote the self-dual and anti-self-dual parts of the Weyl tensor, $\text{Ric}_0$ denotes the trace-free part of the Ricci tensor, and $s$ denotes the scalar curvature of the metric $g$.

**Definition 2.2.** A Riemannian manifold $(X,g)$ is self-dual if the anti-self-dual part of the Weyl tensor vanishes:

$W^- = 0$.

From now on we fix $X$ to be $S^4$ with its standard (self-dual) conformal structure. For simplicity we take $G = \text{SU}(2) \cong \text{Sp}(1)$, although it is straightforward to generalise our results to other groups.

### 2.1. Instanton numbers and index theorem results.

**Lemma 2.3.** [BPST, AHS] Let $\pi : E \to S^4$ be a rank-2 Hermitian complex vector bundle associated to a principal $\text{SU}(2)$ bundle $P \to S^4$. Such bundles are characterised by the second Chern class

$c_2(E) := -\frac{1}{8\pi^2} \int_X \|F_A\|^2 d\text{vol}_X$,

where $A$ is any connection on $E$ and $F_A \in \Omega^2(S^4, \text{End}(E))$ is its curvature. If $\int_X \|F_A\|^2 d\text{vol}_X < \infty$, then $p_2(E) = -k$ where $k$ is an integer, referred to as the instanton number. If the connection $A$ is a self-dual, then $k \geq 0$, with equality if and only if the connection is flat.

There is a natural action of the group of gauge transformations (i.e. maps $S^4 \to \text{SU}(2)$) on the space of connections. We therefore define the moduli space of $k$-instanton solutions:

$\mathcal{M}_k := \{(E, A) : F_A = *F_A, c_2(E) = -k\}$

subject to gauge transformations.

**Theorem 2.4.** [AHS] $\mathcal{M}_k$ is a manifold (possibly with singularities corresponding to reducible connections) of dimension $8k - 3$. 
2.2. The ADHM construction. The ADHM construction [ADHM, At1] allows us to construct the \((8k-3)\)-parameter family of solutions of the self-dual Yang-Mills equations on \(S^4\) using quaternionic linear algebra. Viewing \(S^4\) as \(\mathbb{H}P^1\) then a point \(p \in S^4\) corresponds to a quaternionic line \(\Sigma \subset \mathbb{H}^2\). Choosing a complex structure on \(\mathbb{H}^2\), we can identify it with \(\mathbb{C}^4\), with \(\Sigma\) corresponding to a complex surface. Since, as a subspace of \(\mathbb{H}^2\), \(\Sigma\) is invariant under right multiplication by the unit quaternion \(j\), this implies that as a subspace of \(\mathbb{C}^4\) it is invariant under the corresponding anti-linear anti-involution

\[ \sigma : \mathbb{C}^4 \to \mathbb{C}^4 : (z_1, z_2, z_3, z_4) \mapsto (-z_2, z_1, -z_4, z_3). \]

Under the natural projection \(\mathbb{C}^4 \to \mathbb{C}P^3\) the surface \(\Sigma\) projects to a rational curve which we denote by \(\sigma(p) \cong \mathbb{C}P^1 \subset \mathbb{C}P^3\). Curves that arise in this way are called real lines in \(\mathbb{C}P^3\). The involution on \(\mathbb{C}P^3\) induced by \(\sigma\) leaves the real lines invariant, and acts as the anti-podal map on each real \(\mathbb{C}P^1\).

**Theorem 2.5.** [Wa] There is a bijective correspondence between a). Solutions of the self-dual Yang-Mills equations on \(S^4\) and b). holomorphic vector bundles on \(\mathbb{C}P^3\) that are (holomorphically) trivial when restricted to each real line. We refer to this correspondence as the Ward correspondence.

**Remark.** The holomorphic bundle over \(\mathbb{C}P^3\) will carry additional structures depending on the particular group \(G\) that we are considering. For the case \(G = SU(2)\), we require that the determinant bundle \(\text{det } E\) is trivial and that \(E\) admits a positive real form.

The ADHM construction uses methods from algebraic geometry to construct holomorphic vector bundles on \(\mathbb{C}P^3\) and therefore, via the Ward correspondence, self-dual Yang-Mills connections on \(S^4\). For each \(z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4\), we define a linear map

\[ A(z) : W \to V, \]

where \(W, V\) are complex vector spaces of dimension \(k, 2k+2\) respectively, which is of the form

\[ A(z) = \sum_{i=1}^{4} z_i A_i. \]
The space $W$ is assumed to admit an anti-linear involution $\sigma_W : W \to W$. The space $V$ is assumed to have a non-degenerate, skew-symmetric form $(\cdot,\cdot)$ and an anti-linear anti-involution $\sigma_V : V \to V$ that is compatible with the form in the sense that

$$(\sigma_V u, \sigma_V v) = (u, v), \quad \forall u, v \in V.$$ 

We then require that the map $A(z)$ satisfies the compatibility condition that

$$\sigma_V (A(z)w) = A(\sigma(z))\sigma_W(w), \quad \forall w \in W$$  

and impose the following conditions:

- For all $z \in \mathbb{C}^4$, the space $U_z := A(z)(W) \subset V$ is of dimension $k$;
- For all $z \in \mathbb{C}^4$, $U_z$ is isotropic with respect to $(\cdot,\cdot)$ i.e. $U_z \subseteq U_z^\perp$, where $^\perp$ denotes the complement with respect to the form $(\cdot,\cdot)$.

If we then define the quotient $E_z := U_z^\perp/U_z$, then the dimension and isotropy constraints on $U_z$ imply that the collection of $E_z$ defines a holomorphic, rank-2 complex vector bundle $E \to \mathbb{C}P^3$ with structure group $\text{SL}(2, \mathbb{C})$. The reality condition (2.1) then imply that the bundle is trivial on restriction to any real line and that the structure group reduces from $\text{SL}(2, \mathbb{C})$ to $\text{SU}(2)$.

The power of the ADHM construction is that all $k$-instanton solutions of the self-dual Yang-Mills equations arise in this fashion.

2.3. Patching matrix description. Given a finite point $x := (x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \subset S^4$, then we may view 

$$x := x^1 + ix^2 + jx^3 + kx^4 = u + jv \in \mathbb{H}$$

as an affine coordinate on $\mathbb{R}^4 \cong \mathbb{H} \subset \mathbb{H}P^1$, where

$$u := x^1 + ix^2, \quad v := x^3 - ix^4.$$ 

As mentioned above, the point $x$ therefore defines a quaternionic line $\Sigma(x) \subset \mathbb{H}^2$ and thence a real line $\sigma(x) \cong \mathbb{C}P^1 \subset \mathbb{C}P^3$ which takes the form

$$\sigma(x) = \left\{ [z_1, z_2, z_1 u - z_2 \overline{v}, z_1 v + z_2 \overline{u}] : (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\} \right\}.$$
On a fixed real line $\sigma(x)$, $x \in \mathbb{R}^4$, we introduce the affine coordinate $z = z_2/z_1 \in \mathbb{C}P^1$. The image of point $\infty \in S^4$ is the real line

$$l_\infty := \{(0, 0, z_3, z_4) : (z_3, z_4) \in \mathbb{C}^2 \setminus \{(0, 0)\}\}.$$ 

Given an instanton solution on $S^4$, we may consider the restriction of the solution to $\mathbb{R}^4 \subset S^4$. (Uhlenbeck's theorem [U] implies that there is no loss of information in doing so.) We may therefore consider the corresponding restriction of the bundle $E|_{\mathbb{C}P^3 \setminus l_\infty}$ which, for convenience, we denote by $E \to \mathbb{C}P^3 \setminus l_\infty$. We may split the region $\mathbb{C}P^3 \setminus l_\infty$ into coordinate regions

$$S_0 := \{((u, v), z) \in \mathbb{C}^2 \times \mathbb{C}P^1 : z \neq \infty\} = \mathbb{C}^2 \times U_0,$$

$$S_\infty := \{((u, v), z) \in \mathbb{C}^2 \times \mathbb{C}P^1 : z \neq 0\} = \mathbb{C}^2 \times U_\infty,$$

where

$$U_0 := \{[z_1, z_2] \in \mathbb{C}P^1 : z_1 \neq 0\}, \quad U_\infty := \{[z_1, z_2] \in \mathbb{C}P^1 : z_2 \neq 0\}.$$ 

Since $S_0, S_\infty \cong \mathbb{C}^3$, a result of Grauert (see e.g. [OSS]) implies that the bundle $E$ restricted to either of these regions is trivial. Therefore the bundle $E$ is characterised by the transition functions $G : S_0 \cap S_\infty \to \text{SL}(2, \mathbb{C})$.

We may construct the map $G$ directly from the ADHM data. With respect to bases on $V, W$, $A(z)$ can be viewed as a $(2k + 2) \times k$ matrix with complex coefficients. The columns of $A(z)$ then define a set of $k$ vectors $v_1(z), \ldots, v_k(z) \in \mathbb{C}^{2k+2}$ that span the space $U_z$. These vectors obey the reality condition

$$\sigma_V(v_i(z)) = v_i(\sigma(z)), \quad i = 1, \ldots, k.$$ 

Since $U_z$ is isotropic with respect to the symplectic form $(\cdot, \cdot)$, we deduce that

$$(v_i(z), v_j(z)) = 0.$$ 

We now restrict to a real line $\sigma(x) = \{x\} \times (U_0 \cup U_\infty) \subset \mathbb{C}P^3 \setminus l_\infty$. On the subset $\{x\} \times U_0$ the annihilator $U_z^\perp$ is spanned by $\{v_i(z)\}$ along with two vectors $\{e_A(z) : A = 1, 2\}$ that span $U_z^\perp / U_z$. We therefore have

$$(v_i(z), e_A(z)) = 0,$$
and may, without loss of generality, assume that
\[(e_1(z), e_2(z)) = -(e_2(z), e_1(z)) = 1. \tag{2.2}\]

We may also define a basis \(\{f_A(z) : A = 1, 2\}\) for \(U_z^\perp / U_z\) on the region \(\{x\} \times U_\infty\) by the relations
\[f_1(z) := -\sigma_V (e_2(\sigma(z))), \quad f_2(z) := \sigma_V (e_1(\sigma(z))).\]

This basis has the property that
\[(f_1(z), f_2(z)) = 1, \quad (v_i(z), f_A(z)) = 0.\]

Given that \(\{e_A(z)\}\) and \(\{f_A(z)\}\) are both bases for \(U_z^\perp / U_z\) for \(z \in U_0 \cap U_\infty\), there exist functions \(G_A^B(z), \lambda_A^i(z)\) with the property that
\[f_A(z) = G_A^B(z)e_B(z) + \lambda_A^i(z)v_i(z). \tag{2.3}\]

The matrix \(G\) is then the transition function of our bundle \(E\). The definition of the vectors \(\{f_A\}\) in terms of the \(\{e_A\}\) implies that
\[G^* = G, \quad \det G = 1,\]
where \(G^*(z) := \overline{G(\sigma(z))}\).

Finally note that since the bundle \(E\) is holomorphic over \(\mathbb{C}P^3 \setminus l_\infty\) the transition functions are holomorphic with respect to the natural complex structure on \(\mathbb{C}P^3\) restricted to \(\mathbb{C}P^3 \setminus l_\infty\).

In terms of the coordinates introduced above this implies that
\[G = G \,(u - z\overline{v}, v + z\overline{u}, z).\]

3. Symmetries of the self-dual Yang-Mills equations

In terms of the complex coordinates \(u, v\) introduced on \(\mathbb{C}^2 \cong \mathbb{R}^4\) above, the standard flat metric on \(\mathbb{R}^4\) takes the form
\[g = \frac{1}{2} (du \otimes d\overline{u} + d\overline{u} \otimes du + dv \otimes d\overline{v} + d\overline{v} \otimes dv).\]

The corresponding volume form is then
\[\epsilon = dt \wedge dx \wedge dy \wedge dz = \frac{1}{4} du \wedge d\overline{u} \wedge dv \wedge d\overline{v}.\]
In terms of these coordinates, the self-dual Yang-Mills equations for a connection, $A$, correspond to the following conditions on the components of the curvature tensor, $F_A$, of the connection:

\begin{align}
F_{uv} &= 0; \\
F_{u\bar{\nu}} + F_{\bar{\nu}u} &= 0; \\
F_{\bar{\nu}v} &= 0.
\end{align}

(3.1a) (3.1b) (3.1c)

Introducing the vector fields

$$X(z) := \partial_{\overline{\nu}} + z \partial_u, \quad Y(z) := \partial_{\overline{\nu}} - z \partial_v,$$

which depend on an arbitrary parameter $z \in \mathbb{C} \cup \infty = \mathbb{C}P^1$, then the self-dual Yang-Mills equations (3.1) are equivalent to the condition that

$$F_A(X(z), Y(z)) = 0, \quad \forall z \in \mathbb{C}P^1.$$

3.1. Non-local symmetries of the self-dual Yang-Mills equations. An important property of the self-dual Yang-Mills equations is that they are the compatibility condition for the following overdetermined equations \cite{Ch,BZ,T,Cr}

\begin{align}
(\partial_{\overline{\nu}} + z A_u) \psi(x,z) &= -(A_{\overline{\nu}} + z A_u) \psi(x,z), \\
(\partial_{\overline{\nu}} - z A_v) \psi(x,z) &= -(A_{\overline{\nu}} - z A_v) \psi(x,z),
\end{align}

(3.2a) (3.2b)

for a map $\psi : \mathbb{R}^4 \times U \to \text{SL}(2, \mathbb{C})$, where $U \subset \mathbb{C}P^1$ is a suitable domain. In particular we may find a solution $\psi_0 : \mathbb{R}^4 \times U_0 \to \text{SL}(2, \mathbb{C})$ that is analytic in $z$ for $z \neq \infty$. Given such a solution we may then construct a solution $\psi_\infty(x,z) : \mathbb{R}^4 \times U_\infty \to \text{SL}(2, \mathbb{C})$ that is analytic in $z$ for $z \neq 0$ by taking

$$\psi_\infty(x,z) = (\psi_0(x, \sigma(z))^{-1})^\dagger,$$

where

$$\sigma(z) := \frac{1}{z}.$$

Note that $\sigma$ may be viewed as the anti-podal map on $\mathbb{C}P^1$ viewed in terms of affine coordinates.
Equations (3.2) for $\psi_0$ and $\psi_\infty$ imply that we may write the components of the connection in the form

$$A_u = - (\partial_u \psi_\infty(\infty)) \psi_\infty(\infty)^{-1}, \quad A_v = - (\partial_v \psi_\infty(\infty)) \psi_\infty(\infty)^{-1},$$

$$A_\pi = - (\partial_\pi \psi_0(0)) \psi_0(0)^{-1}, \quad A_\tau = - (\partial_\tau \psi_0(0)) \psi_0(0)^{-1}. \quad (3.3)$$

If we define $J := \psi_\infty(\infty)^{-1} \cdot \psi_0(0)$ then the remaining part of the self-dual Yang-Mills equations imply that $J$ obeys the Yang-Pohlmeyer equation

$$\partial_u \left( J \partial_u J^{-1} \right) + \partial_v \left( J \partial_v J^{-1} \right) = 0. \quad (3.5)$$

It is known that the only local symmetries of the self-dual Yang-Mills equations on flat $\mathbb{R}^4$ are gauge transformations and those generated by the action of the conformal group. On the other hand, there exists a non-trivial family of non-local symmetries of the self-dual Yang-Mills equations [Ch, D]. Let $J(t)$ denote a 1-parameter family of solutions of (3.5) that is assumed to depend smoothly on the parameter $t \in I$, where $I$ is an open subinterval of the real line that contains the origin. Taking the derivative of (3.5) with respect to $t$ we find that we require

$$\partial_u \left( J \partial_u \left( J^{-1} \frac{\partial J}{\partial t} \right) J^{-1} \right) + \partial_v \left( J \partial_v \left( J^{-1} \frac{\partial J}{\partial t} \right) J^{-1} \right) = 0. \quad (3.6)$$

The construction of [Ch, D] proceeds as follows. Let $\chi(z)$ be a solution of the system

$$\left[ (\partial_\pi - J \partial_\pi J^{-1}) + z \partial_u \right] \chi(x, z) = 0,$$

$$\left[ (\partial_\tau - J \partial_\tau J^{-1}) - z \partial_v \right] \chi(x, z) = 0.$$

If we then define

$$\frac{\partial J}{\partial t} = \chi(x, z) T(x, z) \chi(x, z)^{-1} \cdot J, \quad (3.7)$$

where the function $T$ obeys the relations

$$(\partial_\pi + z \partial_u) T = (\partial_\pi - z \partial_v) T = 0, \quad (3.8)$$

then $\partial J/\partial t$ is a solution of the linearisation (3.6). Without loss of generality, we may take

$$\chi(x, z) = \psi_\infty(\infty)^{-1} \cdot \psi_\infty(z),$$

which is analytic for $z \neq 0$, and has the property that $\chi(x, \infty) = \text{Id}$. If we expand the right-hand-side of (3.7) as a Laurent series in $z$ then the coefficients in the expansion define a (generally infinite) family of solutions of the linearisation equations for $J$. Moving from one
coefficient in this expansion to the next defines a map between solutions of the linearisation
equations which, in integrable systems terminology, defines the recursion operator of the
self-dual Yang-Mills equations and its related integrable hierarchy [MW].

Such symmetries of the self-dual Yang-Mills equations with gauge group \( G = \text{GL}(n, \mathbb{C}) \) have
been studied within the Sato/Segal-Wilson approach to integrable systems by Takasaki [T],
and thus have the interpretation of an infinite-dimensional family of projective transforma-
tions on an infinite-dimensional Grassmannian. It is not clear, however, how to implement
this approach for \( G = \text{SU}(n) \), as the transformations generally do not preserve the unitarity
of field \( J \). Takasaki’s approach has, however, been used to study dressing actions on harmonic
maps, where the reality conditions are more straightforward [AJS-A].

In order to maintain unitarity of \( J \), we will consider the symmetry

\[
\frac{\partial J}{\partial t} = \chi(x, z) T(x, z) \chi(x, z)^{-1} \cdot J + J \cdot (\chi(x, z)^{-1})^\dagger T(x, z)^\dagger \chi(x, z)^\dagger
\]

\[
= \psi_\infty(\infty)^{-1} \left[ \psi_\infty(z) T(x, z) \psi_\infty(z)^{-1} + \psi_0(\sigma(z)) T(x, z)^\dagger \psi_0(\sigma(z))^{-1} \right] \psi_0(0), \quad (3.9)
\]

where \( T(x, z) \) obeys the condition (3.8).

If we now consider the symmetry (3.9) with \( T(x, z) \) a constant element of \( g \otimes \mathbb{C} \), then the
algebra of such symmetries is isomorphic to the Kac-Moody algebra of \( g \otimes \mathbb{C} \). The natural
question is whether there is a corresponding group action on the space of solutions. A partial
solution to this problem was given by Crane [Cr], who showed that taking \( T \) to be constant,
one could define an action of the (analytic) loop group of \( G_\mathbb{C} \) (i.e. the group of analytic maps
from \( S^1 \) to \( G_\mathbb{C} \)) on the space of solutions of the self-dual Yang-Mills equations. As Crane
showed by example, however, this action does not preserve the instanton condition that the
curvature of the connection be \( L^2 \). More generally, for a holomorphic bundle on \( \mathbb{C}P^3 \setminus l_\infty \)
defined by the patching matrix \( G(u - z \overline{v}, v + z \overline{u}, z) \), a general \( T(u - z \overline{v}, v + z \overline{u}, z) \) generates
a transformation of the form

\[
G(u - z \overline{v}, v + z \overline{u}, z) \mapsto g(u - z \overline{v}, v + z \overline{u}, z) \cdot G(u - z \overline{v}, v + z \overline{u}, z) \cdot g^*(u - z \overline{v}, v + z \overline{u}, z),
\]

where

\[
g^*(u - z \overline{v}, v + z \overline{u}, z) := g \left( u + \overline{v} \frac{v}{z}, v - \overline{u} \frac{v}{z}, -\frac{1}{z} \right)^\dagger.
\]

The function \( g \) is an arbitrary holomorphic functions of its arguments. If \( g \) is analytic for
\( z \neq \infty \), this transformation is simply a holomorphic change of basis on the bundle over \( \mathbb{C}P^3 \),
and leaves the self-dual Yang-Mills field unchanged. However, if $g$ has poles at finite values of $z$, it will have a non-trivial effect on the connection and generates a distinct solution of the self-dual Yang-Mills equations. This action has been given a cohomological description by Park (see [Pa] and references therein), which has been further investigated by Popov and Ivanova (see [Po, Iv] and references therein).

4. One-parameter families of ADHM data

We now consider a one-parameter family of ADHM data $A(t, z) : W \to V$, where $t \in I$ is a parameter with values in an open subset $I \subseteq \mathbb{R}$ that contains the origin. We assume that $A(t, z)$ is a smooth, continuous function of $t$. Such a one-parameter family of data defines a one-parameter family of holomorphic vector bundles $E(t) \to \mathbb{C}P^3$ and hence a one-parameter family of instanton solutions of the self-dual Yang-Mills equations on $S^4$. We now wish to investigate how the elements of the explicit constructions of the previous sections depend on $A(t, z)$.

The image $A(t, z)(W)$ is now spanned by the vectors $\{v_i(t, z)\}$. On a fixed real line the vector space $U_z^\perp \oplus U_z$ is spanned by $\{e_A(t, z)\}$ for each $z \in U_0$, which we assume are normalised such that (2.2) is satisfied for each $t \in I$. Constructing the vectors $\{f_A(t, z)\}$, we then define the patching matrix $G(t, z)$ and the functions $\lambda_{Ai}(t, z)$ as in equation (2.3).

A short calculation (see [Gr1]) implies that the $t$-derivative of the patching matrix obeys the relation

$$\frac{\partial}{\partial t} G(t, z) = d(t, z) G(t, z) + G(t, z) d(t, z)^*, \quad (4.1)$$

where $d(t, z)$ is a matrix with components

$$d_A^B(t, z) = \sum_{C=1}^2 \epsilon^{BC} \left( \frac{\partial}{\partial t} f_A(t, z), f_C(t, z) - \sum_{i=1}^k \lambda_{Ci}(t, z) v_i(t, z) \right),$$

where $\epsilon^{BC} = -\epsilon^{CB}$ and $\epsilon^{12} = 1$ are the components of the volume form on the bundle and we have defined $d^*(t, z) = d(t, \sigma(z))^\dagger$.

At the formal level, it follows from equation (4.1) that

$$G(t, z) = \alpha(t, z) G(0, z) (\alpha(t, z))^*, \quad (4.2)$$

where $\alpha(t, z)$ satisfies the first order ordinary differential equation

$$\dot{\alpha}(t, z) = d(t, z) \alpha(t, z), \quad \alpha(0 : z) = \text{Id}.$$
The form of \( G(t, z) \) given in equation (4.2) is then precisely of the form given in [Cr]. More precisely, the differential equation (4.1) satisfied by \( G(t, z) \) is a flow generated by symmetries of the form (3.9). Therefore given any smooth path in \( \gamma : I \to \mathcal{M}_k \), where \( I \) is an open sub-interval of \( \mathbb{R} \), then the tangent vector to \( \gamma \) at any point is the fundamental vector field corresponding to a symmetry of the form given in [Ch, D].

At the global level the transformations generated by (3.9) are generally only locally defined and therefore form a pseudo-group (or groupoid) rather than a Lie group. Since the moduli spaces \( \mathcal{M}_k \) are connected [Do], the above calculations imply that this pseudo-group acts transitively on each \( \mathcal{M}_k \). Since the moduli spaces are finite-dimensional, the tangent space at each point will be generated by a finite-dimensional sub-algebra of the algebra of symmetries (3.9) at each point. An implicit description of this sub-algebra may be derived from the ADHM data via equation (4.1). Whether this finite-dimensional sub-algebra gives rise to a transitive action of a finite-dimensional group on each moduli space \( \mathcal{M}_k \), and whether the transitive action of a pseudo-group on the \( \mathcal{M}_k \) leads to any information concerning the global structure of the moduli spaces is currently under investigation.

5. Open problems

Outside of clarifying some of the technical issues surrounding our results, we mention some natural ways in which the current work could be extended. A reformulation of the ADHM construction due to Atiyah [At2] and Donaldson [Do] shows that there is a 1–1 correspondence between (framed) instantons solutions of instanton number \( k \) and based holomorphic maps \( f : \mathbb{C}P^1 \to \Omega G \) of degree \( k \). Given that such maps are in many ways analogous to harmonic maps from \( \mathbb{C}P^1 \to \mathbb{C}P^N \), it seems likely that the dressing action on instanton moduli spaces may be most clearly understood within this formalism.

Although the generalisation from \( G = SU(2) \) to arbitrary \( G \) is essentially trivial, a natural question is whether the results we have found hold for other four-manifolds. In the case where the base manifold is \( \mathbb{C}P^2 \) with \( G = SU(2) \) we can analyse the one-instanton moduli space, \( \mathcal{M}_1(\mathbb{C}P^2) \). This moduli space is isomorphic to a cone on \( \mathbb{C}P^2 \), with the vertex of the cone corresponding to the unique, homogeneous, reducible \( SU(2) \) connection on the bundle \( L \oplus L^{-1} \). Preliminary investigations [Gr2] indicate that in this case the corresponding symmetry group action has two, distinct orbits, the first consisting of the irreducible connections and the second
consisting of the reducible connection which is a fixed point of the group action. Given that on a general four-manifold the space of reducible connections is central to Donaldson’s theory of four-manifolds [DK], this might suggest an interesting correspondence between topological field theory and group actions in integrable systems.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ABERDEEN, ABERDEEN, AB24 3UE, SCOTLAND

ADDRESS FROM 1 MARCH 2005: FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, 1090 WIEN, AUSTRIA

E-MAIL ADDRESS: james.grant@univie.ac.at
URL: http://www.jdegrant.co.uk
Paper 5

*Multidimensional integrable systems and deformations of Lie algebra homomorphisms,*

with M. Dunajski and I.A.B. Strachan,

I. INTRODUCTION

A dispersionless limit of partial differential equations (PDEs) is taken by rescaling the independent variables $X^a \to X^a/\varepsilon$ and taking the limit $\varepsilon \to 0$. This is a delicate procedure, as the limit of the solutions of a given PDE does not usually correspond to solutions of the limiting dispersionless equation. Moreover, inequivalent PDEs may have the same dispersionless limit, so the problem

\begin{itemize}
  \item Recover the original PDE from its dispersionless limit
\end{itemize}

is, of course, ill posed. Some progress can nevertheless be made if the dispersionless equation is integrable, and one insists that its dispersive analog is also integrable. In the next section, we shall explain how dispersionless limits of solitonic PDEs are equivalent to the Wentzel-Kramers-Brillouin (WKB) quasiclassical approximation of the associated linear problems. This suggests that the reconstruction of the dispersive solitonic system should involve a quantization of some kind.

Such a quantization procedure has been developed in the seminal work of Kupershmidt.\textsuperscript{15} This procedure is based on the Moyal product and works well if the Lie algebra underlying the dispersionless linear problem is the algebra $\text{sdiff}(\Sigma^2)$ of divergence-free vector fields on a two-surface $\Sigma^2$. This is the case for the dispersionless Kadomtsev-Petviashvili (dKP) and $\text{SU}(\infty)$ Toda equations in $2+1$ dimensions. Similar progress can also be made in higher dimensions and, indeed, one of us has constructed integrable deformations of Plebanski’s first heavenly equation\textsuperscript{29} by replacing the underlying Poisson bracket with the Moyal bracket.

The idea of deforming integrable systems while retaining the integrability of the resulting equation has now been studied from a number of different points of view:

\begin{itemize}
  \item Takasaki studied properties of the deformed heavenly equations and described how solutions may be described in terms of a Riemann-Hilbert splitting in a Moyal algebra valued loop
\end{itemize}
Extensions of this led to Moyal-KP hierarchies\(^{34}\) and deformations of the self-dual Yang-Mills equations.\(^{32}\) The deformed Riemann-Hilbert procedure was recently fully developed by Formanski and Przanowski.\(^{9,10}\)

- Nekrasov and Schwarz introduced instantons on noncommutative space-time.\(^{22}\) This led to the development of noncommutative soliton equations. These may be viewed as a deformation of the standard, commutative, soliton equations. Many of these may be studied as reductions of the noncommutative self-dual Yang-Mills equations.\(^{12,16}\)
- Associated with any Frobenius manifold is a hierarchy of integrable equations of hydrodynamic type. Integrable deformations of these equations arise naturally when one studies the genus expansion in the corresponding topological quantum field theories.\(^{3}\)

In the present paper, deformations of multidimensional integrable systems are based on the algebra\(^{1}\) \(\text{Diff}(\Sigma)\), the Lie algebra of vector fields on \(\Sigma\), where \(\Sigma \equiv S^1\) or \(\mathbb{R}\). It turns out, however, that this algebra admits no nontrivial deformations.\(^{17}\) However, an alternative method of deforming these integrable systems may be developed. This method is based on the approach of Ovsienko and Rogers\(^{24}\) where a homomorphism from \(\text{Diff}(\Sigma)\) to the Poisson algebra on \(\mathcal{T}^*\Sigma\) can be used to construct nontrivial deformations. We shall use this idea to construct integrable deformations of various equations associated with the algebra \(\text{Diff}(\Sigma)\).

It should be pointed out that the original, undeformed, equations have a natural interpretation in terms of the twistor theory, via the nonlinear graviton construction and its variants. It would seem desirable to develop a “deformed” version of the twistor theory that would encode solutions in terms of the twistor theory, via the nonlinear graviton construction and its variants. It would be differential operators on \(\mathbb{R}\) with coefficients depending on local coordinates \(X^a = (X,Y,T)\) on \(\mathbb{R}^3\). The overdetermined linear system,

\[
\Psi_Y = A \left( \frac{\partial}{\partial X} \right) \Psi, \quad \Psi_T = B \left( \frac{\partial}{\partial X} \right) \Psi,
\]

admits a solution \(\Psi(X,Y,T)\) on a neighborhood of initial point \((X,Y_0,T_0)\) for arbitrary initial data \(\Psi(X,Y_0,T_0) = f(X)\) if and only if the integrability conditions \(\Psi_{YT} = \Psi_{TY}\) or

\[
A_T - B_Y + [A,B] = 0 \quad (2.1)
\]

are satisfied. Nonlinear system (2.1) for \(a_1,\ldots,a_n,b_1,\ldots,b_m\) can be solved by the inverse scattering transform (IST). Integrable systems which admit a Lax representation [Eq. (2.1)] will be referred to as solitonic or dispersive.

The dispersionless limit\(^{37}\) is obtained by substituting

\(^{1}\)In the remainder of this paper, the superscript, denoting the dimension of the manifold, will be dropped.
\[
\frac{\partial}{\partial X^\nu} = \frac{\partial}{\partial x^\nu}, \quad \Psi(X^\nu) = \exp(\phi(x^\nu/\epsilon))
\]
and taking the limit \( \epsilon \to 0 \). In the limit, the commutators of differential operators are replaced by the Poisson brackets of their symbols according to the relation
\[
\frac{\partial^k}{\partial \lambda^k} \Psi \to (\psi_k)^k \Psi, \quad [A,B] \to \frac{\partial A}{\partial \lambda} \frac{\partial B}{\partial \lambda} - \frac{\partial B}{\partial \lambda} \frac{\partial A}{\partial \lambda} = \{A,B\}, \quad \lambda = \psi_x,
\]
where \( A,B \) are polynomials in \( \lambda \), with coefficients depending on \( x^\nu = (x,y,t) \). The dispersionless limit of system (2.1) is
\[
A_x - B_y + \{A,B\} = 0.
\]
Nonlinear differential equations of the form of Eq. (2.2) are called dispersionless integrable systems. One motivation for studying dispersionless integrable systems is their role in constructing partition functions in topological field theories.\(^{14}\)

A natural approach to solving Eq. (2.2) would be an attempt to take a quasiclassical limit of the IST which linearizes Eq. (2.1). This does not yield the expected result, as the quasiclassical limit of the Lax representation for Eq. (2.1) is the system of Hamilton-Jacobi equations
\[
\psi_t = A(\psi,x^e), \quad \psi_x = B(\psi,x^e),
\]
with “two times” \( t \) and \( y \), and the initial value problem for Eq. (2.2) would require a reconstruction of a potential from the asymptotic form of the Hamiltonians. This classical inverse scattering problem is so far open.

There are alternative methods of solving Eq. (2.2).\(^{13,8,35,6}\) In particular, the minitwistor approach of Ref. 6 works as follows. System (2.2) is equivalent to the integrability \( [L,M]=0 \) of a two-dimensional distribution of vector fields
\[
L = \partial_t - B_x \partial_x + B_y \partial_y, \quad M = \partial_y - A_x \partial_x + A_y \partial_y,
\]
on \( \mathbb{R}^3 \times \mathbb{R}^1 \). Assume that \( L,M \) are real analytic and complexify \( \mathbb{R}^3 \) to \( \mathbb{C}^3 \). The minitwistor space \( Z \) is the two complex dimensional quotient manifold
\[
Z = \mathbb{C}^3 \times \mathbb{CP}^1(L,M), \quad \lambda \in \mathbb{CP}^1, \quad x^e \in \mathbb{C}^3.
\]
That is to say that the local coordinates on \( Z \) lift to functions on \( \mathbb{C}^3 \times \mathbb{CP}^1 \) constant along \( L,M \).

The minitwistor space is equipped with a three parameter family of certain rational curves. All solutions to Eq. (2.2) can in principle be reconstructed from a complex structure of the minitwistor space.

In fact, the twistor approach outlined above is capable of solving a wider class of equations. We shall therefore generalize the notion of the dispersionless integrable systems by allowing distributions of vector fields more general than Eq. (2.3). The derivatives \( A_\lambda, A_x, B_x, B_y \) of the symbols \( (A,B) \) of operators can be replaced by independent polynomials \( A_1,A_2,B_1,B_2 \) in \( \lambda \) with coefficients depending on \( (x,y,t) \),
\[
L = \partial_t - B_1 \partial_x + B_2 \partial_y, \quad M = \partial_y - A_1 \partial_x + A_2 \partial_y.
\]
If \( A_1,B_1 \) are linear in \( \lambda \) and \( A_2,B_2 \) are at most cubic in \( \lambda \), then the rational curves in \( Z \) have normal bundle \( O(2) \) (the line bundle over \( \mathbb{CP}^1 \) with transition functions \( \lambda^{-2} \) from the set \( \lambda \neq \infty \) to \( \lambda \neq 0 \), i.e., Chern class 2) and the three-dimensional moduli space of such curves in \( Z \) can be parametrized by \( (x,y,t) \). Allowing polynomials of higher degrees would lead to hierarchies of dispersionless equations. We take the integrability of this generalized distribution [Eq. (2.4)] as our definition of the dispersionless integrable system. The definition is intrinsic in a sense that it does not refer to an underlying dispersive equation.
III. Diff(S¹) DISPERSIONLESS INTEGRABLE SYSTEMS

In this section, two integrable systems associated with the gauge group Diff(S¹) will be given. The first has been extensively studied in Refs. 26, 8, 4, 19, 7, 18, and 25 so only a new gauge-theoretic description will be given—the reader is referred to these earlier papers for more details. The second system, which arises from a Nahm-type system, is new and this system is discussed in more detail.

A. A (2+1) dimensional dispersionless integrable system

An example of a dispersionless system which is integrable in the sense of the outlined twistor correspondence is given by the following distribution:

$$L = \partial_t - w \partial_k - \lambda \partial_y, \quad M = \partial_y + u \partial_x - \lambda \partial_z.$$  \hspace{1cm} (3.1)

A linear combination of this distribution leads to a special case of Eq. (2.4) with $A_2 = B_2 = 0$. Its integrability leads to the pair of quasilinear PDEs,

$$u_t + w_z + uw_x - wu_x = 0, \quad u_x + w_z = 0,$$

for two real functions $u = u(x,y,t)$, $w = w(x,y,t)$. This system of equations has recently been studied in Refs. 26, 8, 4, 19, 7, 18, and 25 in connection with the Einstein-Weyl geometry, hydrodynamic chains, and symmetry reductions of anti-self-dual Yang-Mills equations. From the twistor point of view, Eq. (3.2) is invariantly characterized by requiring that the minitwistor space $Z$ fibers holomorphically over $\mathbb{C}P^1$. The second equation can be used to introduce a potential $H$ such that $u = H_x$, $w = -H_y$. The first equation then gives

$$H_{xt} - H_{yy} + H_x H_{yx} - H_y H_{xy} = 0.$$  \hspace{1cm} (3.3)

System (3.2) arises as a symmetry reduction of the anti-self-dual Yang-Mills equations in signature (2,2) with the infinite-dimensional gauge group Diff(S¹) and two commuting translational symmetries exactly one of which is null. This combined with the embedding of SU(1,1) $\subset$ Diff(S¹) gives rise to explicit solutions to Eq. (3.3) in terms of solutions to the nonlinear Schrödinger equation and the Korteweg–de Vries equation.

The Lie algebra of the group of diffeomorphisms Diff(S¹), where $\Sigma = S¹$ or $\mathbb{R}$, is isomorphic to the infinite-dimensional Lie algebra of functions on $\Sigma$ with the Wronskian,

$$\langle f, g \rangle := fg_x - f_x g,$$

as the Lie bracket, where $f, g \in C^\infty(\Sigma)$ and $x$ is a local coordinate on $\Sigma$. An alternative gauge-theoretic interpretation can be given to Eq. (3.2). Observe that the first equation in Eq. (3.2) can be interpreted as the flatness of a gauge connection on $\mathbb{R}^2$, where the gauge group is Diff(S¹). Indeed, choose local coordinates $(t,y)$ on $\mathbb{R}^2$ and consider $\mathcal{A} \in \Lambda^1(\mathbb{R}^2) \otimes C^\infty(\Sigma)$ of the general form

$$\mathcal{A} = -w dt + u dy,$$

where $u, w: \mathbb{R}^2 \rightarrow C^\infty(\Sigma)$ depend on $(x,y,t)$.

The flatness of this connection yields

$$d \mathcal{A} + \mathcal{A} \wedge \mathcal{A} = (u_t + w_x + \langle u, w \rangle) dt \wedge dy = 0,$$

as claimed. Therefore, the connection is a pure gauge and can be written as $\mathcal{A} = g^{-1} dg$, where $g = g(x,y,t) \in \text{Map}(\mathbb{R}^2, \text{Diff}(\Sigma))$ and

$$w = -g^{-1} g_t, \quad u = g^{-1} g_y.$$

The second equation in Eq. (3.2) yields the following system:
\[(g^{-1}g_y)_y - (g^{-1}g)_x = 0, \quad (3.5)\]

where \(g = \exp(\Delta)\) is a finite diffeomorphism of \(\Sigma\) and terms like \(g^{-1}g\) should be understood as

\[g^{-1}g = \Delta - \langle \Delta , \Delta \rangle + \frac{1}{2} \langle \Delta , \langle \Delta , \Delta \rangle \rangle + \cdots .\]

**B. A (3+1) dimensional dispersionless integrable system**

In this section, we shall present another example of an integrable system associated with the Lie algebra of \(\text{Diff}(S^1)\). We shall first write it as a Nahm system,

\[\dot{e}_i = \frac{1}{2} e_{ijk} [e_j, e_k], \quad i = 1, 2, 3, \quad (3.6)\]

where \(e_i\) are vector fields on an open set in \(\mathbb{R}^4\) given by

\[e_i = \frac{\partial}{\partial y^i} - N_i(x, y^j) \frac{\partial}{\partial x^i},\]

and \((x, y^i)\) are local coordinates. Rewrite Eq. (3.6) as

\[\partial_i N_j + e_{ijk} \partial_j N_k - \frac{1}{2} e_{ijk} \langle N_j, N_k \rangle = 0. \quad (3.7)\]

We shall now discuss the origin and possible applications of Eq. (3.7).

1. Any solution to Eq. (3.7) defines a hyper-Hermitian conformal structure represented by the metric

\[g = n^2 + \delta_{ij} dy^i dy^j, \quad (3.8)\]

where

\[n = dx + N_i dy^i.\]

The three complex structures \(I_i, i = 1, 2, 3\) satisfying the algebra of quaternions,

\[I_i I_j = - \delta_{ij} 1 + e_{ijk} I_k,\]

are given by

\[I_i(n) = dy^i.\]

These formulas together with the algebraic relations satisfied by \(I_j\) determine the complex structures uniquely, e.g.,

\[I_j(dy^i) = - \delta_{ij}(n) + e_{ijk} dy^k.\]

One way to impose integrability of the complex structures is to use the explicit form of the complex structures on the basis \((dy^1, dy^2, dy^3, n)\) and demand that the space \(\Lambda^{(1,0)}\) is closed under exterior differentiation. We begin by defining a basis of self-dual two forms,

\[\Sigma^i = n \wedge dy^i + \frac{1}{2} e^{ijk} dy^j \wedge dy^k.\]

The integrability of the complex structures is then equivalent to the anti-self-duality of the two-form \(dn\),

\[\Sigma^i \wedge dn = 0. \quad (3.9)\]
This condition is equivalent to Eq. (3.7).

A dual formulation leads to the Lax pair of vector fields, which is a special form of the hyper-Hermitian Lax pair. To see it, set \( e_4 = \partial_4 \) and define complex vector fields

\[
\begin{align*}
  w &= e_1 - ie_2, \\
  z &= e_3 - ie_4.
\end{align*}
\]

System (3.7) is equivalent to the commutativity of the Lax pair,

\[
[w - \lambda z, z + \lambda \bar{w}] = 0,
\]

for all values of the parameter \( \lambda \).

(2) System (3.7) arises as a symmetry reduction of the anti-self-dual Yang-Mills equations on \( \mathbb{R}^4 \) with the infinite-dimensional gauge group \( \text{Diff}(S^1) \) and one translational symmetry. In fact, any such symmetry reduction is gauge equivalent to (3.7). To see it, consider the flat metric on \( \mathbb{R}^4 \), which in double null coordinates \( w = y^1 + iy^2, \ z = y^3 + iy^4 \) takes the form

\[
ds^2 = dzd\bar{z} + dwd\bar{w},
\]

and choose the volume element \( dw \wedge dw \wedge dz \wedge d\bar{z} \). Let \( A \in \mathcal{T}^* \mathbb{R}^4 \otimes \mathfrak{g} \) be a connection one-form and let \( F \) be its curvature two form. Here, \( \mathfrak{g} \) is the Lie algebra of some (possibly infinite dimensional) gauge group \( G \). In a local trivialization, \( A = A_\mu dy^\mu \) and

\[
F = (1/2)F_{\mu \nu} dy^\mu \wedge dy^\nu,
\]

where \( F_{\mu \nu} = [D_\mu, D_\nu] \) takes its values in \( \mathfrak{g} \). Here, \( D_\mu = \partial_\mu + A_\mu \) is the covariant derivative. The connection is defined up to gauge transformations \( A \rightarrow b^{-1}Ab - b^{-1}db \), where \( b \in \text{Map}(\mathbb{R}^4, G) \). The anti-self-dual Yang-Mills (ASDYM) equations on \( A_\mu \) are

\[
F_{wz} = 0, \quad F_{w\bar{w}} + F_{z\bar{z}} = 0, \quad F_{w\bar{z}} = 0.
\]

These equations are equivalent to the commutativity of the Lax pair,

\[
L = D_w - \lambda D_z, \quad M = D_z + \lambda D_{\bar{w}},
\]

for every value of the parameter \( \lambda \).

We shall require that the connection possesses a symmetry which in our coordinates is given by \( \partial / \partial y^4 \). Choose a gauge such that the Higgs field \( A_4 \) is a constant in \( \mathfrak{g} \). Now choose \( G = \text{Diff}(S^1) \), so that the components of the one-form \( A \) become vector fields on \( S^1 \). We can choose a local coordinate \( x \) on \( S^1 \) such that \( A_4 = \partial_4 \) and \( A_i = -N_i \partial_4 \), where \( N_i = N_i(x, y^4) \) are smooth functions on \( \mathbb{R}^4 \). The Lax pair [Eq. (3.11)] is identical to [Eq. (3.10)] and the ASDYM equations reduce to the first order PDEs [Eq. (3.7)].

(3) Example: An ansatz \( \mathbf{N}(x, y^4) = f(x) \mathbf{A}(y^4) \), where \( \mathbf{N} = (N_1, N_2, N_3)^T \), reduces Eq. (3.7) to a pair of linear equations

\[
\dot{f} = cf, \quad c\mathbf{A} + \nabla \wedge \mathbf{A} = 0,
\]

where \( c \) is a constant. If \( c = 0 \), then \( \mathbf{N} \) may be absorbed into a redefinition of the coordinate \( x \) in the metric [Eq. (3.8)]. Therefore, we assume that \( c \neq 0 \). We set \( c = 1 \) by rescaling \( y^4 \) and solve for \( f = \exp(x) \), reabsorbing another constant of integration into \( \mathbf{A} \). Now define a new coordinate \( \rho = \exp(-x) \). Rescaling the metric [Eq. (3.8)] yields

\[
\dot{\hat{g}} = \rho dy^2 + \rho^{-1}(d\rho - \mathbf{A} \cdot dy)^2.
\]

This metric is hyper-Hermitian if and only if the vector \( \mathbf{A}(y^4) \) satisfies the Beltrami equation

\[
\mathbf{A} + \nabla \wedge \mathbf{A} = 0.
\]

This is a slight improvement of the result of Ref. 36 where it is shown that Eq. (3.12) is ASD if and only iff Eq. (3.13) holds.
The Beltrami equation implies that $A$ is divergence-free and satisfies $\triangle A + A = 0$, where $\triangle = \nabla^2$ is the scalar Laplacian on $\mathbb{R}^3$ acting on components of $A$. Existence of solutions of Eq. (3.13), at least in the analytic case, can be proved by an application of the Cartan-Kähler theorem [c.f. Example 3.7 in Chap. III of Ref. 1].

(4) System (3.7) can be put in the hydrodynamic form

$$\partial_1 N = M \operatorname{curl} N,$$

where

$$M = \begin{pmatrix} 1 & N_3 & -N_2 \\ -N_3 & 1 & N_1 \\ N_2 & -N_1 & 1 \end{pmatrix}^{-1}.$$

A different analytic continuation of Eq. (3.7) can be obtained at the level of the hyper-Hermitian geometry. This comes down to looking for conformal structures [Eq. (3.8)] of signature $++-$. To achieve this, we regard $y_1, y_2, N_1, N_2$ as imaginary and define

$$Y_1 = iy_1, \quad Y_2 = iy_2, \quad Y_3 = y_3, \quad N_1 = iN_1, \quad N_2 = iN_2, \quad N_3 = N_3.$$

The desired system for $N_j = N_j(Y', x)$ arises from Eq. (3.7).

Clearly, there are many further properties of these dispersionless systems that may be studied. We now turn our attention to the construction of nontrivial integrable deformations of Eqs. (3.2) and (3.7).

IV. DISPERSIVE DEFORMATIONS

Given a dispersionless integrable system, it is natural to ask whether it arises as a limit of some dispersive (or solitonic) system. One would expect the reconstruction of a solitonic system to involve a quantization of some kind because taking a dispersionless limit of Eq. (2.1) was equivalent to a quasiclassical limit of the wave function $\Psi(X)$. This is indeed the case, and the paradigm example is provided by the connection between the Kadomtsev-Petviashvili (KP) equation and its dispersionless limit dKP. One can reconstruct KP from dKP by expressing the latter in the form of Eq. (2.2) and replacing the Poisson brackets by the Moyal bracket. The infinite series involved in a Moyal product truncates in this case because the symbols $A$ and $B$ are polynomials in momentum $\lambda$. The deformation parameter can then be set to 1 and can be removed from the construction. It seems, however, that this beautiful example is rather exceptional and that the reconstruction of dispersive systems (if at all possible) is in general nonunique and can lead to systems which involve a formal power series.

Generalizing the definition of the dispersionless systems to non-Hamiltonian distributions $(L, M)$ such as Eq. (3.1) makes things even worse, as the Poisson bracket is not present, and the connection with known quantization procedures of a classical phase space has been lost. It could be argued that Eq. (3.3) should be regarded as its own deformation as it admits a dual (classical and quantum) description. It is a solitonic system [Eq. (2.1)] with

$$A = H_x \frac{\partial}{\partial X}, \quad B = H_y \frac{\partial}{\partial X},$$

or a dispersionless limit [Eq. (2.2)] with $A = \lambda H_x, B = \lambda H_y$.

One attempt to find a dispersive analog of Eq. (3.2) would be to use the centrally extended Virasoro algebra in place of $\text{diff}(\mathbb{R})$. Recall that such a procedure has been used to produce dispersive systems from dispersionless systems in a different context. Namely, one can view the periodic Monge equation $u_t = uu_x$ as the equation for affinely parametrized geodesics with respect to the right-invariant metric on $\text{Diff}(S^1)$ constructed from the $L^2$ inner product on the Lie algebra. Going to the central extension, one finds that affinely parametrized geodesics on the Virasoro-Bott
group correspond to solutions of the Korteweg de Vries (KdV) equation (for a recent review of such constructions, see Ref. 21). In the current situation, we view a general element of the extended algebra as a pair

\[(f, a) := f(x) \frac{d}{dx} - iac,\]

where \(a \in \mathbb{R}\) does not depend on \(x\) and \(c\) is a constant. Assuming that \(x\) is a periodic variable, the modified commutation relation is

\[\{f \partial_x, g \partial_x\} = \langle f, g \rangle \partial_x + \frac{ic}{48 \pi} \int (f_{xxx}g - f g_{xxx}) dx,\]

and we see that the central term is a function of \((t, y)\) only. Applying this procedure to Eq. (3.3) with

\[H(x, y, t) = \sum_k h_k(y, t)L^k,\]

where \(L^k\) are generators of the centrally extended Virasoro algebra satisfying

\[[L^k, L^m] = (k - m)L^{k+m} + \frac{c}{12}k(k^2 - 1)\delta_{k,-m},\]

would modify only one equation in the infinite chain of PDEs for the functions \(h_k\).

In the remaining part of this paper, we shall present a construction\(^2\) which leads to nontrivial dispersive analogs of Eq. (3.2), or its equivalent form Eq. (3.3), and Eq. (3.7).

### A. Deforming Lie algebra homomorphisms

To find a nontrivial deformation, one would wish to deform the Lie algebra of vector fields on \(\Sigma\), but this algebra is known not to admit any nontrivial deformations.\(^17\)

We shall choose a different route\(^24\) and deform the standard homomorphism between \(\text{diff}(\Sigma)\) and the Poisson algebra on \(T^*\Sigma\), the point being that the homomorphisms between Lie algebras can admit nontrivial deformations even if one of the algebras is rigid. A deformation of Eq. (3.3) is achieved in two steps, each introducing a parameter. In the first step, we shall deform the embedding of \(\text{diff}(\Sigma)\) into the Lie algebra of volume-preserving vector fields on \(T^*\Sigma\). This introduces the first parameter \(\mu\). The second step is a deformation quantization of the first one. The Poisson algebra on \(T^*\Sigma\) is the quasiclassical limit of the Lie algebra of pseudodifferential operators on \(\Sigma\), so (working at the level of symbols) one quantizes the deformed homomorphism by using the deformed associative product of symbols of pseudodifferential operators rather than a pointwise commutative product of functions. This introduces the second parameter \(\epsilon\). In what follows, we shall be interested in the polynomial deformations rather than the formal ones.

\(^2\)An alternative approach which we have not explored would be to consider a quantum deformation of the Virasoro algebra as in Ref. 28 and its free boson realization,

\[\{T_m, T_n\} = -\sum_{r=1}^{\infty} f(r) (1 - r^{-1}) \frac{1}{1 - s} (s^n - s^m) \delta_{m,n}.\]

where \(s = qt^{-1}\) and coefficients \(f(r)\) are given by

\[f(c) = \sum_{n=0}^{\infty} f^n = \exp \left( \sum_{n=1}^{\infty} \frac{1 - q^n(1 - r^n)}{1 + s^n} c^n \right).\]

The ordinary Virasoro algebra [Eq. (4.2)] is recovered as \(a \rightarrow 1\). Applying Eq. (4.3) to Eq. (4.1) would lead to a \(q\)-deformed analog of Eq. (3.3)
The standard embedding \( \pi: \text{vect}(\Sigma) \to C^\infty(T^*\Sigma) \) is given by contracting a vector field \( X_f = f(x)\partial_x \) with a canonical one-form \( \Theta \) on \( T^*\Sigma \). In our case, \( T^*\Sigma = \mathbb{R} \times \Sigma \) and the Lie algebras \( C^\infty(S^1) \) (with the Wronskian bracket) and \( \text{vect}(S^1) \) (with the Lie bracket) are isomorphic so we can regard \( \pi \) as defined on \( C^\infty(\Sigma) \).

If \( \lambda \) is a local coordinate on the fibers of \( T^*\Sigma \) and \( \Theta = \lambda dx \), the map \( \pi \) is explicitly given by

\[
(\pi(f))(\lambda, x) := \lambda f(x).
\]

It is a Lie algebra homomorphism as

\[
\{\pi(f), \pi(g)\} = \pi(fg).
\]

Given \( \mu \in \mathbb{R} \), define\(^24\)

\[
(\pi_\mu(f))(\lambda, x + \mu/\lambda) = \lambda f(x + \mu/\lambda) = \lambda \left( f(x) + \frac{\mu f'(x)}{\lambda} + \frac{1}{2!} f''(x) \left( \frac{\mu}{\lambda} \right)^2 + \cdots \right).
\]

Note that \( \{\pi_\mu(f), \pi_\mu(g)\} = \pi_\mu(fg) \), so that \( \pi_\mu \) is also a Lie algebra homomorphism between \( \text{diff}(\Sigma) \) and \( \text{diff}(T^*\Sigma) \).

The next step is motivated by the canonical quantization \( \lambda \to \partial/\partial \lambda \). For any functions \( F, G \in C^\infty(T^*\Sigma) \) which are also allowed to depend on a parameter \( \mu \), define the Kupershmidt-Manin product

\[
F \star G = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \frac{\partial F}{\partial \lambda^k} \frac{\partial G}{\partial \lambda^k}
\]

(this is equivalent, under an \( \epsilon \)-valued change of variable to the Moyal product) and set

\[
\{F, G\}_\epsilon = \frac{1}{\epsilon} (F \star G - G \star F).
\]

The Poisson bracket is recovered in the limiting procedure

\[
\lim_{\epsilon \to 0} \{F, G\}_\epsilon = \{F, G\},
\]

but the deformed bracket is equal to the Poisson bracket for all \( \epsilon \) if \( F, G \) are linear in \( \lambda \). This is why the first deformation parameter \( \mu \) is needed.

We are now ready to propose the dispersive analog of the dispersionless equations Eqs. (3.3) and (3.7).

1. Let \( \hat{H}(\lambda, x, y, t; \mu, \epsilon) = \pi_\mu(H) \) take values in an algebra of formal power series in \( \epsilon \) with an associative product defined by Eq. (4.4). The deformed analog of Eq. (3.3) is

\[
\hat{H}_{xt} - \hat{H}_{yy} - \{\hat{H}, \hat{H}\}_\epsilon = 0.
\]

2. Let \( \hat{N}(\lambda, x^i; \mu, \epsilon) = \pi_\mu(N) \) take values in an algebra of formal power series in \( \epsilon \) with an associative product defined by Eq. (4.4). The deformed analog of Eq. (3.7) is

\[
\partial_{N_i} \hat{N}_j + \partial_{x^i} \partial_{x^j} \hat{N}_k - \frac{1}{2} \partial_{x^i} \partial_{x^j} \hat{N}_k = 0.
\]

Given a solution \( \hat{H} \) of Eq. (4.6) such that \( (\mu \partial_{\lambda} - \lambda \partial_{\lambda})(\lambda^{-1} \hat{H}) = 0 \) and \( \lambda^{-1} \hat{H} \) is smooth in \( (\mu/\lambda) \), we can construct \( H(x, y, t) \) satisfying Eq. (3.3) by taking any of the two limits \( \mu \to 0, \epsilon \to 0 \), and similar remarks hold for Eq. (4.7). Conversely, formal power series (in the deformation parameters) solution may be constructed from a solution to the original, undeformed, equation in an
analogue manner to the way developed in Ref. 29 The extent to which such formal series converge in a suitable space of functions is as yet, however, unclear.

These deformed equations formally retain their integrability; the various manipulations hold at the level of the Lax pair as well as at the level of the equations themselves. However, as remarked in the Introduction, a direct twistor theory correspondence for these equations is lacking, though one should be able to adopt the methods developed by Takasaki13 and Formanski-Przanowski7,10 to study the geometry of the corresponding Riemann-Hilbert problem (in some suitable Moyal algebra valued loop group).

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Paper 6

The ADHM construction and non-local symmetries of the self-dual Yang-Mills equations.

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The ADHM Construction and Non-local Symmetries of the Self-dual Yang–Mills Equations

James D. E. Grant

Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria. E-mail: james.grant@univie.ac.at

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To Nicola Ramsay

Abstract: We consider the action on instanton moduli spaces of the non-local symmetries of the self-dual Yang–Mills equations on $\mathbb{R}^4$ discovered by Chau and coauthors. Beginning with the ADHM construction, we show that a sub-algebra of the symmetry algebra generates the tangent space to the instanton moduli space at each point. We explicitly find the subgroup of the symmetry group that preserves the one-instanton moduli space. This action simply corresponds to a scaling of the moduli space.

1. Introduction

The self-dual Yang–Mills equations have been investigated from two rather distinct points of view in the last few decades. The first direction is in the study of the topology of four-manifolds, and the work of Donaldson (see, e.g., [13,14]). In this approach, a fundamental role is played by the analysis of moduli spaces of solutions of the self-dual Yang–Mills equations with $L^2$ curvature (so-called “instanton solutions”) on given four-manifolds. The analysis of such moduli-spaces then yields powerful information concerning differentiable structures on the underlying four-manifold. The second, seemingly unrelated, development is in the theory of integrable systems. In particular, it has been shown that many known integrable systems of differential equations may be derived as symmetry reductions of the self-dual Yang–Mills equations (see, e.g., [21]).

The purpose of the current paper, and its companion [15] which studies reducible connections, is to investigate whether properties of the self-dual Yang–Mills equations that follow from its integrable nature may be used to yield global information about instanton moduli spaces. In particular, it is known that the self-dual Yang–Mills equations on $\mathbb{R}^4$ admit an infinite-dimensional algebra of non-local symmetries [6–8]. In this paper, we investigate the action of these symmetries on the instanton-moduli spaces on $\mathbb{R}^4$ and, in particular, investigate, on the one-instanton moduli space, the sub-algebra of symmetries that preserve the $L^2$ condition on the curvature of the connection. Thinking of
such symmetries as generating flows on the moduli space of all self-dual connections, \( \mathcal{M} \), and of the \( k \)-instanton moduli space, \( \mathcal{M}_k \), as a subspace of \( \mathcal{M} \), then it is known that the non-local symmetries in general do not lie tangent to the subspaces \( \mathcal{M}_k \) and, therefore, do not preserve the \( L^2 \) nature of the curvature (see, e.g., [9, Chap. V], but also Remark 4.4 below). Our results are rather double-edged. In Theorem 4.1, we show that the full tangent space to the moduli spaces \( \mathcal{M}_k \) is generated by the fundamental vector fields of the symmetry algebra acting on the moduli space of self-dual connections \( \mathcal{M} \). When we attempt to “exponentiate” these tangent vectors into a group action on \( \mathcal{M}_k \), however, our conclusion is that the family of transformations that preserves the \( L^2 \) nature of the connection is rather small. In particular, the symmetries have orbits of rather high codimension in the moduli spaces. More specifically, in Theorem 5.1, we deduce that the only symmetries of the self-dual Yang–Mills equations that act on five-dimensional one-instanton moduli space \( \mathcal{M}_1 \) correspond to a scaling of the instanton solutions. Such a collapse to orbits of large codimension is not unfamiliar from the theory of harmonic maps into Lie groups [1,17,20,28], where one has similar non-local symmetry algebras [10].

The paper is organised as follows. In Sect. 2, we summarise the relevant background material that we require from both the integrable systems approach to the self-dual Yang–Mills equations and the ADHM approach to the instanton problem. Since our considerations are aimed at making a connection between the local aspects of the self-dual Yang–Mills equations and the global aspects, and the literature in these fields generally have completely different notation, it was deemed necessary to give an integrated, relatively detailed description of both approaches in a consistent notation. This accounts for the length of this section\(^\dagger\). In Sect. 3, we show how the ADHM construction may be used to yield explicit patching matrices for holomorphic bundles over subsets of \( \mathbb{C}P^3 \), to which we may apply the results of [9] concerning the action of the symmetry algebra of the self-dual Yang–Mills equations. In Sect. 4, we show that one-parameter families of ADHM data yield transformations that fall into the category of transformations considered in [9], with the important proviso that these transformations are significantly restricted by the assumption that they are generated by flows on the full moduli space of self-dual connections. In Sect. 5, we show that our constructions can be carried out explicitly on the one-instanton moduli space, and that the only symmetries (consistent with a particular technical assumption) that have a well-defined action on the one-instanton moduli space are scalings. Finally, in an Appendix, we give a direct derivation of the action of the non-local symmetries of the self-dual Yang–Mills equations on the twistorial patching matrix from the action on the self-dual connection.

Finally, note that the symmetries that we investigate can also be constructed, by the same methods, on hyper-complex manifolds. Since we wish to consider symmetries that generalise to manifolds other than \( \mathbb{R}^4 \), and there exist hyper-complex manifolds with no continuous (conformal) isometries, we will not consider symmetries (such as those discussed in [26]) that follow from the existence of a non-trivial conformal group on our manifold.

\(^\dagger\) For more information on the local aspects of the self-dual Yang–Mills equations that are relevant to us, see [9, Chaps. II & III]. For more information concerning the ADHM formalism see either [4] or [2, Chaps. II-IV].
2. Preliminaries

2.1. Quaternions and twistor spaces. We will deal exclusively with the self-dual Yang–Mills equations on $\mathbb{R}^4$ and $S^4$, and make constant use of isomorphisms $\mathbb{R}^4 \cong \mathbb{C}^2 \cong \mathbb{H}$, which we first fix. As such, let $x := (x^1, x^2, x^3, x^4) \in \mathbb{R}^4$. We may then view

$$u := (x^1 + i x^2), \quad v := (x^3 - i x^4)$$

as coordinates on $\mathbb{C}^2$ and defining an isomorphism $\mathbb{R}^4 \rightarrow \mathbb{C}^2; x \mapsto (u, v)$. In terms of these coordinates, the flat metric on $\mathbb{R}^4$ takes the form

$$g = \frac{1}{2} (du \otimes d\bar{u} + d\bar{u} \otimes du + dv \otimes d\bar{v} + d\bar{v} \otimes dv),$$

and the corresponding volume form is

$$\epsilon = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = \frac{1}{4} du \wedge d\bar{u} \wedge dv \wedge d\bar{v}.$$

Let $P \rightarrow \mathbb{R}^4$ be a principal $SU_2$ bundle over $\mathbb{R}^4$, and $E \rightarrow \mathbb{R}^4$ the rank-two complex vector bundle associated to $P$ via the fundamental representation of $SU_2$. (We will switch between the principal bundle and vector bundle pictures without comment.) An $SU_2$ connection on $E$, $A \in \Omega^1(\mathbb{R}^4, su_2)$ is a solution of the self-dual Yang–Mills equations if its curvature satisfies

$$\star F = F.$$

In terms of the complex coordinates introduced above, this equation is equivalent to

$$F_{uv} = 0, \quad F_{u\bar{u}} + F_{v\bar{v}} = 0, \quad F_{\bar{u}v} = 0. \quad (2.1)$$

We introduce complex vector fields $X(z), Y(z) \in C^\infty(\mathbb{R}^4, TM \otimes \mathbb{C})$ depending on a parameter $z \in \mathbb{C} \cup \{\infty\} \equiv \mathbb{C}P^1$:

$$X(z) := \partial_{z\bar{u}} - z \partial_v, \quad Y(z) := \partial_{\bar{z}} + z \partial_u. \quad (2.2)$$

Then Eqs. (2.1) are equivalent to the requirement that

$$F(X(z), Y(z)) = 0, \quad \forall z \in \mathbb{C}P^1. \quad (2.3)$$

Since $\mathbb{R}^4 \subset \mathbb{R}^4 \cup \{\infty\} \cong S^4 \cong \mathbb{H}P^1$, a point $x \in \mathbb{R}^4 \cong \mathbb{C}^2$ determines a quaternionic line in $\mathbb{H}^2$. In particular, we define $x := u + jv \in \mathbb{H}$. In terms of homogeneous coordinates $(p, q) \in \mathbb{H}^2$ on $\mathbb{H}P^1$, the point $x$ determines the quaternionic line

$$l_x := \{(q, p) \in \mathbb{H}^2 \mid q = xp\}$$

in $\mathbb{H}^2$. Now, let $p = z_1 + jz_2$, $q = z_3 + jz_4$ with $z := (z_1, \ldots, z_4) \in \mathbb{C}^4$, and view $z$ as homogeneous coordinates on $\mathbb{C}P^3$. Right-multiplication by $j$ on $\mathbb{H}^2$ defines an anti-linear anti-involution

$$\sigma: \mathbb{C}^4 \rightarrow \mathbb{C}^4; \quad (z_1, z_2, z_3, z_4) \mapsto (-\overline{z_2}, \overline{z_1}, -\overline{z_4}, \overline{z_3}), \quad (2.4)$$

2 We restrict to $SU_2$ for simplicity. All of our considerations are equally valid for any classical Lie group.
which descends to define an involution on the projective space:

$$\sigma : \mathbb{C}P^3 \to \mathbb{C}P^3; \quad [z_1, z_2, z_3, z_4] \mapsto [-\overline{z}_2, \overline{z}_1, -\overline{z}_4, \overline{z}_3].$$

The image of the quaternionic line $l_x$ in $\mathbb{C}P^3$ corresponding to $x \in \mathbb{R}^4$ is given by the embedded projective line

$$L_x \equiv L_{(u,v)} = \left\{ [z_1, z_2, z_3, z_4] \in \mathbb{C}P^3 \mid z_3 = z_1u - z_2\overline{v}, z_4 = z_1v + z_2\overline{u} \right\}. \quad (2.5)$$

Similarly, the embedded line corresponding to the point $\infty \in S^4$ is

$$L_{\infty} := \left\{ [0, 0, z_3, z_4] \mid (z_3, z_4) \in \mathbb{C}^2 \setminus \{(0, 0)\} \right\}.$$

The lines $L_p$, for $p \in S^4$, are invariant under the action of $\sigma$, and are referred to as real lines. We will make particular use of the projection

$$\pi : \mathbb{C}P^3 \setminus L_{\infty} \to \mathbb{R}^4; \quad L_x \to x.$$

On a fixed real line, $L_x$, $x \in \mathbb{R}^4$, we introduce the affine coordinate $z = z_2/z_1 \in \mathbb{C}P^1$.

Finally, on the subset $U_1 := \{ [z_1, z_2, z_3, z_4] \in \mathbb{C}P^3 \mid z_1 \neq 0 \}$, we may introduce coordinates $w_1 := z_3/z_1$, $w_2 := z_4/z_1$, $w_3 := z_2/z_1 = z$. By definition, the coordinates $(w_1, w_2, w_3)$, viewed as functions on $U_1$, are holomorphic with respect to the complex structure that $U_1$ inherits as an open subset of $\mathbb{C}P^3$. From Eq. (2.5), the intersection $L_x \cap U_1$ consists of the set of points with $(w_1, w_2, w_3) = (u - z\overline{v}, v + z\overline{u}, z)$. We will therefore often view $(u, v, z)$ as coordinates on the set $U_1 \cong \mathbb{C}^2 \times \mathbb{C}$, with the functions $(u - z\overline{v}, v + z\overline{u}, z)$ being holomorphic with respect to the complex structure on $U_1$. One may then check that, in this coordinate system, a basis for anti-holomorphic vector fields on the set $U_1$ is given by the vector fields $\{X(z), Y(z), \overline{\sigma}\}$, with $X(z), Y(z)$ as in Eq. (2.2). A similar argument may be carried out on the set $U_2 := \{ [z_1, z_2, z_3, z_4] \in \mathbb{C}P^3 \mid z_2 \neq 0 \}$. In practice, the construction on $U_2$ means that we may use the formulae for $X(z), Y(z)$ with $z$ taking values in $\mathbb{C}P^1$. As such, we will often, henceforth, identify the set $\mathbb{C}P^3 \setminus L_{\infty}$ with the set $\mathbb{C}^2 \times \mathbb{C}P^1$. When we speak of a function being, for example, “holomorphic” on $\mathbb{C}^2 \times \mathbb{C} \subset \mathbb{C}^2 \times \mathbb{C}P^1$, we will mean holomorphic on the set $U_1$ with the induced complex structure mentioned above.\footnote{From a $\mathbb{C}P^1$ point of view, we are viewing $U_1 \cup U_2 = \mathbb{C}P^3 \setminus L_{\infty}$ as the total space of the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ of the rational curve $L_0 \subset \mathbb{C}P^3 \setminus L_{\infty}$, where $0$ denotes the origin in $\mathbb{R}^4$. $X$ and $Y$ are then linearly independent sections of this normal bundle. Unfortunately, this picture is not particularly well-suited to the calculations that we wish to perform.}

**Notation.** Given $\epsilon > 0$, we define the following open subsets of $\mathbb{C}P^1$:

$$\mathcal{V}_\epsilon^0 := \left\{ z \in \mathbb{C}P^1 \mid |z| < 1 + \epsilon \right\}, \quad \mathcal{V}_\epsilon^\infty := \left\{ z \in \mathbb{C}P^1 \mid |z| > \frac{1}{1 + \epsilon} \right\},$$

and their intersection

$$\mathcal{V}_\epsilon := \mathcal{V}_\epsilon^0 \cap \mathcal{V}_\epsilon^\infty = \left\{ z \in \mathbb{C}P^1 \mid \frac{1}{1 + \epsilon} < |z| < 1 + \epsilon \right\}.$$
Given any subset \( V \subset \mathbb{CP}^1 \) and a map \( g : V \to SL_2(\mathbb{C}) \), we define a corresponding map \( g^* : \sigma(V) \to SL_2(\mathbb{C}) \) by

\[
g^*(z) := (g(\sigma(z)))^{\dagger}.
\]

(Throughout, \( \dagger : SL_2(\mathbb{C}) \to SL_2(\mathbb{C}) \) will denote complex-conjugate transpose.) Similarly, given any map \( f : U \times V \to SL_2(\mathbb{C}) \), we define a corresponding map \( f^* : U \times \sigma(V) \to SL_2(\mathbb{C}) \) by

\[
f^*(x, z) := (f(x, \sigma(z)))^{\dagger}.
\]

2.2. Holomorphic bundles. An important property of the self-dual Yang–Mills equations is that they are the compatibility condition for the following overdetermined system of equations [9, Theorem 1]:

\[
(\partial_{\overline{\epsilon}} - z \partial_v) \Psi(x, z) = -(A_{\overline{\epsilon}} - z A_v) \Psi(x, z), \quad (2.6a)
\]

\[
(\partial_{\overline{\epsilon}} + z \partial_u) \Psi(x, z) = -(A_{\overline{\epsilon}} + z A_u) \Psi(x, z), \quad (2.6b)
\]

\[
\partial_{\overline{\epsilon}} \Psi(x, z) = 0, \quad (2.6c)
\]

for a map \( \Psi : \mathbb{R}^4 \times V \to SL_2(\mathbb{C}) \), where \( V \) is a subset of \( \mathbb{CP}^1 \). In particular, given \( \epsilon > 0 \), there exists a solution, \( \Psi_0 : \mathbb{R}^4 \times V_\epsilon^0 \to SL_2(\mathbb{C}) \) that is analytic in \( z \) for \( z \in V_\epsilon^0 \). This solution is unique up to right multiplication

\[
\Psi_0(x, z) \to \bar{\Psi}_0(x, z) := \Psi_0(x, z) R(u - z \overline{v}, v + z \overline{u}, z),
\]

where \( R : \mathbb{C}^2 \times V_\epsilon^0 \to SL_2(\mathbb{C}) \) is holomorphic with respect to the complex structure that \( \mathbb{C}^2 \times V_\epsilon^0 \) inherits as a subset of \( U_1 \). Given \( \Psi_0(x, z) \), we define

\[
\Psi_\infty(x, z) := \left( \Psi_0^*(x, z) \right)^{-1}.
\]

It is straightforward to check that \( \Psi_\infty(x, z) \) is also a solution of (2.6) that is analytic in \( z \) on \( \mathbb{R}^4 \times V_\infty \). Defining the fields

\[
\psi_0(x) := \psi_0(x, 0), \quad \psi_\infty(x) := \psi_\infty(x, \infty),
\]

then Eqs. (2.6) imply that we may write the components of the connection in the form

\[
A_u = -(\partial_u \psi_\infty(x)) \psi_\infty(x)^{-1}, \quad A_v = -(\partial_v \psi_\infty(x)) \psi_\infty(x)^{-1},
\]

\[
A_{\overline{\epsilon}} = -(\partial_{\overline{\epsilon}} \psi_0(x)) \psi_0(x)^{-1}, \quad A_{\overline{\epsilon}} = -(\partial_{\overline{\epsilon}} \psi_0(x)) \psi_0(x)^{-1}.
\]

We then define the Yang \( J \)-function \( J : \mathbb{R}^4 \to SL_2(\mathbb{C}) \) by

\[
J := \psi_\infty(x)^{-1} \cdot \psi_0(x).
\]

Noting that, from the definition of \( \psi_\infty \), we have \( \psi_\infty(x) = \left( \psi_0(x) \right)^{\dagger} \), it then follows that \( J^{\dagger} = J \). A short calculation shows that the remaining part of the anti-self-dual part of the curvature may be written in the form

\[
F_{u\overline{\epsilon}} + F_{v\overline{\epsilon}} = -\psi_\infty \left[ \partial_u \left( J_{\overline{\epsilon}} J^{-1} \right) + \partial_v \left( J_{\overline{\epsilon}} J^{-1} \right) \right] \psi_\infty^{-1}
\]

\[
= -\psi_0 \left[ \partial_{\overline{\epsilon}} \left( J^{-1} J_u \right) + \partial_{\overline{\epsilon}} \left( J^{-1} J_v \right) \right] \psi_0^{-1}.
\]
If the connection, \( A \), satisfies the self-dual Yang–Mills equations it therefore follows that the field \( J \) satisfies the Yang–Pohlmeyer equation
\[
\partial_u \left( J_{\pi} J^{-1} \right) + \partial_v \left( J_{\pi} J^{-1} \right) = 0.
\]
Conversely, given \( J : \mathbb{R}^4 \rightarrow \text{SL}_2(\mathbb{C}) \) that satisfies the Yang–Pohlmeyer equation and admits a splitting of the form (2.8) for some \( \psi_0 \) and \( \psi_\infty \) such that \( \psi_\infty = \left( \psi_0^{-1} \right)^\dagger \), then the connection constructed as in Eqs. (2.7) will satisfy the self-dual Yang–Mills equations.

Finally, the quantity
\[
G(x, z) := (\psi_\infty(x, z))^{-1} \cdot \psi_0(x, z),
\]
will be referred to as the \textit{patching matrix}. It defines a holomorphic map \( \mathbb{C}^2 \times \mathcal{V}_\epsilon \subset \mathbb{C} P^3 \setminus \mathbb{R}_\infty \rightarrow \text{SL}_2(\mathbb{C}) \), and hence the transition function of a holomorphic vector bundle over \( \mathbb{C} P^3 \setminus \mathbb{R}_\infty \). The splitting (2.10) implies that this bundle is trivial on restriction to real lines. The above is an explicit version of the Ward correspondence [30], which defines a \( 1 \rightarrow 1 \) correspondence between self-dual Yang–Mills fields and holomorphic bundles over appropriate subsets of \( \mathbb{C} P^3 \) that are trivial on restriction to real lines.\(^4\) Given such a bundle, the transition functions necessarily admit a splitting of the form (2.10), and the connection may then be reconstructed from the resulting fields \( \psi_0, \psi_\infty \) via Eqs. (2.7).

2.3. \textit{Non-local symmetries of the self-dual Yang–Mills equations.} If we consider a one-parameter family of solutions, \( J(t) \), of (2.9), depending in a \( C^1 \) fashion on a parameter \( t \in (-\epsilon, \epsilon) \), then we deduce that \( \dot{J} := \frac{d}{dt} J(t) \) must satisfy the linearisation of (2.9):
\[
\partial_u \left( J_{\pi} \left( J^{-1} \dot{J} \right) J^{-1} \right) + \partial_v \left( J_{\pi} \left( J^{-1} \dot{J} \right) J^{-1} \right) = 0.
\]
Such a \( \dot{J} \) defines a symmetry of the self-dual Yang–Mills equations. It is known that the only local symmetries\(^5\) of the self-dual Yang–Mills equations on flat \( \mathbb{R}^4 \) are gauge transformations and those generated by the action of the conformal group (see, e.g., [25]). On the other hand, there exists a non-trivial family of non-local symmetries of the self-dual Yang–Mills equations [6–8]. To describe these, we define the auxiliary maps
\[
\chi_0 : \mathbb{R}^4 \times \mathcal{V}_\epsilon^0 \rightarrow \text{SL}_2(\mathbb{C}), \quad \chi_\infty : \mathbb{R}^4 \times \mathcal{V}_\epsilon^\infty \rightarrow \text{SL}_2(\mathbb{C}),
\]
\[
\chi_0(x, z) := \psi_0(x)^{-1} \cdot \psi_0(x, z), \quad \chi_\infty(x, z) := \psi_\infty(x)^{-1} \cdot \psi_\infty(x, z),
\]
which are analytic in \( z \) for \( z \in \mathcal{V}_\epsilon^0 \) and \( z \in \mathcal{V}_\epsilon^\infty \), respectively. These maps have the property that \( \chi_0(x, 0) = \chi_\infty(x, \infty) = \text{Id} \).

The following result, based on the results of [6–8], may then be extracted from Sect. III.A of [9]:

**Proposition 2.1.** Let \( T : \mathbb{R}^4 \times S^1 \rightarrow \text{SL}_2(\mathbb{C}) \) be a map that extends continuously to a map \( T : \mathbb{R}^4 \times \mathcal{V}_\epsilon \rightarrow \text{SL}_2(\mathbb{C}) \) (for some \( \epsilon > 0 \)) that is analytic in \( z \) for \( z \in \mathcal{V}_\epsilon \) and satisfies
\[
(\partial_{\pi} + z \partial_u) T(x, z) = (\partial_{\pi} - z \partial_v) T(x, z) = 0
\]
\(^4\) See [9] for more details of the Ward correspondence from this point of view.
\(^5\) i.e. depending only on the connection and its derivatives pointwise
for \((x, z) \in \mathbb{R}^4 \times \mathcal{V}_\epsilon\). Then, given any \(\lambda \in \mathcal{V}_\epsilon\), the quantity
\[
\dot{J} := \chi_\infty(x, \lambda) T(x, \lambda) \chi_\infty(x, \lambda)^{-1} \cdot J + J \cdot \chi_0(x, \sigma(\lambda)) T(x, \lambda)^\dagger \chi_0(x, \sigma(\lambda))^{-1}
\]
\[
= \Psi_\infty(x)^{-1} \left[ \Psi_\infty(x, \lambda) T(x, \lambda) \Psi_\infty(x, \lambda)^{-1} + \Psi_0(x, \sigma(\lambda)) T(x, \lambda)^\dagger \Psi_0(x, \sigma(\lambda))^{-1} \right] \psi_0(x)
\]
(2.12)
is a solution of the linearisation (2.11).

In the case where the function \(T\) is independent of \(x\), it defines an element of the loop group \(\text{ASL}_2(\mathbb{C})\) that admits a holomorphic extension to an open neighbourhood of \(S^1\) in \(\mathbb{C}^*\). The algebra of symmetries generated by such \(T\) is then isomorphic to the Kac-Moody algebra of \(\mathfrak{sl}_2(\mathbb{C})\).

The action of such symmetries on the patching matrix is given by the following result:

**Theorem 2.1.** Let \(T: \mathbb{R}^4 \times S^1 \rightarrow \text{SL}_2(\mathbb{C})\) be as in the previous proposition. The induced flow on the patching matrix of the corresponding bundle over \(\mathbb{C}P^3 \setminus L_\infty\) is given by
\[
\dot{G}(x, z) = -T(x, z) G(x, z) - G(x, z) T^*(x, z) + \rho_\infty(x, z) G(x, z) + G(x, z) \rho_0(x, z),
\]
(2.13)
for \((x, z) \in \mathbb{R}^4 \times \mathcal{V}_\epsilon\). In this equation, \(\rho_0: \mathbb{R}^4 \times \mathcal{V}_\epsilon^0 \rightarrow \mathfrak{sl}_2(\mathbb{C})\) and \(\rho_\infty: \mathbb{R}^4 \times \mathcal{V}_\epsilon^\infty \rightarrow \mathfrak{sl}_2(\mathbb{C})\) are analytic functions of \(z\) on the respective regions and satisfy
\[
(\partial_{\overline{\gamma}} + z \partial_{\gamma}) \rho_0(x, z) = (\partial_{\overline{\gamma}} - z \partial_{\gamma}) \rho_0(x, z) = 0,
\]
\[
(\partial_{\overline{\gamma}} + z \partial_{\gamma}) \rho_\infty(x, z) = (\partial_{\overline{\gamma}} - z \partial_{\gamma}) \rho_\infty(x, z) = 0.
\]
Moreover, the functions \(\rho_0, \rho_\infty\) may be absorbed into holomorphic changes of bases on the regions \(\mathcal{V}_\epsilon^0\) and \(\mathcal{V}_\epsilon^\infty\). When this absorption process is carried out, the transformation (2.13) takes the simpler form
\[
\dot{G}(x, z) = -T(x, z) G(x, z) - G(x, z) T^*(x, z).
\]
(2.14)

**Remark 2.1.** These transformations have been investigated from the viewpoint of twistor-theory and have a natural sheaf-theoretic interpretation [18,19,24,25]. In the literature, it is standard to assume (2.13) (and the group-theoretic version (2.15) below) as the transformation law of the patching matrix, and to work backwards to derive the transformation law of \(J\) and the connection (see, e.g., [18,19,25]). Since a direct proof of (2.13), starting from (2.12), does not appear in the literature, we have included a proof in Appendix A.

**Remark 2.2.** The transformation (2.14) is independent of the parameter \(\lambda\) that appears in Eq. (2.12). As such, the transformation depends only on the function \(T\). In Eq. (2.13), the functions \(\rho_0, \rho_\infty\) will generally depend on the parameter \(\lambda\), but the corresponding dependence of \(G\) on \(\lambda\) may be removed by a holomorphic change of frame. This situation is different from that in, for example, the theory of harmonic maps from a domain \(X \subseteq \mathbb{R}^2\) to a Lie group \(G\). In this case, one has a similar family of non-local symmetries [10] depending on a function \(T(\lambda)\). There, however, the transformation properties of the extended harmonic map depends explicitly on the value of the parameter \(\lambda\) (see, e.g., [28, §3]). Power-series expanding in \(\lambda\) then gives a family of flows acting on the extended solution, and hence on the space of harmonic maps.
The exponentiated form of the transformation law (2.14) is given by the following:

**Theorem 2.2** [9, Chap. IV.C]. Let \( g : \mathbb{R}^4 \times S^1 \rightarrow \text{SL}_2(\mathbb{C}) \) be a smooth map that admits a continuous extension to a holomorphic map \( g : \mathbb{C}^2 \times \mathbb{V}_\epsilon \subset \mathbb{C}P^3 \setminus L_\infty \rightarrow \text{SL}_2(\mathbb{C}) \), for some \( \epsilon > 0 \). Then we define the action of \( g \) on the patching matrix \( G(x, z) \) by the law

\[
G(x, z) \mapsto g(x, z) \cdot G(x, z) \cdot g^*(x, z).
\]

(2.15)

If \( g \) extends holomorphically to \( \mathbb{R}^4 \times \mathbb{V}_0^0 \), then the corresponding transformation is a holomorphic change of basis on the bundle over \( \mathbb{C}P^3 \setminus L_\infty \), which leaves the self-dual connection, \( A \), unchanged.

**Remark 2.3.** The infinitesimal form of (2.15), where \( g(x, z) = \exp(-tT(x, z)) \) is Eq. (2.14).

2.4. **The ADHM construction.** We wish to study the action of the symmetries mentioned above on the moduli spaces of instanton solutions of the self-dual Yang–Mills equations on \( \mathbb{R}^4 \) or, equivalently, \( S^4 \). As such, we are concerned with connections whose curvatures are \( L^2 \), in which case we have

\[
\int_{\mathbb{R}^4} |F|^2 \, d^4x = -8\pi^2 k,
\]

where \( k \in \mathbb{N}_0 \) is the second Chern number, \( c_2(E) \), of the bundle \( E \) (also called the instanton number of the connection). A self-dual connection with \( L^2 \) curvature on \( \mathbb{R}^4 \) necessarily extends to a self-dual connection on \( S^4 \) [29]. We will refer to such connections, defined on either \( \mathbb{R}^4 \) or \( S^4 \) as an instanton. The moduli space of instanton solutions, with instanton number \( k \), modulo gauge transformations is a manifold of dimension \( (8k - 3) \) (away from singularities due to reducible connections), which we denote by \( \mathcal{M}_k \). For later considerations, it will be important to think of \( \mathcal{M}_k \) as being finite-dimensional submanifolds of the (infinite-dimensional) space of all self-dual connections on \( \mathbb{R}^4 \), not necessarily having \( L^2 \) curvature, which we denote by \( \mathcal{M} \). The symmetries of the self-dual Yang–Mills equations that we consider may be viewed as defining flows on the space \( \mathcal{M} \), and our main question is when these flows preserve the sub-manifolds \( \mathcal{M}_k \).

Via the Ward correspondence [5,30], self-dual connections of instanton number \( k \) correspond to holomorphic bundles over \( \mathbb{C}P^3 \) that are trivial on real lines. All such bundles may be constructed directly in terms of quaternionic linear algebra by the ADHM construction [4], which we now briefly recall.

For each \( z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \), we define a linear map

\[
A(z) : W \rightarrow V,
\]

between complex vector spaces \( W, V \) of dimension \( k, 2k + 2 \) respectively, which is of the form

\[
A(z) = \sum_{i=1}^{4} z_i A_i.
\]

The space \( W \) is assumed to admit an anti-linear involution \( \sigma_W : W \rightarrow W \). The space \( V \) is assumed to have a symplectic form \( (\cdot, \cdot) \) and an anti-linear anti-involution \( \sigma_V : V \rightarrow V \) that are compatible in the sense that

\[
(\sigma_V u, \sigma_V v) = (u, v), \quad \forall u, v \in V.
\]
We require that the map $A(z)$ satisfies the compatibility condition

$$\sigma_V(A(z)) = A(\sigma(z)) \sigma_W(w), \quad \forall w \in W, \quad \forall z \in \mathbb{C}^4,$$

(2.16)

where $\sigma : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ is as in Eq. (2.4). Finally, we impose the following conditions:

- For all $z \in \mathbb{C}^4$, the space $U_z := A(z)(W) \subset V$ is of dimension $k$;
- For all $z \in \mathbb{C}^4$, $U_z$ is isotropic with respect to $(\cdot, \cdot)$ i.e. $U_z \subseteq U_z^\perp$, where $\perp$ denotes the complement with respect to the form $(\cdot, \cdot)$.

If we then define the quotient $E_z := U_z^\perp / U_z$, then the collection of $E_z$ defines a holomorphic, rank-2 complex vector bundle $E \rightarrow \mathbb{C}P^3$ with structure group $\text{SL}_2(\mathbb{C})$. The reality condition (2.16) then implies that the bundle is trivial on restriction to any real line and that the self-dual connection on $S^4$ determined by the Ward correspondence is an $\text{SU}_2$ connection.

3. Patching Matrix Description of ADHM Construction

In order to make contact between the action of the non-local symmetries of the self-dual Yang–Mills equations in the form of (2.13) and the ADHM construction, we first need to reformulate the ADHM construction in terms of patching matrices.

We assume given an instanton solution of the self-dual Yang–Mills equations on $S^4$, with corresponding holomorphic bundle $E \rightarrow \mathbb{C}P^3$. We then consider (without any loss of information [29]) the restriction of this solution to $\mathbb{R}^4 \subset S^4$ and the restriction of the bundle $E$ to $\pi^{-1}(\mathbb{R}^4) \equiv \mathbb{C}P^3 \setminus L_\infty$, which, for convenience, we denote by $E \rightarrow \mathbb{C}P^3 \setminus L_\infty$. We split the set $\mathbb{C}P^3 \setminus L_\infty$ as the union of two regions

$$S_0 := \{(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid |z| < 1 + \epsilon\} = \mathbb{C}^2 \times \mathbb{V}^0_\epsilon,$$

$$S_\infty := \{(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid |z| > \frac{1}{1 + \epsilon}\} = \mathbb{C}^2 \times \mathbb{V}^\infty_\epsilon.$$

Since $S_0, S_\infty \cong \mathbb{C}^3$, the bundle $E$ restricted to either of these regions is holomorphically trivial [9,23]. The bundle $E$ is therefore characterised by the transition function $G : S_0 \cap S_\infty \rightarrow \text{SL}_2(\mathbb{C})$, which is the patching matrix from Sect. 2.2.

The map $G$ may be constructed directly from the ADHM data, at the expense of fixing bases on the spaces $V$ and $W$. In particular, let $\{a_i\}_{i=1}^k$ be a basis of vectors in $W$ that are real with respect to $\sigma_W$, in the sense that

$$\sigma_W(a_i) = a_i, \quad i = 1, \ldots, k.$$

(So, in practice, we are looking on $W$ as being the complexification of the fixed point set of $\sigma_W$.) The vectors

$$v_i(z) := A(z)a_i \in V, \quad i = 1, \ldots, k$$

define a collection of $k$ vectors that span the space $U_z \subset V$. Due to the reality of the vectors $a_i$, these vectors obey the reality condition

$$\sigma_V(v_i(z)) = v_i(\sigma(z)), \quad i = 1, \ldots, k, \quad \forall z \in \mathbb{C}^4.$$  

(3.1)

Since $U_z$ is isotropic with respect to the symplectic form, we deduce that

$$\langle v_i(z), v_j(z) \rangle = 0, \quad i, j = 1, \ldots, k, \quad z \in \mathbb{C}^4.$$  

(3.2)
We now view \( z \) as homogeneous coordinates on \( \mathbb{C}P^3 \), and construct bases for \( \{ z \} \in S_0 \subset \mathbb{C}P^3 \setminus L_\infty \). Given \( \{ z \} \in S_0 \), the annihilator \( U_z^\perp \) is spanned by \( \{ v_i(z) \} \) along with two vectors \( \{ e_A(z) \mid A = 1, 2 \} \) that span \( U_z^\perp / U_z \). We therefore have

\[
(v_i(z), e_A(z)) = 0, \quad i = 1, \ldots, k, \quad A = 1, 2, \quad [z] \in S_0, \tag{3.3}
\]

and, without loss of generality, may assume that

\[
(e_1(z), e_2(z)) = -(e_2(z), e_1(z)) = 1. \tag{3.4}
\]

Although not strictly necessary, it will sometimes be useful to extend the vectors \( \{ e_A(z), v_i(z) \} \) to a full basis for \( V \) by adding a set of vectors \( \{ w^i(z) \mid i = 1, \ldots, k \} \) with the property that

\[
(w^i(z), w^j(z)) = 0, \quad (v_i(z), w^j(z)) = \delta^i_j, \quad (w^i(z), e_A(z)) = 0. \tag{3.5}
\]

We may also define a basis \( \{ f_A(z) \mid A = 1, 2 \} \) for \( U_z^\perp / U_z \) for \( [z] \in S_\infty \) by the relations

\[
f_1(z) := -\sigma_V(e_2(\sigma(z))), \quad f_2(z) := \sigma_V(e_1(\sigma(z))).
\]

This basis automatically has the property that

\[
(f_1(z), f_2(z)) = 1, \quad (v_i(z), f_A(z)) = 0. \tag{3.6}
\]

Given that \( \{ e_A(z) \} \) and \( \{ f_A(z) \} \) are both bases for \( U_z^\perp / U_z \) for \( [z] \in S_0 \cap S_\infty \), there exist functions \( G_A^B(z), \lambda_A^i(z) \) defined on this region with the property that

\[
f_A(z) = G_A^B(z) e_B(z) + \lambda_A^i(z) v_i(z). \tag{3.7}
\]

(From now on, the summation convention will be assumed over repeated indices.) The matrix \( G(z) \), defined for \( [z] \in S_0 \cap S_\infty \) is then the transition function of our bundle \( E \).

Before deriving some properties of the patching matrix that we will require, we define the \( SL_2(\mathbb{C}) \)-invariant tensor \( \epsilon \) by \( \epsilon^{AB} = -\epsilon^{BA} \) with \( \epsilon^{12} = 1 \) and the \( SO_2(\mathbb{C}) \)-invariant tensor \( \delta \) with components

\[
\delta_{AB} = \begin{cases} 1 & A = B, \\ 0 & A \neq B. \end{cases}
\]

**Proposition 3.1.** The patching matrix, \( G \), as defined above obeys the conditions

\[
det G(z) = 1, \quad G^*(z) = G(z),
\]

for \( [z] \in S_0 \cap S_\infty \), where \( G^*(z) := G(\sigma(z))^\dagger \). The functions \( \lambda_A^i \) obey the reality condition

\[
\lambda_A^i(\sigma(z)) = \delta_{AB} G_{C}^{B} \epsilon^{CD} \lambda_D^i(z), \quad \lambda_A^{i}(z) = -G_{A}^{B} \delta_{BC} \epsilon^{CD} \lambda_D^{i}(\sigma(z))
\]

for \( [z] \in S_0 \cap S_\infty \).
Proof. Firstly, we have
\[ 1 = (f_1(z), f_2(z)) = \left( G^1_B(z) e_B(z) + \lambda_1^i(z) v_i(z), G^2_B(z) e_B(z) + \lambda_2^i(z) v_i(z) \right) \]
\[ = \left( G^1_B(z) G^2_B(z) - G^1_B(z) G^2_B(z) \right) (e_1(z), e_2(z)) = \det G(z), \]
where the four equalities follow from Eqs. (3.6), (3.7), (3.3) and (3.4), respectively. Therefore, \( \det G(z) = 1 \), as required.

The definition of the vectors \( f_A(z) \) may be rewritten in the form
\[ f_A(z) = -\delta_{AB} \epsilon^{BC} \sigma_V (e_C(\sigma(z))). \] (3.8)

We now apply \( \sigma_V \) to this equation, substitute Eqs. (3.7) and (3.1), and use the anti-linear, anti-involution nature of \( \sigma_V \). After some manipulation of \( \delta \) and \( \epsilon \) tensors, and using the fact that \( \det G = 1 \), we then find that
\[ f_A(z) = \left( G^*(z) \right)_A^B \left[ e_B(z) - \delta_{BC} \epsilon^{CD} \lambda_D^i(\sigma(z)) v_i(z) \right]. \]
Comparing with (3.7) then gives the required equalities. \( \square \)

Remark 3.1. We will be primarily interested in Eq. (3.7) when it is restricted to a real line \( L_x \subset \mathbb{C}^1 \setminus L_\infty \). Since the patching matrix, \( G \), defined above is holomorphic on \( \mathbb{C}P^1 \setminus L_\infty \), when restricted to a neighbourhood of the line \( L_x \equiv L_{(u,v)} \), then \( G \) will restrict to a function (which we denote by \( G(x,z) \)) that is holomorphic in \( (u - z\overline{v}, v + z\overline{u}, z) \) for \( z \in V_{\epsilon} \), for some \( \epsilon > 0 \).

4. One-Parameter Families of ADHM Data

We now consider a one-parameter family of ADHM data \( A(t : z) := A(t : z) \), with \( t \in I \) a parameter, \( I \) a sub-interval of the real line containing the origin. We assume that \( A(t : z) \) is a \( C^1 \) function of \( t \).

We wish to investigate how the elements of the above explicit construction depend on \( A(t : z) \). The image \( A(t : z)(W) \) is now spanned by the vectors \( \{v_i(t : z)\} \), and \( U^\perp z / U_z^\perp \) is spanned by \( \{e_A(t : z)\} \), which we assume normalised such that (3.4) is satisfied for each \( t \in I \). Constructing the vectors \( \{f_A(t : z)\} \), we then define the patching matrix \( G_A^B(t : z) \) and the functions \( \lambda_A^i(t : z) \) as in (3.7).

Proposition 4.1. Given a one-parameter family of ADHM data, \( A(t : z) \), and patching matrices as defined in (3.7), then there exists a matrix-valued function \( d(t : z) \) with the property that
\[ \hat{G}(t : z) = d(t : z)G(t : z) + G(t : z)d^*(t : z). \] (4.1)

Proof. To investigate the \( t \)-dependence of these quantities, we consider their derivatives with respect to \( t \). The derivatives of the relevant vectors are given as follows: \(^6\)
\[ v_i = A_i^j v_j + B_i^j w_j + \epsilon^{AB} s_{Ai} e_B, \]
\[ w^j = C_i^j v_j - A_i^j w_j - \epsilon^{AB} r_{Ai}^j e_B, \]
\[ e_A = c_A^B e_B + r_{Ai}^B v_i + s_{Ai}^j w_i, \] (4.2a) (4.2b) (4.2c)

\(^6\) Everything depends on \((t : z)\), but we drop explicit mention of this dependence for the moment.
where $A_i^j, \ldots sA_i$ are functions of $(t, z)$, that satisfy the relationships

$$B_{ij} = B_{ji}, \quad C_{ij} = C_{ji}, \quad c_A^A = 0.$$  

It is straightforward to check that these are the most general forms of $\dot{v}_i, \dot{w}_i, \dot{e}_A$ that preserve the relations (3.2), (3.3), (3.4) and (3.5).

We also define functions that characterise the time-dependence of the vector fields $f_A$:

$$\dot{f}_A = d_A B f_B + t_A^i v_i + u_{Ai} w^i. \quad (4.3)$$

From this expression and Eq. (3.7), we deduce that

$$\dot{G}_A^B = d_A^C G_C^B - G_A^C C_B^B + \lambda_A^i \epsilon^{BC} s_{Ci}, \quad (4.4)$$

along with the relations

$$\dot{\lambda}_A^i = d_A^C \lambda_C^i + t_A^i - G_A^B r_B^i - \lambda_A^j A_j^i,$$

$$u_{Ai} = G_A^B s_{Bi} + \lambda_A^j B_{ji}.$$  

Also, equating $\dot{f}_1$ with $-\dot{e}_2$, and $\dot{f}_2$ with $\dot{e}_1$, we find that

$$d_A^B = u_{Ai} \lambda^i B + \epsilon_{AC} \epsilon^{CD} \delta_{DE} \delta_{EF} \epsilon^{BF},$$

and

$$u_{Ai} = \epsilon_{AB} \delta^{BC} s_{Ci}.$$  

These equations, along with (4.4) imply that the $t$-derivative of the patching matrix obeys the relation (4.1) with

$$d = u_i \otimes \lambda^i + \epsilon_{AC} \epsilon^{CD} \delta_{DE} \delta_{EF} \epsilon^{BF},$$

as required. ~\(\Box\)

**Remark 4.1.** The quantities that occur in Eq. (4.1) may all be constructed directly from the vector fields $e_A, v_i$ since

$$(v_i, \dot{e}_A) = s_{Ai}, \quad (\dot{e}_A, e_B) = \sum_C c_A^C \epsilon_{CB}.$$  

Therefore the construction does not actually require the introduction of the basis vectors $\{w^i\}$.

**Corollary 4.1.** Given a one-parameter family of ADHM data and patching matrix defined as above, then there exists a map $d : I \times \mathbb{C}^2 \times \mathcal{V}_\epsilon \to \text{SL}_2(\mathbb{C})$ that is holomorphic in $(u - z \overline{v}, v + z \overline{u}, z)$ for $z \in \mathcal{V}_\epsilon$ such that the restriction of the patching matrix to real-lines $L_x$ evolves according to

$$\dot{G}(t : x, z) = d(t : x, z) G(t : x, z) + G(t : x, z) d^{*}(t : x, z), \quad (4.5)$$

for $(x, z) \in \mathbb{C}^2 \times \mathcal{V}_\epsilon$.  

**Proof.** Restrict (4.1) to $L_x$. ~\(\Box\)
Remark 4.2. Let $\alpha(t : x, z)$ satisfy the first order ordinary differential equation
\[
\dot{\alpha}(t : x, z) = d(t : x, z) \alpha(t : x, z), \quad \alpha(0 : x, z) = \text{Id}.
\]

Given an initial patching matrix $G(x, z)$, it follows that the one-parameter family of patching matrices
\[
G(t : x, z) := \alpha(t : x, z) G(x, z) \alpha^*(t : x, z) \tag{4.6}
\]
satisfies Eq. (4.1) with initial conditions $G(0 : x, z) = G(x, z)$. Conversely, by uniqueness of solutions of (4.1), it follows that $G(t : x, z)$, as defined in Eq. (4.6), is the unique one-parameter family of patching matrices determined by the flow (4.1) with initial data $G(x, z)$.

Note that these transformations (4.5) and (4.6) are of the same form as those generated by the symmetries of the self-dual Yang–Mills equations given in Eq. (2.13) and Theorem 2.2, with the important proviso that the function $d(t : x, z)$ occurring in (4.5) depends explicitly on the parameter $t$. The symmetries (2.13) should be viewed as defining a flow on the space, $M$, of self-dual connections defined by the map $T$. In solving (2.13), we are simply constructing the integral curves of this flow, with $t$ a parameter along the integral curve. As such, in (2.13), it is important that the function $T(x, z)$ is independent of the parameter $t$.

Viewing the function $T$ as defining a flow on $M$ and the instanton moduli spaces $M_k$ as submanifolds of $M$, we directly deduce:

**Theorem 4.1.** Let $A \in M_k$ be a $k$-instanton self-dual connection (modulo gauge transformation) on $\mathbb{R}^4$, with $M_k$ viewed as a submanifold of the space, $M$, of all self-dual connections on $\mathbb{R}^4$. Then for each vector $v \in T_A M_k$, there exists a function $T$ such that the fundamental vector field on $M$ corresponding to $T$ via Eq. (2.13) coincides with $v$ at the point $A \in M_k$.

**Proof.** Any element $v \in T_A M_k$ is generated by a one-parameter family of ADHM data, $A(t : z)$, with $A(0 : z)$ corresponding to the connection $A$. This one-parameter family of ADHM data then gives rise to a one-parameter family of patching matrices $G(t : x, z)$ evolving according to (4.5), where $G(0 : x, z)$ is the patching matrix corresponding to $A$ and $G(0 : x, z)$ corresponds to the tangent vector $v$. Taking $T(x, z) := -d(0 : x, z)$ gives a symmetry that, via (2.13) (with $\rho_0 = \rho_\infty = 0$) generates the tangent vector $v$. \hfill $\square$

Remark 4.3. Theorem 2.1 states that, given a function $T$, there is a corresponding fundamental vector field on $M$, the space of self-dual connections, corresponding to $T$. We shall denote this fundamental vector field by $X_T$. Theorem 4.1 states that, given a connection $A \in M_k$ and a tangent vector $v \in T_A M_k$, then there exists such a function $T$ such that $X_T|_A = v$. It is important to note, however, that the integral curve of $X_T$ starting at $A \in M_k$ will, generally, not remain within the sub-manifold $M_k$ of $M$. In order to determine which one-parameter groups of symmetries gives flows that remain in the moduli space $M_k$, we need to determine which transformations of the form (4.5) are generated by transformations of the form (2.13), with $T(x, z)$ independent of $t$.

From the form of (4.5) and (2.13), it appears natural to identify $d(t : x, z)$ with $-T(x, z) + \rho_\infty(t, x, z)$. We impose that $T$ is independent of $t$. The map $\rho_\infty$ simply generates a change of holomorphic frame for $z \in V^\infty_z$. At this point, we should recall that we have partially fixed our holomorphic frames in deriving our patching matrix from the
ADHM data. As such, if we wish to employ our approach with one-parameter families of ADHM data, we must allow for one-parameter families of changes of frame in order to compensate for this fixing of frames. As such, we should allow \( \rho_\infty \) to be \( t \)-dependent (i.e. \( \rho_\infty = \rho_\infty(t, x, z) \)). Note that such a \( t \)-dependent change of frame does not affect the corresponding self-dual connection \( A(t) \).

As such, we may use \( \rho_\infty \) to absorb any part of \( d(t : x, z) \) that is holomorphic on \( \mathbb{C}^2 \times V_\infty^\epsilon \), leaving an irreducible part of \( d(t : x, z) \), denoted \( d_0(t : x, z) \), that has singularities in the region \( V_\infty^\epsilon \) that cannot be removed by absorption into \( \rho_\infty \). In order to arise from a symmetry of the self-dual Yang–Mills equations, \( d_0(t : x, z) \) must then be independent of \( t \). Since \( d(t : x, z) \) is determined by first \( t \)-derivatives of the ADHM data, \( A(t : z) \), imposing that \( d_0(t : x, z) \) is constant in \( t \) will impose conditions on the first \( t \)-derivatives of the \( A(t : z) \) data that must be satisfied in order for this one-parameter family of data (and corresponding self-dual connections) to arise from a symmetry of the self-dual Yang–Mills equations. Explicit calculations, in the next section, suggest that these conditions are quite restrictive.

**Remark 4.4.** The fact that the flow on the moduli space does not generally preserve the \( L^2 \) nature of the curvature is well-known (see, e.g., [6–8] where this effect is mentioned). In [9, Chap. V], an explicit example of a transformation acting on a one-instanton patching matrix is given to demonstrate this phenomenon. In the notation of (4.6), this transformation takes the form

\[
\alpha(t : x, z) = \frac{1}{\sqrt{1-t^2}} \begin{pmatrix} 1 & t/z \\ t/z & 1 \end{pmatrix}.
\]

From this expression, we deduce that

\[
d(t : x, z) = \frac{1}{(1-t^2)^{3/2}} \begin{pmatrix} t & 1/z \\ z & t \end{pmatrix}.
\]

Following the programme of the previous remark, we then isolate the part of \( d \) that has singularities in the region \( z \in V_\infty^\epsilon \), namely

\[
d_0(t : x, z) = \frac{1}{(1-t^2)^{3/2}} \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}.
\]

Since \( d_0 \) depends explicitly on \( t \), we deduce that the counterexample provided in [9, Chap. V] falls outside of the class of transformations generated by transformations (2.13) with \( T \) independent of \( t \).

**Remark 4.5.** If one drops the reality condition that our connections are \( \text{SU}_2 \) connections, rather than \( \text{SL}_2(\mathbb{C}) \) connections, then Takasaki has argued [27] that the action of the non-local symmetry group generated by transformations of the form

\[
\dot{J}(x) = \chi_\infty(x, \lambda) T(x, \lambda) \chi_\infty(x, \lambda)^{-1} \cdot J
\]

is transitive on the space of \( \text{SL}_2(\mathbb{C}) \) solutions of the self-dual Yang–Mills equations. If, as here, we restrict to symmetries of the form (2.12) that explicitly preserve the \( \text{SU}_2 \) nature of the connection, then the symmetry group need not act transitively on the moduli space of solutions, even though the symmetries have been shown to generate the tangent space at each point. Moreover, if we explicitly impose that we only consider symmetries that preserve the \( L^2 \) nature of the connection, then the explicit calculations carried out in the next section for the one-instanton moduli space suggest that the orbits of the symmetry group are actually of high codimension in the moduli space.
5. The One-Instanton Solution

In the case of a one-instanton solution, it is straightforward to carry out the ADHM construction and the construction of deformations explicitly. We find that the one-parameter subgroups of ADHM data with \( d(t : x, z) \) of the form \(-T(x, z) + \rho_\infty(t, x, z)\) are rather small.

First, we fix some notation. In the case \( k = 1 \), then we may write

\[
v(z) := A(z) = \begin{pmatrix} A_1(z) \\ A_2(z) \\ A_3(z) \\ A_4(z) \end{pmatrix} \in \mathbb{C}^4,
\]

where \( A_i(z) = \sum_{j=1}^4 A_j^i z_j \), \( i = 1, \ldots, 4 \). Letting

\[
\sigma_V \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} := \begin{pmatrix} -\beta \\ \alpha \\ -\delta \\ \gamma \end{pmatrix},
\]

then (3.1) implies that the functions \( A_i(z) \) must satisfy the reality conditions:

\[
\overline{A_1(\sigma(z))} = -A_2(z), \quad \overline{A_2(\sigma(z))} = A_1(z),
\]
\[
\overline{A_3(\sigma(z))} = -A_4(z), \quad \overline{A_4(\sigma(z))} = A_3(z).
\]

In particular, using the symmetry transformations inherent in the ADHM construction [2, Chap. II], we may fix

\[
A_1(z) = \lambda z_1, \quad A_2(z) = \lambda z_2, \quad A_3(z) = \alpha z_1 - \beta z_2 - z_3, \quad A_4(z) = \beta z_1 + \alpha z_2 - z_4,
\]

where \( \lambda \) is a positive, real number and \( \alpha, \beta \) are complex numbers. Finally, we may take the symplectic form on \( V \cong \mathbb{C}^4 \) to be

\[
(a, b) = a^1 b^2 - a^2 b^1 + a^3 b^4 - a^4 b^3, \quad a, b \in \mathbb{C}^4.
\]

**Theorem 5.1.** The only transformations of the ADHM data \((\lambda, \alpha, \beta)\) that arise from a non-local symmetry of the self-dual Yang–Mills equations (2.12) according to (4.5) with \( d(t : x, z) \) of the form \(-T(x, z) + \rho_\infty(t : x, z)\) are of the form

\[
\lambda \mapsto \lambda(t) := \frac{\lambda}{\sqrt{1 - k\lambda^2 t}}, \quad \alpha, \beta \text{ constant},
\]

where \( k \in \mathbb{R} \) is a real constant.

**Proof.** On a region with \( A_1(z) \neq 0 \) (and hence \( A_2(z) \neq 0 \)), then we find that \( U_z = v(z)^\perp/v \) is spanned by the vectors

\[
e_1(z) = \begin{pmatrix} 0, A_4(z) \\ A_1(z) \end{pmatrix}, \quad e_2(z) = \begin{pmatrix} 0, -A_3(z) \\ A_1(z) \end{pmatrix},
\]
which have the property that \((e_1, e_2) = 1\). Such a basis, including the normalisation property, is unique up to a translation \(e_A \mapsto e_A + \lambda_A v\), and an \(\text{SL}_2(\mathbb{C})\) rotation of the vectors \(e_A(z)\). Taking the conjugates of these vectors, we find that

\[
\mathbf{f}_1(z) = -e_2(z) = \left( -\frac{A_4(z)}{A_2(z)}, 0, 1, 0 \right), \quad \mathbf{f}_2(z) = e_1(z) = \left( \frac{A_3(z)}{A_2(z)}, 0, 0, 1 \right).
\]

These expressions imply that on the overlap where the two above regions overlap, we have the patching matrix (see \([9, \text{Chap. V}]\))

\[
G = \begin{pmatrix}
1 + \frac{A_3(z)A_4(z)}{A_1(z)A_2(z)} & \frac{A_4(z)^2}{A_1(z)A_2(z)} \\
- \frac{A_4(z)^2}{A_1(z)A_2(z)} & 1 - \frac{A_3(z)A_4(z)}{A_1(z)A_2(z)}
\end{pmatrix}
\]

and

\[
\lambda_1(z) = - \frac{A_4(z)}{A_1(z)A_2(z)}, \quad \lambda_2(z) = \frac{A_3(z)}{A_1(z)A_2(z)}.
\]

We may take the vector \(\mathbf{w}(z)\) to be

\[
\mathbf{w} = \left( 0, \frac{1}{A_1(z)}, 0, 0 \right),
\]

which is unique up to \(\mathbf{w} \mapsto \mathbf{w} + \phi \mathbf{v}\).

If we now let \(\mathbf{v}(z)\) depend smoothly on a parameter \(t \in (-\varepsilon, \varepsilon)\), then we may calculate the parameters of the deformation \(A, B, C, \ldots\) as defined in (4.2) and (4.3). The parameter \(d\) is the one that we primarily require and a straightforward calculation shows that

\[
d(t : z) = \frac{\partial}{\partial t} \left( \frac{A_4(t : z)/A_2(t : z)}{-A_3(t : z)/A_2(t : z)} \right) \times \left( \frac{A_3(t : z)/A_1(t : z)}{A_4(t : z)/A_1(t : z)} \right).
\]

Taking \(A_i(t : z)\) as in (5.1), with \(\lambda\) replaced by \(\lambda(t)\), etc, then, restricted to the line \(L_x\), the deformation parameter that we require takes the form

\[
d(x, z) = \frac{1}{z} \frac{\partial}{\partial t} \left( \frac{(\beta - v) + z(\beta - v)}{(x - u) - z(\beta - v)} \right) \times \left( \frac{(x - u) - z(\beta - v)}{\lambda} \right).
\]

This expression may be written in the form

\[
d(x, z) = \left[ \frac{2}{z} + \frac{D}{z} \right] (u - zv) + \left[ \frac{F}{z} + G \right] (v + zu) + \frac{H}{z} + I + Jz,
\]
where
\[
A = \frac{\lambda}{\lambda^3} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \frac{\lambda}{\lambda^3} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \frac{\lambda}{\lambda^3} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},
\]
\[
D = \frac{1}{\lambda^3} \begin{pmatrix} -\lambda \dot{\beta} + \beta \dot{\lambda} & 0 \\ \lambda \dot{\alpha} - 2 \alpha \dot{\lambda} & -\beta \dot{\lambda} \end{pmatrix}, \quad E = \frac{1}{\lambda^3} \begin{pmatrix} -\lambda \ddot{\alpha} + \bar{\alpha} \ddot{\lambda} & 0 \\ -\lambda \ddot{\beta} + 2 \bar{\beta} \ddot{\lambda} & -\bar{\alpha} \ddot{\lambda} \end{pmatrix},
\]
\[
F = \frac{1}{\lambda^3} \begin{pmatrix} \alpha \dot{\lambda} & -\lambda \dot{\beta} + 2 \beta \dot{\lambda} \\ 0 & \lambda \dot{\alpha} - \alpha \dot{\lambda} \end{pmatrix}, \quad G = \frac{1}{\lambda^3} \begin{pmatrix} -\bar{\beta} \dot{\lambda} & -\lambda \ddot{\alpha} + 2 \bar{\alpha} \ddot{\lambda} \\ 0 & -\lambda \ddot{\beta} + \bar{\beta} \ddot{\lambda} \end{pmatrix},
\]
\[
H = \frac{1}{\lambda^3} \begin{pmatrix} \alpha (\lambda \ddot{\beta} - \beta \ddot{\lambda}) & \beta (\lambda \ddot{\beta} - \beta \ddot{\lambda}) \\ \alpha (-\lambda \ddot{\alpha} + \alpha \ddot{\lambda}) & \beta (-\lambda \ddot{\alpha} + \alpha \ddot{\lambda}) \end{pmatrix},
\]
\[
I = \frac{1}{\lambda^3} \begin{pmatrix} \alpha \lambda \ddot{\alpha} - \bar{\beta} \lambda \ddot{\beta} - \alpha \bar{\alpha} \ddot{\lambda} + \beta \bar{\beta} \ddot{\lambda} & \beta \lambda \ddot{\alpha} + \alpha \bar{\alpha} \ddot{\lambda} - 2 \alpha \beta \ddot{\lambda} \\ \beta \lambda \ddot{\alpha} + \alpha \bar{\alpha} \ddot{\lambda} - 2 \alpha \beta \ddot{\lambda} & \bar{\alpha} (-\lambda \ddot{\alpha} + \alpha \ddot{\lambda}) + \beta (\lambda \ddot{\beta} - \beta \ddot{\lambda}) \end{pmatrix},
\]
\[
J = \frac{1}{\lambda^3} \begin{pmatrix} \beta (-\lambda \ddot{\alpha} + \bar{\alpha} \ddot{\lambda}) & \bar{\alpha} (\lambda \ddot{\alpha} - \bar{\alpha} \ddot{\lambda}) \\ \beta (-\lambda \ddot{\beta} + \bar{\beta} \ddot{\lambda}) & \bar{\alpha} (\lambda \ddot{\beta} - \bar{\beta} \ddot{\lambda}) \end{pmatrix},
\]

where \( \cdot \) denotes differentiation with respect to \( t \).

According to the philosophy of Remark 4.3, we note that the coefficients \( D, F, H \) and \( I \) correspond to terms that are analytic for \( z \in V^\infty_\epsilon \), and therefore may be absorbed into the \( \rho_\infty \) term. The remaining part of the parameter \( d \) is then
\[
d_0(x, z) = \frac{1}{z} \left[ A(u - z \bar{v})^2 + B(u - z \bar{v})(v + z \bar{u}) + C(v + z \bar{u})^2 \right] + E(u - z \bar{v}) + G(v + z \bar{u}) + Jz.
\]
All of the terms in \( d_0 \) have singularities at \( z = \infty \in V^\infty_\epsilon \). In order for such transformations to arise from a \( T \) that is independent of \( t \) with \( d = -T + \rho_\infty \), we therefore require that the remaining coefficients \( A, B, C, E, G \) and \( J \) must be independent of \( t \) (i.e. constant). An analysis of the explicit form of these coefficients given above shows that this condition is only possible if
\[
\frac{\dot{\lambda}}{\lambda^3} = \frac{k}{2}, \quad \dot{\alpha} = \dot{\beta} = 0,
\]
where \( k \) is a constant. Integrating these equations yields (5.2). Therefore the only transformation on the one-instanton moduli space that arises from a symmetry of the form (2.12) with \( d(t : x, z) = -T(x, z) + \rho_\infty(t, x, z) \) is a scaling of the moduli space. \( \square \)

**Remark 5.1.** The group of transformations on the one-instanton moduli space is therefore only one-dimensional. Such a collapse to a finite-dimensional action is familiar from the theory of harmonic maps (see, e.g., [1, 17, 20, 28]), where the orbits of the group action are also, generically, of high codimension.

### 6. Final Remarks

Our first main result is Theorem 4.1, which states that the tangent space to the instanton moduli spaces, \( M_k \), are generated by symmetries of the self-dual Yang–Mills equations. Nevertheless, our second main result, based on an analysis of the one-instanton moduli...
space, is that the subgroup of the symmetry group that preserves the $L^2$ nature of the connection, and hence has orbits that lie in a particular $\mathcal{M}_k$, is rather small. In particular, the orbits of this subgroup on the space $\mathcal{M}_k$ are of high codimension. We have restricted ourselves to one-parameter families of ADHM data that arise from transformations of the form (2.13) and (4.5) with $d(t : x, z) = -T(x, z) + \rho_\infty(t, x, z)$. Note that this is a sufficient, but not necessary, condition for Eqs. (2.13) and (4.5) to be consistent. It is conceivable that there might be a larger group of transformations acting on the moduli spaces, $\mathcal{M}_k$, consistent with these equations, but we have not investigated this possibility.

It is hoped that there is a more elegant way of carrying out the calculations in the previous section. In particular (also regarding the remark in the previous paragraph) one would like to pull the infinitesimal action on the patching matrix (2.13) directly up to the space of ADHM data. An alternative approach to extending our analysis would be to investigate our approach from the point of view of Donaldson’s reformulation of the ADHM construction [12], where one views instantons as defining holomorphic bundles over $\mathbb{C}P^2$. Restricting our constructions to the $\mathbb{C}P^2$ picture is straightforward, but it is again to directly calculate the action of the symmetry transformations on the data. Work of Nakamura [22] concerning dynamical systems defined on the space of data of the Donaldson construction may be relevant in this regard. The approach where one might expect the symmetries to have the simplest form would be within Atiyah’s reformulation [3] of the instanton moduli spaces in terms of holomorphic maps $\mathbb{C}P^1 \rightarrow \Omega G$. In this case, the connection with harmonic map theory is quite strong. In the case of the self-dual Yang–Mills equations, however, one expects the symmetry group to act directly on the map in the Atiyah construction, whereas for harmonic maps the “dressing action” acts purely on the space $\Omega G$. It is also quite difficult to see directly how the action on the patching matrix or ADHM data transfer to the Atiyah picture, due to the non-holomorphic transformations required in passing from the ADHM construction to this approach.

More broadly, thinking of $(\lambda, \alpha, \beta)$ as coordinates on the five-dimensional ball (with $(\alpha, \beta)$ compactified to the four-sphere and $\lambda$ the radial coordinate) then the flow in (5.2) is simply a radial scaling. In particular, for $k > 0$, the flow converges to the fixed point $\lambda = 0$ as $t \rightarrow -\infty$, and diverges to $+\infty$ as $t \rightarrow \left(\frac{1}{k^2}\right)^{-\frac{1}{k}}$. Such flows are, in some respects, reminiscent of Morse flows, and it would be of interest to know whether our approach has a Morse-theoretic interpretation. In addition, it would be interesting to relate our work to other examples of systems where one has a symmetry algebra, but no corresponding group action e.g. Teichmüller theory.\textsuperscript{7}

As mentioned in the Introduction, the original motivation for this work was to determine whether the integrable systems approach to the self-dual Yang–Mills equations could give information about instanton moduli spaces as used in the more topological context of Donaldson theory. In this regard, the results of this paper should be viewed alongside the results of the companion paper [15]. In [15], reducible connections were studied on open subsets of $\mathbb{R}^4$, and were found to bear a strong resemblance to harmonic maps of finite type (see, e.g., [16, Chap. 24]). In particular, all reducible connections lie in the orbit, under flows (2.13), of the flat connection on $\mathbb{R}^4$. Therefore instanton solutions on $\mathbb{R}^4$ and reducible connections (which are necessarily not $L^2$ on $\mathbb{R}^4$) appear to have quite different behaviour under the symmetry group of the self-dual Yang–Mills equations. Since reducible and irreducible connections play a different role in Donalds-\textsuperscript{7} The author is grateful to Prof. K. Ono for this suggestion.
son’s work [11], corresponding to the smooth and singular parts of the moduli space respectively, it is striking that such connections also seem to have different behaviour from the point of view of integrable systems. In this respect, it would be of particular interest to investigate the one-instanton moduli space on $\mathbb{C}P^2$, where one has $L^2$ and reducible connections in the same moduli space.

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A. Action of Symmetries on the Patching Matrix

It appears that the direct derivation of the infinitesimal flow, (2.13), from the flow of the $J$-function, (2.12), has not appeared in the literature. We therefore give a proof of this result here. The closest to our derivation that we have found is the corresponding construction for harmonic maps into Lie groups given in [28, §3-4].

For ease of notation, we define the quantities

\[ \alpha(x, \lambda) := \Psi_\infty(x, \lambda)^T \Psi_\infty(x, \lambda)^{-1}, \]
\[ \alpha(x, \lambda)^\dagger := \Psi_0(x, \sigma(\lambda))^T \Psi_0(x, \sigma(\lambda))^{-1}, \]

and recall the solution of the linearisation equation, (2.12), in this notation:

\[ \dot{J}(x) = \Psi_\infty(x)^{-1} \left[ \alpha(x, \lambda) + \alpha(x, \lambda)^\dagger \right] \psi_0(x). \]

**Proposition A.1.** There exists a function $h_\infty(x, z) \equiv h_\infty(u - z\overline{v}, v + z\overline{u}, z)$ with the property that

\[ \dot{\Psi}_\infty(x, z)\Psi_\infty(x, z)^{-1} - \dot{\psi}_\infty(x)\psi_\infty(x)^{-1} = \frac{\lambda}{\lambda - z} (\alpha(x, \lambda) - \alpha(x, z)) + \frac{1}{1 + z\lambda} \left( \alpha(x, \lambda)^\dagger - \alpha(x, \sigma(z))^\dagger \right) - \Psi_\infty(x, z)h_\infty(z)\Psi_\infty(x, z)^{-1}. \]

(A.3)

for all $z \in \mathbb{C}P^1$ such that $z \neq 0, \lambda, -1/\lambda$. Similarly, there exists a function $h_0(x, z) \equiv h_0(u - z\overline{v}, v + z\overline{u}, z)$ such that

\[ \dot{\Psi}_0(x, z)\Psi_0(x, z)^{-1} - \dot{\psi}_0(x)\psi_0(x)^{-1} = \frac{z}{\lambda - z} (\alpha(x, \lambda) - \alpha(x, z)) - \frac{z\overline{\lambda}}{1 + z\lambda} \left( \alpha(x, \lambda)^\dagger - \alpha(x, \sigma(z))^\dagger \right) + \Psi_0(x, z)h_0(z)\Psi_0(x, z)^{-1}, \]

(A.4)

for all $z \in \mathbb{C}P^1$ such that $z \neq \infty, \lambda, -1/\lambda$.

**Proof.** From (A.2), we deduce that

\[ \dot{\psi}_0(x)\psi_0(x)^{-1} - \dot{\psi}_\infty(x)\psi_\infty(x)^{-1} = \alpha(x, \lambda) + \alpha(x, \lambda)^\dagger. \]

(A.5)
From the defining relations for $\psi_0(x, z), \psi_\infty(x, z)$ we deduce that the derivative of the components of the connection are given by

\[
(\dot{A}_\pi - z \dot{A}_v) = -(D_\pi - z D_v) \left( \dot{\psi}_0(x, z) \psi_0(x, z)^{-1} \right)
= -(D_\pi - z D_v) \left( \dot{\psi}_\infty(x, z) \psi_\infty(x, z)^{-1} \right),
\]

\[
(\dot{A}_\tau + z \dot{A}_u) = -(D_\tau + z D_u) \left( \dot{\psi}_0(x, z) \psi_0(x, z)^{-1} \right)
= -(D_\tau + z D_u) \left( \dot{\psi}_\infty(x, z) \psi_\infty(x, z)^{-1} \right).
\]

This expression implies that

\[
(D_\pi - z D_v) \left[ \dot{\psi}_\infty(x, z) \psi_\infty(x, z)^{-1} \right] = D_\pi \left[ \dot{\psi}_0(x) \psi_0(x)^{-1} \right] - z D_v \left[ \dot{\psi}_\infty(x) \psi_\infty(x)^{-1} \right],
\]

\[
(D_\tau + z D_u) \left[ \dot{\psi}_\infty(x, z) \psi_\infty(x, z)^{-1} \right] = D_\tau \left[ \dot{\psi}_0(x) \psi_0(x)^{-1} \right] + z D_u \left[ \dot{\psi}_\infty(x) \psi_\infty(x)^{-1} \right].
\]

We need to solve these equations for $\Psi_\infty(x, z)$ with the boundary condition that $\dot{\Psi}_\infty(x, z) \rightarrow \dot{\psi}_\infty(x)$ as $z \rightarrow \infty$. These equations may be rewritten in the form

\[
(D_\pi - z D_v) \left[ \dot{\psi}_\infty(x, z) \psi_\infty(x, z)^{-1} - \dot{\psi}_\infty(x) \psi_\infty(x)^{-1} \right] = D_\pi \left[ \alpha(x, \lambda) + \alpha(x, \lambda)^\dagger \right],
\]

\[
(D_\tau + z D_u) \left[ \dot{\psi}_\infty(x, z) \psi_\infty(x, z)^{-1} - \dot{\psi}_\infty(x) \psi_\infty(x)^{-1} \right] = D_\tau \left[ \alpha(x, \lambda) + \alpha(x, \lambda)^\dagger \right].
\]

We now note that

\[
(D_\pi - \lambda D_v) \alpha(x, \lambda) = (D_\tau + \lambda D_u) \alpha(x, \lambda) = 0.
\]

Therefore, for all $z \neq \lambda$,

\[
D_\pi \alpha(x, \lambda) = \frac{\lambda}{\lambda - z} \left( D_\pi - z D_v \right) \alpha(x, \lambda),
\]

\[
D_\tau \alpha(x, \lambda) = \frac{\lambda}{\lambda - z} \left( D_\tau + z D_u \right) \alpha(x, \lambda).
\]

Similarly,

\[
(D_v + \lambda D_\pi) \alpha(x, \lambda)^\dagger = (D_u - \lambda D_\tau) \alpha(x, \lambda)^\dagger = 0,
\]

from which we deduce that, for all $z \neq -1/\lambda$,

\[
D_\pi \alpha(x, \lambda)^\dagger = \frac{1}{1 + \frac{\lambda}{z}} \left( D_\pi - z D_v \right) \alpha(x, \lambda)^\dagger,
\]

\[
D_\tau \alpha(x, \lambda)^\dagger = \frac{1}{1 + \frac{\lambda}{z}} \left( D_\tau + z D_u \right) \alpha(x, \lambda)^\dagger.
\]
Hence,
\[
(D_\pi - zD_v) \left[ \psi_\infty(x, z) \psi_\infty(x, z)^{-1} - \dot{\psi}_\infty(x) \psi_\infty(x)^{-1} \right] = 0,
\]
and, similarly, \((D_\pi + zD_u) [\ldots] = 0\). It then follows that there exists a function \(H_\infty(u - z\overline{v}, v + z\overline{u}, z)\) with the property that
\[
\dot{\psi}_\infty(x, z) \psi_\infty(x, z)^{-1} - \dot{\psi}_\infty(x) \psi_\infty(x)^{-1} = \frac{\lambda}{\lambda - z} \alpha(x, \lambda) - \frac{1}{1 + z\lambda} \alpha(x, \sigma(z)) \psi_\infty(x, z)^{-1}.
\]
Taking
\[
H_\infty(x, z) = h_\infty(x, z) - \psi_\infty(x, z)^{-1} \left[ \frac{\lambda}{\lambda - z} \alpha(x, z) + \frac{1}{1 + z\lambda} \alpha(x, \sigma(z)) \psi_\infty(x, z)^{-1} \right] \psi_\infty(x, z)
\]
cancels the poles in the first two terms in the right-hand-side of (A.6), and yields Eq. (A.3). A similar argument for \(\Psi_0(x, z)\) yields Eq. (A.4). \(\square\)

**Lemma A.1.**
\[
\dot{G}(z) = T(z)G(z) + G(z)T^*(z) + h_\infty(z)G(x, z) + G(x, z)h_0(z).
\]

**Proof.** Firstly,
\[
\dot{G}(z) = \frac{\partial}{\partial s} \left[ \psi_\infty(x, z)^{-1} \cdot \psi_0(x, z) \right] = \psi_\infty(x, z)^{-1} \left[ \dot{\psi}_0(x, z) \cdot \psi_0(x, z)^{-1} - \dot{\psi}_\infty(x, z) \cdot \psi_\infty(x, z)^{-1} \right] \psi_0(x, z).
\]
Now use Eqs. (A.1), (A.3), (A.4) and (A.5). \(\square\)

The left-hand-side of (A.4) is analytic for \(|z| < 1 + \epsilon\). Any singularities in this region that occur in the first two terms on the right-hand-side must therefore be cancelled by corresponding singularities in the function \(h_0\). It turns out that this consideration is enough to determine \(h_0\) up to addition of a function of \((u - z\overline{v}, v + z\overline{u}, z)\) that is holomorphic on the region \(|z| < 1 + \epsilon\). Similar remarks apply to \(h_\infty\) and Eq. (A.3).

**Proposition A.2.** There exists a function \(\rho_0(x, z) \equiv \rho_0(u - z\overline{v}, v + z\overline{u}, z)\), holomorphic for \(|z| < 1 + \epsilon\) with the property that on the region \(1/\epsilon < |z| < 1 + \epsilon\) we have
\[
\dot{\psi}_0(x, z) \psi_0(x, z)^{-1} - \dot{\psi}_0(x) \psi_0(x)^{-1} = \frac{1}{\lambda - z} (z\alpha(x, \lambda) - \lambda\alpha(x, z)) - \frac{1}{1 + z\lambda} \left( z\overline{\lambda} \alpha(x, \lambda) + \alpha(x, \sigma(z)) \psi_0(x, z) \rho_0(z) \psi_0(x, z)^{-1} \right).
\]
Proof. Rearranging Eq. (A.4) yields
\[
h_0(z) = \chi_0(z)^{-1} \chi_0(z) - \frac{z}{\lambda - z} \Psi_0(z)^{-1} (\alpha(x, \lambda) - \alpha(x, z)) \Psi_0(z)
+ \frac{z\lambda}{1 + z\lambda} \Psi_0(z)^{-1} \left( \alpha(x, \lambda)^\dagger - \alpha(x, \sigma(z))^\dagger \right) \Psi_0(z).
\]
Since $\Psi_0$ is analytic for $|z| < 1 + \epsilon$ and the poles at $z = \lambda, \sigma(\lambda)$ have been cancelled, it follows that $h_0$ is analytic for $\frac{1}{1+\epsilon} < |z| < 1 + \epsilon$. We may therefore split $h_0(z) = h_0^0(z) + h_0^\infty(z)$, where $h_0^0$ is analytic for $|z| < 1 + \epsilon$ and $h_0^\infty$ is analytic for $|z| > \frac{1}{1+\epsilon}$. For $|z| > \frac{1}{1+\epsilon}$, we have
\[
h_0^\infty(z) = -\frac{1}{2\pi i} \oint_{\gamma_-} \frac{h_0(w)}{w - z} \, dw,
\]
where $\gamma_- = \{ w \in \mathbb{C} : w = \frac{1}{1+\epsilon'} \}$, where $\epsilon' < \epsilon$ is chosen such that $|z| > \frac{1}{1+\epsilon'}$. Using the fact that $\chi$ and $\Psi_0$ are analytic for $|z| < \frac{1}{1+\epsilon}$, we find that
\[
h_0^\infty(z) = \frac{1}{2\pi i} \oint_{\gamma_-} \frac{1}{w - z} \left[ \frac{w\lambda}{1 + w\lambda} T^*(w) - \frac{w}{\lambda - w} G(w)^{-1} T(w) G(w) \right] \, dw
\]
for $|z| > \frac{1}{1+\epsilon}$. Differentiating under the integral sign, we find that
\[
(\partial_{\tilde{\tau}} - z \partial_v) h_0^\infty(z) = \partial_{\tilde{\tau}} K(x), \quad (\partial_{\tilde{\tau}} + z \partial_u) h_0^\infty(z) = \partial_{\tilde{\tau}} K(x),
\]
where
\[
K(x) := \frac{1}{2\pi i} \oint_{\gamma_+} \left[ \frac{1}{w - \sigma(\lambda)} T^*(w) + \frac{1}{w - \lambda} G(w)^{-1} T(w) G(w) \right] \, dw.
\]
Note that this expression is independent of $z$. In order to construct the function $h_0$, we must find a function $h_0^0$, holomorphic (in $z$) for $|z| < 1 + \epsilon'$ with the property that
\[
(\partial_{\tilde{\tau}} - z \partial_v) h_0^0(z) = -\partial_{\tilde{\tau}} K(x), \quad (\partial_{\tilde{\tau}} + z \partial_u) h_0^0(z) = -\partial_{\tilde{\tau}} K(x).
\]
To construct such a function, we define the contour $\gamma_+ = \{ w \in \mathbb{C} : |w| = 1 + \epsilon' \}$ and deduce that
\[
K(x) = \frac{1}{2\pi i} \oint_{\gamma_+} \left[ \frac{1}{w - \sigma(\lambda)} T^*(w) + \frac{1}{w - \lambda} G(w)^{-1} T(w) G(w) \right] \, dw
\]
\[
- T^*(\sigma(\lambda)) - G(\lambda)^{-1} T(\lambda) G(\lambda).
\]
We then find that, for $|z| < 1 + \epsilon'$,
\[
-\partial_{\tilde{\tau}} K(x) = -\frac{1}{2\pi i} \oint_{\gamma_+} \left[ \frac{1}{w - \sigma(\lambda)} \partial_{\tilde{\tau}} T^*(w) + \frac{1}{w - \lambda} \partial_{\tilde{\tau}} \left( G(w)^{-1} T(w) G(w) \right) \right] \, dw
\]
\[
+ \partial_{\tilde{\tau}} T^*(\sigma(\lambda)) + \partial_{\tilde{\tau}} \left( G(\lambda)^{-1} T(\lambda) G(\lambda) \right)
\]
\[
= (\partial_{\tilde{\tau}} - z \partial_v) \Phi(x, \lambda, z),
\]
where

\[ \Phi(x, \lambda, z) := -\frac{1}{2\pi i} \oint_{\gamma} \frac{w}{w - z} \left[ \frac{1}{w - \sigma(\lambda)} T^*(w) + \frac{1}{w - \lambda} \left( G(w)^{-1} T(w) G(w) \right) \right] dw + \frac{\sigma(\lambda)}{\sigma(\lambda) - z} T^*(\sigma(\lambda)) + \frac{\lambda}{\lambda - z} G(\lambda)^{-1} T(\lambda) G(\lambda), \]

with a similar expression for \(-\partial_\gamma K(x)\). Again cancelling the poles at \(z = \lambda, \sigma(\lambda)\), we deduce that, for \(|z| < 1 + \epsilon\), we may take

\[ h^{(0)}_0(z) = \rho_0(z) + \frac{\lambda}{\lambda - z} \left( G(\lambda)^{-1} T(\lambda) G(\lambda) - G(z)^{-1} T(z) G(z) \right) \]

\[ -\frac{1}{2\pi i} \oint_{\gamma} \frac{w}{w - z} \left[ \frac{1}{w - \sigma(\lambda)} T^*(w) + \frac{1}{w - \lambda} \left( G(w)^{-1} T(w) G(w) \right) \right] dw + \frac{\sigma(\lambda)}{\sigma(\lambda) - z} \left[ T^*(\sigma(\lambda)) - T^*(z) \right], \]

where \(\rho_0 = \rho_0(u - z\overline{v}, v + \overline{u}, z)\) is analytic for \(|z| < 1 + \epsilon\). Finally, we note that, in the region \(\frac{1}{1+\epsilon} < |z| < 1 + \epsilon\) we have

\[ h_0(z) = h^{(0)}_0(z) + h^{(\infty)}_0(z) = \frac{z}{\lambda - z} G(z)^{-1} T(z) G(z) - \frac{\overline{z} \overline{\lambda}}{1 + z\overline{z}} T^*(z) + \rho_0(z). \quad (A.8) \]

Substituting this expression into (A.4) yields (A.7). \(\square\)

**Theorem A.1.** On the region \(\frac{1}{1+\epsilon} < |z| < 1 + \epsilon\) we have

\[ \dot{G}(z) = -T(z)G(z) - G(z)T^*(z) + \rho_\infty(z)G(x, z) + G(x, z)\rho_0(z). \]

**Proof.** The reality conditions for \(\Psi_0\) and \(\Psi_\infty\) imply that \(h_\infty(z) = h^*_0(z)\). The result then follows from Lemma A.1 and Eq. (A.8). \(\square\)

**Remark A.1.** Since the functions \(\rho_0, \rho_\infty\) are holomorphic in \((u - z\overline{v}, v + \overline{u}, z)\) and analytic for \(|z| < 1 + \epsilon, |z| > \frac{1}{1+\epsilon}\), respectively, they simply generate holomorphic changes of basis on these regions. As such, modulo holomorphic changes of basis, the symmetry (2.12) generates the flow

\[ \dot{G}(z) = -T(z)G(z) - G(z)T^*(z) \]

for the patching matrix. Since \(T\) is independent of \(t\), the corresponding one-parameter group of transformations determined by \(T\) with initial conditions the patching matrix \(G_0(x, z)\) is of the form

\[ G(t; x, z) = \exp \left( -t T(x, z) \right) G_0(x, z) \exp \left( -t T^*(x, z) \right). \]

In particular, we recover the group action constructed on heuristic grounds by Crane [9]: Given a map \(h : X \times S^1 \to \text{SL}_2(\mathbb{C})\) that extends to a holomorphic map \(\tilde{h} : X \times \mathcal{V}_\epsilon \to \text{SL}_2(\mathbb{C})\) (where holomorphic means with respect to the complex structure \(X \times \mathcal{V}_\epsilon\) as a subset of \(\mathbb{C} P^3\)) then the group action on the patching matrix is of the form

\[ G(x, z) \mapsto (h \cdot G)(x, z) := \tilde{h}(x, z)G(x, z)\tilde{h}^*(x, z). \]
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Reducible connections and non-local symmetries of the self-dual Yang-Mills equations,

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Reducible Connections and Non-local Symmetries of the Self-dual Yang–Mills Equations

James D. E. Grant

Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria. E-mail: james.grant@univie.ac.at

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To David E. Williams

Abstract: We construct the most general reducible connection that satisfies the self-dual Yang–Mills equations on a simply-connected, open subset of flat $\mathbb{R}^4$. We show how all such connections lie in the orbit of the flat connection on $\mathbb{R}^4$ under the action of non-local symmetries of the self-dual Yang–Mills equations. Such connections fit naturally inside a larger class of solutions to the self-dual Yang–Mills equations that are analogous to harmonic maps of finite type.

1. Introduction

Reducible connections play an important rôle in Donaldson’s study of four-manifold topology [11]. In particular, the singular ends of the moduli space of global solutions of the self-dual Yang–Mills equations on a four-manifold are due to the existence of connections on which the group of gauge transformations (modulo its centre) do not act freely. In the current paper, we study reducible connections from a different point of view, that of integrable systems theory.

In this paper and its companion [13] we investigate the non-local symmetry algebra of the self-dual Yang–Mills equations on $\mathbb{R}^4$ discussed in [7–9] and corresponding group actions on spaces of solutions of the self-dual Yang–Mills equations on open subsets of $\mathbb{R}^4$. In [13] we studied instanton moduli spaces, as explicitly described by the ADHM construction [1,2], and group actions that preserved the $L^2$ nature of the curvature of the connection. In the current work, we investigate reducible connections defined on a simply-connected, open subset of $\mathbb{R}^4$. Given that reducible and irreducible connections play a different rôle in Donaldson’s work, our main motivation was to study whether such connections have different properties from an integrable systems point of view. This does, indeed, seem to be the case.

In the case of instanton solutions on $\mathbb{R}^4$, it was argued in [13] that the symmetry group that acts on the moduli space has orbits of high codimension in the moduli space. (In other words, the orbits are quite small.) Our conclusion for reducible connections, however, is quite different. After explicitly constructing the most general reducible self-dual
connection on a simply-connected, open subset of $\mathbb{R}^4$ (there are no reducible self-dual connections on $\mathbb{R}^4$ with $L^2$ curvature), we deduce that all reducible connections lie in the orbit of the flat connection on $\mathbb{R}^4$ under the action of the non-local symmetry group of the self-dual Yang–Mills equations. Also, the reducible connections lie within a larger class of solutions that arise quite naturally from the symmetries of the self-dual Yang–Mills equations. Solutions in this larger family are determined by a holomorphic function, $T$, defined on an open subset of $\mathbb{C}P^3$ that obeys the condition $[T, T^*] = 0$. Such formulae bear a strong resemblance to those arising in the theory of harmonic maps of finite type (see, e.g., [14, Chap. 24]). This result is distinct from the instanton case discussed in [13], which bears more of a resemblance to the theory of harmonic maps of finite uniton number [25].

The organisation of this paper is as follows. In the following section, we set up notation and recall the non-local symmetries of the self-dual Yang–Mills equations on $\mathbb{R}^4$ constructed in [7–9]. We also recall the main results of [10] concerning the twistorial interpretation of these symmetries in terms of their action on the patching matrix of holomorphic bundles over open subsets of $\mathbb{C}P^3$. In sect. 3, we determine the most general reducible self-dual connection on a simply-connected, open subset of $\mathbb{R}^4$. We show that these may be constructed directly from harmonic functions. We then show the patching matrix of such connections may be constructed directly. In Sect. 4, we deduce that all such patching matrices, and therefore all reducible self-dual connections, lie in the orbit of the flat connection on $\mathbb{R}^4$. In particular, we see there is a larger class of patching matrices that appear quite naturally from the group action of [10] that contains all reducible connections. Analogies between this larger class of solutions and harmonic maps of finite type are briefly investigated in Sect. 5. After some final remarks, in an Appendix we study some properties of a simplified version of the group action of [10].

As in the companion paper [13], we study only the non-local symmetries of the self-dual Yang–Mills equations constructed in [7–9], and not symmetries that require the existence of a non-trivial conformal group on our manifold. We also specialise to the case of SU$_2$ structure group, although the generalisation to any classical Lie group is straightforward.

2. Preliminaries

2.1. The self-dual Yang–Mills equations on $\mathbb{R}^4$. Let $U$ be a connected, simply connected open subset of $\mathbb{R}^4$ with its flat metric. From the Cartesian coordinates $(t, x, y, z)$ on $\mathbb{R}^4$, we define complex coordinates

$$u := t + ix, \quad v := y - iz$$

on $\mathbb{R}^4 \cong \mathbb{C}^2$. In terms of these coordinates, the metric on $\mathbb{R}^4$ is

$$g = \frac{1}{2} (du \otimes d\bar{u} + d\bar{u} \otimes du + dv \otimes d\bar{v} + d\bar{v} \otimes dv),$$

and the standard volume form is

$$\epsilon = dt \wedge dx \wedge dy \wedge dz = \frac{1}{4} du \wedge d\bar{u} \wedge dv \wedge d\bar{v}.$$
Let $\pi : P \to U$ be a principal SU$_2$ bundle over $U$. Since we are, essentially, working locally, a connection on $P$ may be represented by an su$_2$-valued one-form $A \in \Omega^1(U, su_2)$, with curvature $F \in \Omega^2(U, su_2)$. In terms of the complex coordinates above, the connection satisfies the self-dual Yang–Mills equations on $U$ if and only if

\begin{align}
F_{uv} &= 0, \\
F_{u\bar{u}} + F_{v\bar{v}} &= 0, \\
F_{\bar{u}\bar{v}} &= 0.
\end{align}

(2.1a) (2.1b) (2.1c)

Since $U$ is simply-connected, (2.1a) and (2.1c) imply the existence of maps $\psi_0, \psi_\infty : U \to SL_2(\mathbb{C})$ with the property that

\begin{align}
A_u &= - (\partial_u \psi_\infty) \psi_\infty^{-1}, \\
A_v &= - (\partial_v \psi_\infty) \psi_\infty^{-1}, \\
A_{\bar{u}} &= - (\partial_{\bar{u}} \psi_0) \psi_0^{-1}, \\
A_{\bar{v}} &= - (\partial_{\bar{v}} \psi_0) \psi_0^{-1}.
\end{align}

(2.2a) (2.2b)

The fields $\psi_0, \psi_\infty$ are determined by Eqs. (2.2) up to transformations

$\psi_0(x) \mapsto \tilde{\psi}_0(x) := \psi_0(x) R(u, v), \quad \psi_\infty(x) \mapsto \tilde{\psi}_\infty(x) := \psi_\infty(x) S(\bar{u}, \bar{v}),$

where $R, S$ are arbitrary analytic functions of $(u, v), (\bar{u}, \bar{v})$ respectively. We may use this freedom to set, without loss of generality, $\psi_\infty(x) = (\psi_0(x)^{-1})^\dagger, \forall x \in U$. The remaining freedom in the choice of $\psi_0, \psi_\infty$ is then of the form

$\psi_0(x) \mapsto \tilde{\psi}_0(x) := \psi_0(x) R(u, v), \quad \psi_\infty(x) \mapsto \tilde{\psi}_\infty(x) := \psi_\infty(x) \left(R(u, v)^{-1}\right)^\dagger.$

(2.3)

In terms of these fields we define the Yang J-function, $J : U \to SL_2(\mathbb{C})$, by

$J(x) := \psi_\infty^{-1}(x) \cdot \psi_0(x), \quad x \in U.$

It follows from the reality properties of $\psi_0, \psi_\infty$ that $J$ is Hermitian

$J(x) = J(x)^\dagger, \quad x \in U,$

and that, under the transformation (2.3), $J$ transforms according to the rule

$J(x) \mapsto \tilde{J}(x) := R(u, v)^\dagger J(x) R(u, v).$

(2.4)

Substituting into Eq. (2.1b), we find that the connection, $A$, satisfies the self-dual Yang–Mills equations if and only if $J$ satisfies the two (equivalent) versions of the Yang–Pohlmeyer equation

\begin{align}
\partial_u \left(J_{\bar{u}} J^{-1}\right) + \partial_v \left(J_{\bar{v}} J^{-1}\right) &= 0, \\
\partial_{\bar{u}} \left(J^{-1} J_u\right) + \partial_{\bar{v}} \left(J^{-1} J_v\right) &= 0.
\end{align}

(2.5a) (2.5b)
2.2. Associated linear problem. Let $\Omega_1$ be a non-empty, open subset of $\mathbb{CP}^1 := \mathbb{C} \cup \{\infty\}$, and consider the following overdetermined system of equations for a map $\Psi: U \times \Omega \to \text{SL}_2(\mathbb{C})$

$$
(\partial_\tau + z \partial_u) \Psi(x, z) = -(A_\tau + z A_u) \Psi(x, z), \quad (2.6a)
$$
$$
(\partial_\bar{\tau} - z \partial_v) \Psi(x, z) = -(A_\bar{\tau} - z A_v) \Psi(x, z), \quad (2.6b)
$$
$$
\partial_z \Psi(x, z) = 0. \quad (2.6c)
$$

An important property of the self-dual Yang–Mills equations is that they are the integrability condition for this system. In particular, if the connection $A$ satisfies the self-dual Yang–Mills equations on $U$, then there exists $\epsilon > 0$ and a solution $\Psi_0: U \times \mathcal{V}_\epsilon^0 \to \text{SL}_2(\mathbb{C})$ of this system that is analytic in $z$ for $z \in \mathcal{V}_\epsilon^0$, where $
\mathcal{V}_\epsilon^0 := \{z \in \mathbb{CP}^1 \mid |z| < 1 + \epsilon \}.$

**Notation.** We define the involution $\sigma: \mathbb{CP}^1 \to \mathbb{CP}^1$ by $\sigma(z) = -1/z$. Given a subset $\mathcal{V} \subset \mathbb{CP}^1$ and a map $g: \mathcal{V} \to \text{SL}_2(\mathbb{C})$, we define a corresponding map $g^*: \sigma(\mathcal{V}) \to \text{SL}_2(\mathbb{C})$ by

$$
g^*(z) := (g(\sigma(z)))^\dagger.
$$

Similarly, given any map $f: U \times \mathcal{V} \to \text{SL}_2(\mathbb{C})$, we define a corresponding map $f^*: U \times \sigma(\mathcal{V}) \to \text{SL}_2(\mathbb{C})$ by

$$
f^*(x, z) := (f(x, \sigma(z)))^\dagger.
$$

Given the solution, $\Psi_0$, of (2.6), we may now construct another solution $\Psi_\infty: U \times \mathcal{V}_\infty^\infty \to \text{SL}_2(\mathbb{C})$, where $\mathcal{V}_\infty^\infty := \{z \in \mathbb{CP}^1 \mid |z| > \frac{1}{1+\epsilon} \}$, by $\Psi_\infty(x, z) := \Psi_0^*(x, z)^{-1}$. The solution $\Psi_\infty$ is analytic in $z$ for $z \in \mathcal{V}_\epsilon^\infty$.

**Remark 2.1.** Note that, for the construction of the connection in Eq. (2.2), we may take $\psi_0(x) := \Psi_0(x, 0)$ and $\psi_\infty(x) := \Psi_\infty(x, 0)$. We will assume, from now on, that $\psi_0$ and $\psi_\infty$ are defined in this way.

**Definition 2.1.** The patching matrix (or clutching function, in Uhlenbeck’s terminology [25]), $G: U \times \mathcal{V}_\epsilon \to \text{SL}_2(\mathbb{C})$ is defined by

$$
G(x, z) = \Psi_\infty(x, z)^{-1} \cdot \Psi_0(x, z), \quad (2.7)
$$

where $\mathcal{V}_\epsilon := \mathcal{V}_\epsilon^0 \cap \mathcal{V}_\infty^\infty = \{z \in \mathbb{CP}^1 \mid \frac{1}{1+\epsilon} < |z| < 1 + \epsilon \}.$

**Remark 2.2.** Viewing $U \times \mathcal{V}_\epsilon$ as a subset of $\pi^{-1}(U) \subseteq \mathbb{CP}^3$, the patching matrix is the transition function of the holomorphic vector bundle over $\mathbb{CP}^3$ corresponding to our self-dual connection $A$ [3,27]. Since $U \times \mathcal{V}_\epsilon^0$ and $U \times \mathcal{V}_\infty^\infty$ are open subsets of $\mathbb{CP}^3$, any holomorphic bundle over them is trivial. As such, the bundle over $\pi^{-1}(U)$ is completely determined by the patching matrix $G$ (see, e.g., [10]). The fact that the patching matrix splits as in (2.7) implies that the bundle is trivial on restriction to a line $\pi^{-1}(x)$, for each $x \in U$ [27]. Since the patching matrix obeys the reality condition $G(t, z) = G^*(t, z)$, the bundle admits a Hermitian structure, and the corresponding self-dual connection is an $\text{SU}_2$ connection, rather than an $\text{SL}_2(\mathbb{C})$ connection.
2.3. Non-local symmetries. In order to study symmetries of the self-dual Yang–Mills equations, we let \( J(\cdot, s): U_s \rightarrow \text{SL}_2(\mathbb{C}) \) be a one-parameter family of solutions of the Yang–Pohlmeyer equations (2.5). Here, \( s \in I \) with \( I \) an open interval in \( \mathbb{R} \) containing the origin, \( J \) is assumed to depend in a \( C^1 \) fashion on the parameter \( s \), and \( U_s \subseteq \mathbb{R}^4 \) is the open subset of \( \mathbb{R}^4 \) on which the solution is well defined (i.e. non-singular). Taking the derivative with respect to \( s \) of (2.5), we find that \( J(\cdot, s) \) must satisfy the linearised equation

\[
\partial_u \left( J \frac{\partial}{\partial \tau} \left( J^{-1} \dot{J} \right) J^{-1} \right) + \partial_v \left( J \frac{\partial}{\partial \sigma} \left( J^{-1} \dot{J} \right) J^{-1} \right) = 0, 
\]

where \( \dot{J} := \partial J / \partial s \). It is known that the only local symmetries of the self-dual Yang–Mills equations on flat \( \mathbb{R}^4 \) are gauge transformations and those generated by the action of the conformal group (see, e.g., [22]). However, there exists a non-trivial family of non-local symmetries of the self-dual Yang–Mills equations [7–9], defined as follows. We define maps \( \chi_0: U \times V_\epsilon^0 \rightarrow \text{SL}_2(\mathbb{C}), \chi_\infty: U \times V_\epsilon^\infty \rightarrow \text{SL}_2(\mathbb{C}) \) by the relations

\[
\chi_0(x, z) := \psi_0(x)^{-1} \cdot \Psi_0(x, z), \quad (x, z) \in U \times V_\epsilon^0,
\]

\[
\chi_\infty(x, z) := \psi_\infty(x)^{-1} \cdot \Psi_\infty(x, z), \quad (x, z) \in U \times V_\epsilon^\infty.
\]

\( \chi_0 \) is analytic in \( z \) for all \( z \in V_0 \), with \( \chi(x, 0) = \text{Id} \), for all \( x \in U \), and is a solution of the system

\[
\begin{align*}
\left( \partial_\tau - J J^{-1} \right) \chi_0(x, z) & = 0, \\
\left( \partial_\sigma - J J^{-1} \right) \chi_0(x, z) & = 0,
\end{align*}
\]

for all \( (x, z) \in U \times V_\epsilon^0 \). Similarly, \( \chi_\infty \) is analytic in \( z \) for all \( z \in V_\epsilon^\infty \), with \( \chi_\infty(x, \infty) = \text{Id} \), for all \( x \in U \). Note that we have

\[
\chi_\infty(x, \lambda) = (\chi_0(x, \sigma(\lambda)))^{-\dagger}, \quad \text{for all } (x, \lambda) \in U \times V_\epsilon^\infty.
\]

Based on the work of [7–9], we have the following result from [10]:

**Proposition 2.1.** Let \( T: U \times V_\epsilon \rightarrow \mathfrak{sl}_2(\mathbb{C}) \) obey the relations

\[
(\partial_\tau + \lambda \partial_\sigma) T(x, \lambda) = (\partial_\tau - \lambda \partial_\sigma) T(x, \lambda) = 0,
\]

and be analytic in \( \lambda \) on a neighbourhood, \( V_\epsilon \), of the unit circle \( S^1 \subset \mathbb{C} \). Then

\[
\dot{J}(x, s) = \chi_\infty(x, \lambda) T(x, \lambda) \chi_\infty(x, \lambda)^{-1} \cdot J + J \cdot \chi_0(x, \sigma(\lambda))^\dagger T(x, \lambda)^\dagger \chi_0(x, \sigma(\lambda))^{-1}
\]

\[
= \psi_\infty(x)^{-1} \left[ \Psi_\infty(x, \lambda) T(x, \lambda) \Psi_\infty(x, \lambda)^{-1} + \Psi_0(x, \sigma(\lambda))^\dagger \Psi_0(x, \sigma(\lambda))^{-1} \right] \psi_0(x) 
\]

(2.9)

is a solution of the linearisation equation (2.8), for all \( x \in U \), and all \( \lambda \in V_\epsilon \).

\(^1\) We will often notationally suppress the dependence of the domain \( U \) on \( s \).
Remark 2.3. In the case where the function $T$ is independent of $(u, v)$, it defines an element of the loop group $\Lambda \text{SL}_2(\mathbb{C})$ with a holomorphic extension to an open neighbourhood of $S^1$ in $\mathbb{C}^*$. The algebra of symmetries generated by such $T$ is then isomorphic to the Kac-Moody algebra of $\mathfrak{sl}_2(\mathbb{C})$ [7–9].

The natural question is how to exponentiate the above algebra into a group action on the space of solutions of the self-dual Yang–Mills equations. A solution to this problem is given by the following result

**Theorem 2.1.** [10, Chap. IV.C]. Let $g : X \times S^1 \to \text{SL}_2(\mathbb{C})$ be a smooth map that admits a continuous extension to a holomorphic map $g : X \times \mathcal{V}_\epsilon \subset \mathbb{C}P^3 \to \text{SL}_2(\mathbb{C})$, for some $\epsilon > 0$. Then the action of $g$ on the patching matrix, $G(x, z)$, is defined by

$$G(x, z) \mapsto (g \cdot G)(x, z) := g(x, z) \cdot G(x, z) \cdot g^*(x, z). \quad (2.10)$$

This equation defines an action of the set of such maps $g$ on the space of self-dual connections on $X$. If $g$ extends holomorphically to $z \in \mathcal{V}_\epsilon^0$, then the corresponding transformation is a holomorphic change of basis on the bundle over $\pi^{-1}(X)$, which leaves the self-dual connection, $A$, unchanged.

Remark 2.4. The above group action on the solution space have been given a cohomological description by Park (see [20] and references therein), which has been further investigated by Popov and Ivanova (see [17,18,22] and references therein).

Remark 2.5. The group action (2.10) is slightly unusual since, in integrable systems theory, it is usually adjoint or coadjoint orbits of Lie groups that turn out to be relevant. If we consider the case where $G$ and $g$ are constant, and study the action of $\text{SL}_2(\mathbb{C})$ on itself by $(g, G) \mapsto g \cdot G := g G g^\dagger$, then the generic orbits of this action are five-dimensional. As such, the orbits do not carry the invariant symplectic structures that one would associate with, for example, coadjoint orbits. A brief investigation of this orbit structure is given in Appendix A.

The connection between Theorem 2.1 and the transformation (2.9) is given by the following:

**Theorem 2.2.** Given $T : U \times \mathcal{V}_\epsilon \to \mathfrak{sl}_2(\mathbb{C})$ as in Proposition 2.1, the corresponding flow on the space of patching matrices is given by

$$\dot{G}(x, z) = -T(x, z)G(x, z) - G(x, z)T^*(x, z) + \rho_\infty(x, z)G(x, z) + G(x, z)\rho_0(x, z) \quad (2.11)$$

for $(x, z) \in \mathbb{R}^4 \times \mathcal{V}_\epsilon$. In this equation, $\rho_0 : \mathbb{R}^4 \times \mathcal{V}_\epsilon^0 \to \mathfrak{sl}_2(\mathbb{C})$ and $\rho_\infty : \mathbb{R}^4 \times \mathcal{V}_\epsilon^\infty \to \mathfrak{sl}_2(\mathbb{C})$ are analytic functions of $z$ on the respective regions and satisfy

$$(\partial_\tau + z\partial_v) \rho_0(x, z) = (\partial_\tau - z\partial_v) \rho_0(x, z) = 0,$n

$$(\partial_\tau + z\partial_v) \rho_\infty(x, z) = (\partial_\tau - z\partial_v) \rho_\infty(x, z) = 0.$n

Remark 2.6. It follows from a similar argument to that given in the proof of Proposition 1 (b) of [10] that the terms $\rho_0$ and $h_\infty$ in the above formula may be absorbed into a change of holomorphic frame on the sets $z \in \mathcal{V}_\epsilon^0$ and $\mathcal{V}_\epsilon^\infty$, respectively.

Remark 2.7. The group action (2.10) was derived in [10], arguing by analogy with the action of dressing transformations in harmonic map theory. It has been investigated further in, for example, [17,18,22]. The first direct derivation of the infinitesimal result (2.11) from the generator (2.9), to my knowledge, appears in [13].
3. Reducible Connections

Recall [12, Chap. 3] that a connection, $D$, on an $SU_2$ bundle $\pi : E \to U$ is reducible if the group of gauge transformations $G := C^\infty(U, SU_2)$ modulo its centre does not act freely on the connection $D$. We now proceed to derive the most general form of a reducible connection on a simply-connected, open subset of $\mathbb{R}^4$. In doing so, we make extensive use of the classical Pauli matrices, which we define as follows:

$$\tau \equiv (\tau_1, \tau_2, \tau_3) = \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

**Proposition 3.1.** Let $U$ be a simply-connected, open subset of $\mathbb{R}^4$, $a \in C^\infty(U, \mathbb{R})$ a harmonic function. We define the connection

$$A = \frac{1}{2} (\partial a - \bar{\partial} a^*) \tau_3 \in \Omega^1(U, su_2).$$

(3.1)

Then the connection, $A$, is reducible and satisfies the self-dual Yang–Mills equations on $U$. Conversely, up to gauge transformation, all reducible self-dual connections on $U$ arise in this way.

**Proof.** A connection is reducible if and only if there exists a parallel section, $\eta$, of the adjoint bundle $\text{ad} su_2$ [12, Theorem 3.1]. In terms of local coordinates $(u, v) \in \mathbb{C}^2 \cong \mathbb{R}^4$, and relative to a local trivialisation of the bundle $E$, this condition implies that

$$\frac{\partial}{\partial x^a} \eta + [A_a, \eta] = 0.$$ 

It follows from Eqs. (2.2) that, for a parallel section $\eta$, the map $A : \mathbb{C}^2 \to sl_2(\mathbb{C})$ defined by the equation $\eta = \psi_0(x) A(u, v) \psi_0(x)^{-1}$ is holomorphic. In a similar fashion, using (2.2) and the relationship between $\psi_0$ and $\psi_\infty$, one may verify that $\eta = -\psi_\infty(x) A(u, v)^* \psi_\infty(x)^{-1}$. These relations imply that

$$JA(u, v) + A(u, v)^* J = 0.$$ 

(3.2)

Note that this equation and the fact that $det J \neq 0$ implies that $det A(u, v) = det A(u, v)^*$. Therefore $det A(u, v)$ is real. Since $A$ is holomorphic in $u$ and $v$, it follows that $det A$ is a real constant.

Let $A = (R + i I) \cdot \tau$, with $R, I : U \to \mathbb{R}^3$. The fact that $A$ depends holomorphically on $(u, v)$ implies that

$$\partial_x R = \partial_y I, \quad \partial_y R = -\partial_x I, \quad \partial_y R = -\partial_z I, \quad \partial_z R = \partial_y I.$$ 

(3.3)

Moreover, we find that

$$det A = |I|^2 - |R|^2 - 2i (R, I).$$

Since $det A$ is real, we deduce that $(R, I) = 0$. We now let $J = \Lambda + \lambda \cdot \tau$, with $\lambda : U \to \mathbb{R}^3$ and the condition that $det J = 1$ implies that we require $\Lambda^2 = 1 + |\lambda|^2$. Imposing (3.2), we find that we require

$$\Lambda R = \lambda \times I.$$ 

(3.4)
This equation implies that
\[ \Lambda^2 |R|^2 = |\lambda \times I|^2 = |\lambda|^2 |I|^2 - \langle I, \lambda \rangle^2. \]
Therefore
\[ \det A = |I|^2 - |R|^2 = \frac{1}{|\lambda|^2} \left( \Lambda^2 |R|^2 + \langle I, \lambda \rangle^2 \right) - |R|^2 \]
\[ = \frac{1}{|\lambda|^2} \left( |R|^2 + \langle I, \lambda \rangle^2 \right) \geq 0. \]

For \( \lambda \neq 0 \), equality occurs in this inequality if and only if \( R = 0 \) and \( \langle I, \lambda \rangle = 0 \). From (3.4), it follows that, in this case, \( R = I = 0 \), so \( A = 0 \). Moreover, if \( \lambda = 0 \), then (3.2) implies that \( R = 0 \), so \( \det A = |I|^2 \), which is, again, strictly positive unless \( I = 0 \) and, hence, \( A = 0 \).

To summarise, the fact that \( \det A \) is real, along with (3.2), implies that \( \det A \) is a non-negative constant. Moreover, \( \det A = 0 \) if and only if \( A = 0 \). Since we are assuming \( A \neq 0 \) and since any constant multiple of a parallel section is also parallel we may, without loss of generality, assume that \( \det A = 1 \). In this case, it follows that the eigenvalues of \( A \) are \( \pm i \).

Note that we still have the freedom to rotate the \( \psi \)’s, as given in (2.3). It follows that \( A \) transforms under the adjoint action of \( R^{-1} \):
\[ A(u, v) \mapsto \tilde{A}(u, v) := R(u, v)^{-1} A(u, v) R(u, v) = \text{Ad}_{R^{-1}} A. \]
(3.5)

We now write \( A \) in the form
\[ A(u, v) = \begin{pmatrix} a(u, v) & b(u, v) \\ c(u, v) & -a(u, v) \end{pmatrix}, \]
where \( a, b, c \) are holomorphic functions of \( (u, v) \). On a neighbourhood of any point \( (u, v) \) at which \( a(u, v) \neq -i \), the holomorphic change of frame
\[ R_+(u, v) = \begin{pmatrix} a + i & b \\ c & a + i \end{pmatrix} \]
has the property that
\[ R_+(u, v)^{-1} A(u, v) R_+(u, v) = i \tau_3. \]
Similarly, for \( a(u, v) \neq +i \),
\[ R_-(u, v) = \begin{pmatrix} -b & a - i \\ a - i & c \end{pmatrix} \]
gives a holomorphic change of frame with the property that
\[ R_-(u, v)^{-1} A(u, v) R_-(u, v) = i \tau_3. \]
As such, given any point \( p \in X \), there exists a neighbourhood of \( p \) and a holomorphic frame such that \( A = i \tau_3 \) in that frame.
From (3.2), it follows that there exist real functions $\alpha, \beta$ such that $J = \alpha \text{Id} + \beta \tau_3$. Since $\det J = 1$, we have $\alpha^2 = 1 + \beta^2$. Since $J$ is continuous, $\alpha$ will have constant sign, so we assume that $\alpha > 0$. Therefore, since $U$ is assumed simply-connected, we may consistently define a real-valued function $a$ with the property that $\alpha = \cosh a, \beta = \sinh a$. It then follows that

$$J = \exp (a \tau_3).$$

(3.6)

We therefore have

$$J^{-1} J_u = a_u \tau_3, \quad J^{-1} J_v = a_v \tau_3.$$

Imposing the Yang–Pohlmeyer equation implies that $a$ is harmonic:

$$(\partial_u \partial_\tau + \partial_v \partial_\tau) a = 0.$$

From (3.6), we see that, up to a gauge transformation, we may take

$$\psi_0 = \exp \left( \frac{a}{2} \tau_3 \right), \quad \psi_\infty = \exp \left( - \frac{a}{2} \tau_3 \right).$$

The form of the connection given in Eq. (3.1) then follows from Eq. (2.2). The parallel section, $\eta$, is equal to $i \tau_3$. \quad \Box

**Example 1.** The case

$$J(x) = \exp \left\{ (|u|^2 - |v|^2) \tau_3 \right\}$$

corresponds to $a = |u|^2 - |v|^2$ and, therefore, defines a reducible connection. In this case, the connection is non-singular on $\mathbb{R}^4$. However, the curvature is not $L^2$, and therefore the connection cannot be extended to $S^4$ [26].

In this particular case the connection is algebraically special, in the sense that, in addition to satisfying the self-dual Yang–Mills equations (2.1), the curvature satisfies

$$F_{u\bar{v}} = 0, \quad F_{v\bar{u}} = 0.$$

It can be shown that all algebraically special connections arise in this way, and are thus reducible.

Recall [16] that there is a $1-1$ correspondence between harmonic functions on $U \subseteq \mathbb{R}^4$ and sheaf cohomology classes in $H^1(\mathbb{C}P^3, \mathcal{O}(-2))$, where $\mathbb{C}P^3$ is a subset of $\mathbb{C}P^3$. In the present case, such a cohomology class may be represented by a holomorphic function\(^2\) $f : U \times \mathbb{C}^* \to \mathbb{C}$. In terms of homogeneous coordinates on $\mathbb{C}P^3$, we have $f(\lambda z) = \lambda^{-2} f(z)$. The corresponding harmonic function on $U \subseteq \mathbb{R}^4$ is then given by the contour integral

$$a(x) = \frac{1}{2\pi i} \oint_\gamma f(u - w\bar{v}, v + w\bar{u}, w) dw,$$

(3.7)

where $\gamma := \{ w \in \mathbb{C} \subseteq \mathbb{C}P^1 : |w| = 1 \}$.\(^2\) By holomorphic, we mean with respect to the complex structure induced on $U \times \mathbb{C}^*$ as a subset of $\mathbb{C}P^3$.\(^2\)
Proposition 3.2. Given the connection as in Proposition 3.1, the patching matrix for the holomorphic bundle on \( \hat{U} \) may be taken as

\[
G(x, z) = \exp \left[ \frac{1}{2} \left( F(x, z) + F^*(x, z) \right) \tau_3 \right],
\]

where

\[
F(x, z) := \frac{1}{2\pi i} \oint_{\gamma} \frac{w + z}{w - z} f(u - w\overline{v}, v + w\overline{u}, w) dw,
\]

and the holomorphic function \( f \) is a representative of the cohomology class in \( H^1(\hat{U}, \mathcal{O}(-2)) \) corresponding to the harmonic function \( a \).

Proof. We assume that there exists \( \Psi_0(x, z) \) of the form \( \exp \left( \frac{1}{2} F(x, z) \tau_3 \right) \), with \( F(x, 0) = a(x) \). From (2.6) for \( \Psi \) and the explicit form of the connection, we deduce that \( F \) must be analytic in \( z \) and satisfy the relations

\[
(\partial_{\overline{\pi}} - z\partial_v) F(x, z) = (\partial_{\overline{\pi}} + z\partial_v) a(x), \quad (\partial_{\overline{\pi}} + z\partial_u) F(x, z) = (\partial_{\overline{\pi}} - z\partial_u) a(x).
\]

From (3.7) we then calculate

\[
(\partial_{\overline{\pi}} + z\partial_v) a(x) = (\partial_{\overline{\pi}} + z\partial_v) \frac{1}{2\pi i} \oint_{\gamma} f(u - w\overline{v}, v + w\overline{u}, w) dw
\]

\[
= \frac{1}{2\pi i} \oint_{\gamma} (w + z) (\partial_2 f) (u - w\overline{v}, v + w\overline{u}, w) dw
\]

\[
= \frac{1}{2\pi i} \oint_{\gamma} \frac{w + z}{w - z} (w - z) (\partial_2 f) (u - w\overline{v}, v + w\overline{u}, w) dw
\]

\[
= \frac{1}{2\pi i} \oint_{\gamma} \frac{w + z}{w - z} (\partial_{\overline{\pi}} - z\partial_v) f(u - w\overline{v}, v + w\overline{u}, w) dw
\]

\[
= (\partial_{\overline{\pi}} - z\partial_v) \frac{1}{2\pi i} \oint_{\gamma} \frac{w + z}{w - z} f(u - w\overline{v}, v + w\overline{u}, w) dw.
\]

Along with a similar calculation for \( (\partial_{\overline{\pi}} - z\partial_u) a(x) \), we have the above candidate for \( F \) given in (3.8). By its definition as a contour integral, it follows that \( F \) is analytic in \( z \), for \( z \) in the interior of \( \gamma \). Since the function \( \frac{w + z}{w - z} f(u - w\overline{v}, v + w\overline{u}, w) \) is continuous for \( w \in \gamma \), and \( \gamma \) is compact, the Dominated Convergence Theorem implies that \( \lim_{z \to 0} F(x, z) = a(x) \). As such, \( F \) has all of the required properties. \( \square \)

Remark 3.1. Note that the patching matrix in Proposition 3.2 takes the form \( G(x, z) = \exp (\varphi(x, z) \tau_3) \), where \( \varphi \) is a holomorphic function of \( (u - z\overline{v}, v + z\overline{u}, z) \) that satisfies \( \varphi^*(x, z) = \varphi(x, z) \).

Example 2. In the case of our algebraically special connection, where \( a = |u|^2 - |v|^2 \), we may take \( f(x, z) = \frac{1}{z} (u - z\overline{v})(v + z\overline{u}) \). We then find that \( F(x, z) = |u|^2 - |v|^2 - 2u\overline{v} \) and \( G(x, z) = \exp \left[ \frac{1}{z} (u - z\overline{v})(v + z\overline{u}) \tau_3 \right] \).
4. Orbit of the Flat Connection

Let $T : U \times S^1$ be analytic on a neighbourhood of $U \times S^1$ in $U \times \mathbb{C}^* \subset \mathbb{C}P^3$, with the property that

$$[T, T^*] = 0. \quad (4.1)$$

Given a solution of the self-dual Yang–Mills equations, described by patching matrix $G(x, z)$, we may consider the one-parameter family of connections generated by $T$ via the flow (2.11). Since the functions $\rho_0, \rho_\infty$ can be removed by a holomorphic change of basis on the regions $U \times V_0^\epsilon$ and $U \times V_\infty^\epsilon$ we may, without loss of generality, fix the holomorphic bases by setting $\rho_0 = \rho_\infty = 0$. The unique solution of (2.11) with initial conditions $G(x, z)$ is then

$$G_t(x, z) = \exp(-tT(x, z)) G(x, z) \exp(-tT(x, z)^*).$$

In general, one would perform a Birkhoff splitting of $G_t$, which would yield connections $A_t$ generated from the connection, $A$, corresponding to the patching matrix $G$. (Generally, there will be jumping points at which $G$ does not admit a splitting of the form (2.7), so we will need to shrink the set $U$ accordingly. The set of such points will, generically, be of strictly positive codimension in $U$.)

A case of particular interest to us is when the initial connection is flat, in which case we may take $G(x, z) = 1$. We then have

$$G_t(x, z) = \exp(-t \left(T(x, z) + T(x, z)^*\right))$$

as the patching matrix of connections that lie in this orbit of the flat connection. As a special case of this construction, letting $T = -\frac{1}{2}\phi(x, z)\tau_3$, where $\phi : U \times V_\epsilon \to \text{SL}_2(\mathbb{C})$ satisfies

$$(\partial_{\bar{u}} - z\partial_v) \phi = (\partial_{\bar{u}} + z\partial_u) \phi = 0$$

and is analytic in $z \in V_\epsilon$ for some $\epsilon > 0$. Then, assuming that $\epsilon$ is chosen sufficiently small that $G(x, z)$ is analytic on $U \times V_\epsilon$, we deduce that

$$G_t(x, z) = \exp(t\phi(x, z)\tau_3), \quad (x, z) \in U \times V_\epsilon.$$  

In light of this construction, and the classification of patching matrices arising from reducible connections in the previous section, we deduce:

Theorem 4.1. Let $A$ be a reducible connection on an open subset $U \subset \mathbb{R}^4$. Then $A$ lies on the orbit of the flat connection on $U$ under the action of the non-local symmetry group of the self-dual Yang–Mills equations.

Remark 4.1. As mentioned earlier, the set $U$ on which the self-dual connection is defined will generally shrink under the action of the symmetry group. In the case of reducible connections, where one may start from the flat connection on $\mathbb{R}^4$, then there do exist reducible connections defined on the whole of $\mathbb{R}^4$. (Our algebraically special connection is an example of such.) Since there are no non-trivial reducible connections on $S^4$, however, Uhlenbeck’s theorem [26] implies that the curvature of such a connection cannot be $L^2$. (This property may also be checked directly from the explicit form of the connection.)
Remark 4.2. Takasaki [24] has argued that, if we drop the reality conditions on self-dual connections and consider $\text{SL}_2(\mathbb{C})$ connections rather than $\text{SU}_2$ ones, then the group action generated by transformations of the form

$$J(x, s) = \chi_\infty(x, \lambda) T(x, \lambda) \chi_\infty(x, \lambda)^{-1} \cdot J$$  \hspace{1cm} (4.2)

is transitive on the space of local solutions of the $\text{SL}_2(\mathbb{C})$ self-dual Yang–Mills equations. In the current context, we are explicitly restricting ourselves (via the form of Eq. (2.9)) to transformations that preserve the $\text{SU}_2$ nature of the connection, in which case there is no reason to believe that the group action should be transitive. In an analogous situation in the theory of harmonic maps into Lie groups, one can show that transformations analogous to (4.2) map real extended harmonic maps to real harmonic maps if and only if the action is trivial [4, Prop. 3.4] (i.e. $g \cdot \Phi = \Phi$). It is, similarly, expected that transformations of the form (4.2) will map $\text{SU}_2$ connections to $\text{SU}_2$ connections if and only if the connections coincide.

5. Harmonic Maps of Finite Type

It is clear from the discussion in the previous section that the reducible connections on a simply-connected, open subset $U \subseteq \mathbb{R}^4$ are a special case of a more general type of connection. In particular, given a map $T : U \times \mathcal{V}_\epsilon \rightarrow \text{SL}_2(\mathbb{C})$ satisfying the commutator condition (4.1), then the patching matrix

$$G(x, z) = \exp \left( T(x, z) + T(x, z)^* \right)$$ \hspace{1cm} (5.1)

will generate a solution of the self-dual Yang–Mills equations on a subset of $U$.

The forms of condition (4.1) and the patching matrix (5.1) are reminiscent of formulæ that appear when one considers harmonic maps of finite type into Lie groups (see, e.g., [14, Chap. 24] and [5,6] for harmonic maps into $k$-symmetric spaces). Recall that, in this context, we consider Lie groups $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ such that $\mathcal{G} = \mathcal{G}_1 \cdot \mathcal{G}_2$ (in the sense that, given $g \in \mathcal{G}$, there exist unique $g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2$ such that $g = g_1 g_2$). At the Lie algebra level, we have a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$, and we denote the projections onto the two summands by $\pi_1, \pi_2$. In the case where $\text{Ad}_{\mathcal{G}_1} \mathfrak{g}_2 \subseteq \mathfrak{g}_2$, then various Lax flows on $\mathfrak{g}$ can be solved explicitly. In particular, let $\mathbf{J}_1, \mathbf{J}_2$ be invariant vector fields on $\mathfrak{g}^3$ and consider the Lax equations

$$\begin{align*}
\partial_s X(s, t) &= [X(s, t), (\pi_1 \circ \mathbf{J}_1) (X(s, t))] , \\
\partial_t X(s, t) &= [X(s, t), (\pi_1 \circ \mathbf{J}_2) (X(s, t))] ,
\end{align*} \hspace{1cm} (5.2a, b)$$

for a map $X : \mathbb{R}^2 \mapsto \mathfrak{g}$ with initial conditions

$$X(0, 0) = \mathbf{V} \in \mathfrak{g}.$$ 

These equations are compatible, and the solution to this problem may be written in the form

$$X(s, t) = \text{Ad}_{F(s, t)^{-1}} \mathbf{V},$$

where $F : \mathbb{R}^2 \rightarrow \mathcal{G}_1$ takes the form

$$F(s, t) = \exp \left( s (\pi_1 \circ \mathbf{J}_1)(\mathbf{V}) + t (\pi_1 \circ \mathbf{J}_2)(\mathbf{V}) \right), \hspace{1cm} (s, t) \in \mathbb{R}^2.$$ 

3 I.e. $\mathbf{J}_1, \mathbf{J}_2 : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfy $\mathbf{J}_1(\text{Ad}_g \mathbf{v}) = \text{Ad}_g \mathbf{J}_1(\mathbf{v})$ for all $g \in \mathcal{G}$ and $\mathbf{v} \in \mathfrak{g}$ and similarly for $\mathbf{J}_2$. 

\[\chi_\infty(x, \lambda) T(x, \lambda) \chi_\infty(x, \lambda)^{-1} \cdot J \]
The connection with harmonic maps arises if we let $G$ be a compact Lie group and use the standard loop-group decompositions (see, e.g., [14, Chap. 12] and [23, Chap. 8]) to take $G := \Lambda G^C, G_1 := \Omega G, G_2 := \Lambda_+ G^C$. If we now impose that the initial conditions for the Lax equation (5.2) correspond to an element of the loop group of $G$ (rather than $G^C$) and that their Laurent expansion lies between degrees $-d$ and $d$:

$$V = \left[ \lambda \mapsto f(\lambda) \equiv \sum_{n=-d}^{d} \alpha_n \lambda^n \right] \in \Lambda G^C,$$

then it turns out that the map $X$ also has Laurent expansion that lies between degrees $-d$ and $d$. Moreover, $F : \mathbb{R}^2 \to \Omega G$ is automatically the extended solution corresponding to a harmonic map $\varphi : \mathbb{R}^2 \to G$. (In particular, $\varphi(s, t) = F(s, t)|_{\lambda=-1}$.) Such harmonic maps are of finite type.

In the context of self-dual Yang–Mills connections, the analogue of harmonic maps of finite type for self-dual Yang–Mills fields would appear to be patching matrices of the form

$$G(x, z) = \exp \Phi(x, z),$$

where $\Phi : U \times \mathcal{V}_\epsilon \to \text{SL}_2(\mathbb{C})$ satisfies

1) $(\partial_\pi - z \partial_u) \Phi(x, z) = (\partial_\pi + z \partial_u) \Phi(x, z) = 0;$
2) $\Phi^*(x, z) = \Phi(x, z)$
3) $\Phi(x, z)$ is analytic in $z$ for $z \in \mathcal{V}_\epsilon$ for some $\epsilon > 0$, and there exists $d \in \mathbb{N}_0$ such that $\Phi$ has a finite Laurent expansion of the form

$$\Phi(x, z) = \sum_{n=-d}^{d} a_n(x) z^n, \quad (x, z) \in U \times \mathcal{V}_\epsilon,$$

for some $a_i : U \to g^C, i = -d, \ldots, d$ on this set.

In particular, fixing a point $p \in U$, then the finite Laurent expansion at $p$ is analogous to the initial condition $V$ having finite Laurent expansion. Moreover, Condition 1) above is then the analogue of the Lax equations (5.2) satisfied by the map $X$ in the harmonic map case.

**Definition 5.1.** We will call a solution of the self-dual Yang–Mills equations for which there exists a patching matrix that satisfies the above criteria a self-dual connection of finite type$^4$.

**Remark 5.1.** The conditions above imply that the maps $a_i$ satisfy the conditions

$$\begin{align*}
\partial_\pi a_{-d} &= 0, & \partial_\pi a_{-d} &= 0, \\
\partial_\pi a_{n+1} &= \partial_\nu a_n, & \partial_\pi a_{n+1} &= -\partial_\nu a_n, & n &= -d, \ldots, d-1, \\
\partial_\nu a_d &= 0, & \partial_\nu a_d &= 0.
\end{align*}$$

(5.3a)

(5.3b)

(5.3c)

For $d = 0$, we deduce that $G$ is constant, and therefore the corresponding self-dual connection $A$ is flat. For $d \geq 1$, the algebraic condition (4.1) imposes non-trivial restrictions on the coefficients $a_i$. Note that our algebraically special connection is a self-dual connection of type 1.

$^4$ Or, of type $d$, when we wish to be more specific
Remark 5.2. Let $A$ be a reducible connection defined by a harmonic function $a$ as in (3.1). Letting $a_0(x) := a(x)$ then $A$ is a self-dual connection of finite type if and only if there exists $d > 0$ and functions $a_{-d}, \ldots, a_d$ such that Eqs. (5.3) hold. In particular, this condition implies that

$$\frac{\partial^d a}{\partial u^r \partial \bar{u}^s} = 0,$$

for all $r, s$ such that $r + s = d$.

Therefore, $\Phi$ is necessarily a polynomial of degree less than or equal to $d$ in $(u, \bar{u}, v, \bar{v})$. As such, the space of self-dual connections of type $d$ is necessarily finite-dimensional.

Remark 5.3. The most restrictive case is when we impose that $G$ splits in the form (2.7) with

$$\Psi_0(x, z) = \exp \left( \frac{a_0(x)}{2} + \sum_{n=1}^{d} a_n(x) z^n \right),$$

$$\Psi_\infty(x, z) = \exp \left( -\frac{a_0(x)}{2} - \sum_{n=-d}^{-1} a_n(x) z^n \right).$$

Such a splitting only occurs if $[a_i(x), a_j(x)] = 0$, for all $i, j = -d, \ldots, d$. Since $\text{SL}_2(\mathbb{C})$ is of rank one, it follows that there exists a constant element $\alpha \in \text{SL}_2(\mathbb{C})$ such that $a_i(x) = \varphi_i(x) \alpha$, for functions $\varphi_i: U \mapsto \mathbb{C}$. A change of basis (rotating so that $\alpha \mapsto \tau_3$) implies that such patching matrices give rise to reducible connections when $a_0$ is real.

Remark 5.4. One of the main differences between the integrable systems approach to harmonic maps and the self-dual Yang–Mills equations is the form of the symmetry group action on the solutions. In the case of harmonic map equations from a domain $X \subseteq \mathbb{R}^2$ to a Lie group $G$, one interprets the harmonic map equations as implying the existence of a holomorphic map $E: X \to \Omega G$ into the based loop group of $G$. The “dressing action” on the space of harmonic maps is then induced by the action of various groups on the group $\Omega G$ [15,25]. In particular, the symmetry group acts only on the space where the map $E$ takes its values, rather than on the map $E$ itself. In the case of the self-dual Yang–Mills equations, the object of study is the patching matrix $G: U \times \mathcal{V}_e \to \mathfrak{sl}_2(\mathbb{C})$, and the group action (2.10) acts non-trivially on the map $G$. This difference is the main issue that makes the case of self-dual Yang–Mills equations more complicated. As remarked earlier, the particular form of the group action (2.10) implies that many of the techniques used to study orbits in the harmonic map case have no direct analogue in the self-dual Yang–Mills case.

6. Final Remarks

Our main result is that reducible connections that satisfy the self-dual Yang–Mills equations on simply-connected, open subsets of $\mathbb{R}^4$ lie in the orbit of the flat connection under the action of the non-local symmetry group of these equations found in [7–9]. In particular, such connections lie within a larger class of solutions, discussed in Sect. 4, defined by a holomorphic function $T(x, z)$ with the property that $[T(x, z), T^*(x, z)] = 0$. This condition defines a class of solutions of the self-dual Yang–Mills equations that seem quite natural from the integrable systems point of view,
and suggests a connection with the theory of harmonic maps of finite type. Whether the analogy with such harmonic maps may be extended, and techniques developed in, for example [5,6], may be adapted to the study of our class of self-dual connections, is under investigation.

It is clear that the work here (and in the sister paper [13]) may be extended in several ways. The investigation of the symmetry group on the one-instanton moduli space on the four-manifold $\mathbb{C}P^2$ would be of particular interest, since, in this case, the standard reducible connection is also $L^2$, so we have reducible and irreducible connections in the same moduli space. Such an investigation would yield further information concerning the different behaviour of reducible connections studied here and the instanton connections studied in [13] under the symmetry group. In a different direction, given that the reducible connections and the class discussed in Sect. 4 seem quite a natural family of solutions to investigate from the point of view of integrable systems, it would be of interest to investigate whether there are similar families of self-dual Ricci-flat four-manifolds (for example, those with algebraically special self-dual Weyl tensor) that arise naturally from the symmetries of, for example, Plebanski’s equations [21].

We should also point out that we have exclusively considered the self-dual Yang–Mills equations on Riemannian manifolds, due to the original motivation of Donaldson theory. It is more usual to investigate the integrable systems aspects of the self-dual Yang–Mills equations on manifolds of signature $(- - + +)$ (see, e.g., [19] for an extensive treatment of this topic). It would be of interest to investigate the action of symmetries on, for example, reducible connections in the case of signature $(- - + +)$.

Viewed in conjunction with the results of the companion paper [13], where instanton solutions of the self-dual Yang–Mills equations were investigated, it appears that the action of the non-local symmetry group on the space of solutions of the self-dual Yang–Mills equations is quite different in the two cases. In the case of instanton moduli spaces, evidence was found that the orbits of the symmetry group that preserve the $L^2$ nature of the curvature of the connection are rather small. In the present case, however, all reducible connections are contained in a single orbit. It appears that the distinction between instanton connections and reducible connections for the self-dual Yang–Mills equations are, in this sense, similar to the distinction between harmonic maps of finite uniton number [25] and harmonic maps of finite type. Since the original motivation for the current work (and [13]) was to investigate connections between integrable systems theory and Donaldson’s use of the self-dual Yang–Mills equations in connection with four-dimensional topology [11], it is rather striking that the behaviour of reducible connections and irreducible connections should be so different from the integrable systems point of view. Whether these results point to a deeper relationship between integrable systems theory and topological field theory would certainly seem worthy of further investigation.

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A. Constant Group Action

The group action $G(x, z) \mapsto h(x, z)G(x, z)h^*(x, z)$ is a little unusual. In order to gain some insight into this action, we consider some similar actions on simpler groups, analogous to the case where $G$ and $h$ are constant.
A.1. \( SL_2(\mathbb{R}) \). Consider the action of \( SL_2(\mathbb{R}) \) on itself given by

\[
SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \to SL_2(\mathbb{R}); \quad (h, g) \mapsto h \cdot g := hgh^t,
\]
where \(^t\) denotes transpose. The subgroup \( PSL_2(\mathbb{R}) \cong SO^0_{2,1} \) acts effectively. We decompose \( g \) into symmetric and skew-symmetric parts

\[
g = U + \alpha \epsilon,
\]
where \( U \) is symmetric, \( \alpha \in \mathbb{R} \) and \( \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). It then follows that \( \alpha \) is invariant under the action of \( h \) and \( U \) transforms according to \( U \mapsto hUh^t \). The fact that \( g \) lies in \( SL_2(\mathbb{C}) \) implies that

\[
\det U = 1 - \alpha^2.
\]

Writing

\[
U = \begin{pmatrix} t + x & y \\ y & t - x \end{pmatrix},
\]
then \( \det U = -\|u\|^2 \), where \( u = (t, x, y) \in \mathbb{R}^{2,1} \). The orbits of the group action are then parametrised by \( \alpha \in \mathbb{R} \) and consist of vectors \( u \in \mathbb{R}^{2,1} \) with

\[
\|u\|^2 = \alpha^2 - 1.
\]

Since the restriction on \( u \) is insensitive to the sign of \( \alpha \), we consider the orbits for \( \alpha \geq 0 \):

- \( \alpha = 0 \) Here there are two orbits consisting of symmetric elements of \( SL_2(\mathbb{R}) \). We have \( \|u\|^2 = -1 \), so \( u \) lies on the two-sheeted hyperboloid in \( \mathbb{R}^{2,1} \), with each sheet constituting an orbit. In this case, giving the orbits the induced hyperbolic metric, the group \( SL_2(\mathbb{R}) \) acts isometrically.

- \( 0 < \alpha < 1 \) In this case, there are two orbits, i.e. the two components of the hyperboloid \( \|u\|^2 = -1 + \alpha^2 \in (-1, 0) \) in \( \mathbb{R}^{2,1} \). Again, the group \( SL_2(\mathbb{R}) \) acts isometrically with respect to the induced metric on the orbits.

- \( \alpha = 1 \) In this case, \( \|u\|^2 = 0 \), so either \( u = 0 \) or \( u \) is null. In the first case, the group orbit consists of the point \( u = 0 \). In the latter case, the future and past null-cones of the origin give two distinct group orbits.

- \( \alpha > 1 \) In this case, there is one orbit, consisting of the one-sheeted hyperboloid \( \|u\|^2 = -1 + \alpha^2 \in (1, \infty) \) in \( \mathbb{R}^{2,1} \). In this case, \( SL_2(\mathbb{R}) \) acts isometrically with respect to the induced (Lorentzian) metric on the orbit.

A.2. \( SL_2(\mathbb{C}) \). In particular, we consider the action of \( SL_2(\mathbb{C}) \) on itself given by

\[
SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to SL_2(\mathbb{C}); \quad (h, g) \mapsto h \cdot g := hgh^*,
\]
where \(*\) denotes complex-conjugate transpose. The subgroup \( PSL_2(\mathbb{C}) \cong SO_{3,1} \) acts effectively. It is straightforward to check that

\[
I[g] := \frac{1}{2} \text{tr} \left( g \left( g^{-1} \right)^* \right)
\]

is invariant under the transformation \( g \mapsto h \cdot g \). It is useful to split \( g \) into Hermitian and skew-Hermitian parts:
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\[ g = U + V, \quad \text{where } U^* = U, \quad V^* = -V, \]

and to note that this decomposition is preserved under (A.1) (i.e. \((h \cdot g)_U = h \cdot g_U\), etc.). A straightforward calculation implies that

\[ \det U = \frac{1}{2} (I + 1). \]

In particular, letting \( U = t + x\tau_1 + y\tau_2 + z\tau_3 \) and \( u := (t, x, y, z) \in \mathbb{R}^{3,1} \), we deduce that

\[ \|u\|^2 = -\frac{1}{2} (I + 1). \quad (A.2) \]

Similarly, letting \( V = i (T + X\tau_1 + Y\tau_2 + Z\tau_3) \) and \( v := (T, X, Y, Z) \in \mathbb{R}^{3,1} \), we find that

\[ \det g = -\|u\|^2 - 2i \langle u, v \rangle + \|v\|^2. \]

Since \( g \in \text{SL}_2(\mathbb{C}) \), we therefore deduce that

\[ \|v\|^2 = -\frac{1}{2} (I - 1), \quad (A.3) \]

\[ \langle u, v \rangle = 0. \quad (A.4) \]

If \( I > 1 \) then \( u \) and \( v \) would be non-zero, orthogonal, time-like vectors in \( \mathbb{R}^{3,1} \). Since this cannot occur, we deduce that \( I \leq 1 \). We investigate the distinct cases separately:

\[ I = 1 \]

In this case, \( \|u\|^2 = -1 \) and \( \|v\|^2 = 0 \). As such, \( u \) lies on the two-sheeted hyperboloid in \( \mathbb{R}^{3,1} \). The condition that \( \|v\|^2 = 0 \) and is orthogonal to the non-zero, time-like vector \( u \) then implies that \( v = 0 \). As such, we have two distinct orbits, corresponding to the two components of the two-sheeted hyperboloid. These orbits correspond to the Hermitian elements of \( \text{SL}_2(\mathbb{C}) \). Giving the orbits the hyperbolic metric induced from \( \mathbb{R}^{3,1} \), the group \( \text{SL}_2(\mathbb{C}) \) acts isometrically.

Note that for \( I < 1 \), the vector \( v \) is always space-like, and lies on the one-sheeted hyperboloid \( \Sigma_I := \{ w \in \mathbb{R}^{3,1} : \|w\|^2 = \frac{1}{2} (1 - I) \} \) in \( \mathbb{R}^{3,1} \).

\[ -1 < I < 1 \]

We have \( \|u\|^2 = -(I + 1)/2 \) in \( \mathbb{R}^{3,1} \). Since \( u \) is orthogonal to \( v \), we may view \( u \) as a time-like vector of length \( \sqrt{(I + 1)/2} \) lying in the two-sheeted hyperboloid in \( T_v \Sigma_I \). As such, we have two distinct orbits, consisting of the two components of the two-sheeted hyperboloid bundle in \( T \Sigma_I \). In this case, the group action on the orbit is the action induced by the isometric action of \( \text{SL}_2(\mathbb{R}) \) on the Lorentzian metric induced on the one-sheeted hyperboloid.

Alternatively, we may view \( u \) as a time-like vector lying on the two-sheeted hyperboloid \( \|u\|^2 = -(I + 1)/2 \) in \( \mathbb{R}^{3,1} \). We then view \( v \) as a tangent vector to the hyperboloid of length \( \frac{1}{\sqrt{2}} (1 + |I|) \). Therefore the orbits in this case may be identified with the radius \( \frac{1}{\sqrt{2}} (1 + |I|) \) sphere sub-bundle of the tangent bundle of the hyperbolic space of radius \( \frac{1}{\sqrt{2}} (I + 1) \). Again, there are two orbits corresponding to the two components of the hyperboloid. In this case, the group action on the orbit is the action induced by the isometric action of \( \text{SL}_2(\mathbb{R}) \) on the induced metric on the two-sheeted hyperboloid.

\[ I = -1 \]

In this case, \( \|u\|^2 = 0 \) and \( \|v\|^2 = 1 \). As such, we may view \( u \) as a null vector in \( T_v \Sigma_{-1} \). There are then three distinct orbits. The first consists of \( u = 0 \), and is simply the hyperboloid \( \Sigma_{-1} \). This orbit consists of the skew-Hermitian elements of \( \text{SL}_2(\mathbb{C}) \).
The other orbits consist of the sub-bundle of $\mathcal{T}_{-1}$ consisting of the past and future null cone of the origin in each tangent space. In this case, $\|u\|^2 = (|I| + 1)/2 > 0$ in $\mathbb{R}^{3,1}$. Therefore there is one orbit, consisting of the one-sheeted hyperboloid sub-bundle of $\mathcal{T}_{-1}$. The $\text{SL}_2(\mathbb{C})$ action is that induced by the isometric action on the induced Lorentzian metric on $\Sigma_I$.

References


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Curriculum Vitae

Personal information.

Full name: James Dodds Ellis Grant
Date of Birth: 5 October 1967
Nationality: British

Academic Record.

1989–1990: Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Certificate of Advanced Study in Mathematics (with Distinction).
1985–1989: University of Edinburgh, B.Sc. (Hons.) in Mathematical Physics (First Class), Graduated top of year.

Postdoctoral Career.

• Visiting Professor in Fakultät für Mathematik, Universität Wien (01.10.07–31.01.08).
• Senior postdoctoral position in Fakultät für Mathematik, Universität Wien (01.03.05–30.09.07 and 01.02.08–Present).
• Postdoctoral positions in University of Aberdeen (01.09.03–28.02.05), Università degli Studi dell’Aquila (01.11.02–31.08.03), University of Hull (01.10.97–31.10.02), University of Newcastle (01.10.95–30.09.97), CINVESTAV del IPN, México City (01.09.04–31.08.05), University of Pittsburgh (01.09.03–31.08.04).
• Invited researcher at the California Institute of Technology (1992 and 1993), Keio University (2004), University of Innsbruck (2005), University of L’Aquila (2006), Banff International Research Station (2007), and Paris VI/CNRS (January and April 2009).