Manifolds – Problem Solutions

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These notes are slightly modified from those written by Neil Lambert and Alice Rogers
1. Manifolds

Problem 1.1. Show that the induced topology indeed satisfies the definition of a topology.

Solution:
Let \( U = \{U_i\} \) be the topology of \( Y \) and \( X \subset Y \). The induced topology is \( V = \{U_i \cap X | U_i \in \mathcal{U}\} \).

i) \( \emptyset = \emptyset \cap X \) so since \( \emptyset \in \mathcal{U} \) we have \( \emptyset \in \mathcal{V} \).

ii) \( X = X \cap Y \) so since \( Y \in \mathcal{U} \) we have \( X \in \mathcal{V} \).

iii) Take \( V_i = U_i \cap X \in \mathcal{V} \) then
\[
V_1 \cap V_2 \cap \ldots \cap V_n = (U_1 \cap X) \cap (U_2 \cap X) \cap \ldots \cap (U_n \cap X)
\]
\[
= (U_1 \cap U_2 \cap \ldots \cap U_n) \cap X
\in \mathcal{V}
\] (1.1)

iv) For an arbitrary number of \( V_i \)'s:
\[
\bigcup_i V_i = \bigcup_i (U_i \cap X) = \left( \bigcup_i U_i \right) \cap X \in \mathcal{V}
\] (1.2)

Problem 1.2. Why aren’t closed subsets of \( \mathbb{R}^n \), e.g. a disk with boundary or a line in \( \mathbb{R}^2 \), along with the identity map charts (note that in its own induced topology any subset of \( \mathbb{R}^n \) is an open set)?

Solution: If \( C \subset \mathbb{R}^n \) is closed we may still view it as an open set in its own induced topology. The identity map \( id : C \to \mathbb{R}^n \) is certainly continuous (in inverse image of an open set is open by definition of the induced topology). It is also clearly 1-1 and hence is a bijection onto its image. However consider \( id^{-1} : id(C) \subset \mathbb{R}^n \to C \) this has
\[
(id^{-1})^{-1}(C) = C
\] (1.3)
but since \( C \) is open in its own induced topology and closed in the topology of \( \mathbb{R}^n \) we see that \( id^{-1} \) is not continuous (as \( (id^{-1})^{-1} \) of an open set is closed). Thus \( id \) is not a homeomorphism.

Problem 1.3. What is \( \mathbb{R}P^1 \)?

Solution:
By definition \( \mathbb{R}P^1 = \{(x, y) \in \mathbb{R}^2 - (0, 0)|(x, y) \sim (\lambda x, \lambda y), \lambda \in \mathbb{R} - 0\} \). A point \( (x, y) \in \mathbb{R}^2 - (0, 0) \) defines a line through the origin. The point \( (\lambda x, \lambda y) \) with \( \lambda \neq 0 \) will define the same line as \( (x, y) \). Thus \( \mathbb{R}P^1 \) is the space of lines through the origin.

On the other and a line through the origin is specified by the angle (roughly \( \theta = \arctan(y/x) \)) it makes with the postive x-axis. Since \( (x, y) \) and \( (-x, -y) \) define the same line this angle is identified modulo \( \pi \) rather than \( 2\pi \). Thus \( \mathbb{R}P^1 \) can be identified with a circle.

To make this more precise one should construct a diffeomorphism from \( \mathbb{R}P^1 \) to \( S^1 \). Try this.
Problem 1.4. Show that the following:

\[ U_1 = \{(x, y) \in S^1 | y > 0 \} , \quad \phi_1(x, y) = x \]
\[ U_2 = \{(x, y) \in S^1 | y < 0 \} , \quad \phi_2(x, y) = x \]
\[ U_3 = \{(x, y) \in S^1 | x > 0 \} , \quad \phi_3(x, y) = y \]
\[ U_4 = \{(x, y) \in S^1 | x < 0 \} , \quad \phi_4(x, y) = y \]

are a set of charts which cover \( S^1 \).

Solution:

It should be clear that all the \( U_i \) are open and cover \( S^1 \) and that the \( \phi_i \) continuous with \( \phi_i(U_i) = (-1, 1) \). Their inverses are

\[ \phi_i^{-1}(\theta) = (\theta, \sqrt{1 - \theta^2}) \]

which are continuous for \( \theta \in (-1, 1) \) hence they are homeomorphisms (onto their image).

Next we must check that \( \phi_i \circ \phi_j^{-1} : (-1, 1) \to (-1, 1) \) are \( C^\infty \) for all non-intersecting pairs. Thus we must check that

\[ \phi_1 \circ \phi_3^{-1}(\theta) = \sqrt{1 - \theta^2} , \quad \phi_3 \circ \phi_1^{-1}(\theta) = \sqrt{1 - \theta^2} \]
\[ \phi_1 \circ \phi_4^{-1}(\theta) = -\sqrt{1 - \theta^2} , \quad \phi_4 \circ \phi_1^{-1}(\theta) = \sqrt{1 - \theta^2} \]
\[ \phi_2 \circ \phi_3^{-1}(\theta) = \sqrt{1 - \theta^2} , \quad \phi_3 \circ \phi_2^{-1}(\theta) = -\sqrt{1 - \theta^2} \]
\[ \phi_2 \circ \phi_4^{-1}(\theta) = -\sqrt{1 - \theta^2} , \quad \phi_4 \circ \phi_2^{-1}(\theta) = \sqrt{1 - \theta^2} \]

These are all \( C^\infty \) since \( \theta \in (-1, 1) \).

Problem 1.5. Show that the 2-sphere \( S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\} \) is a 2-dimensional manifold.

Solution:

The hint was to consider stereographic projection. This requires using two charts \( U_S = \{(x, y, z) \in S^2 | z < 1 \} \) and \( U_N = \{(x, y, z) \in S^2 | z > -1 \} \).

these are clearly open and cover \( S^2 \). In each chart one constructs \( \phi_{N/S} : U_{N/S} \to \mathbb{R}^2 \) by taking a straight line through either the south pole \((0, 0, -1)\) or north pole \((0, 0, 1)\) and then through the point \( p \in U_{N/S} \). These lines are defined by the equation

\[ X(\lambda) = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} + \lambda \begin{pmatrix} x \\ y \\ z + 1 \end{pmatrix} \]
so that $X(0)$ is either the north or south pole and $X(1)$ is a point on $S^2$. We define $\phi_{N/S}(p)$ to be the point in the $(x, y)$-plane where the line intersects $z = 0$.

**Figure 1:** Stereographic projection from $U_S = S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{R}^2$

Note that this construction works on all of $U_N$ and on all of $U_S$ respectively but not on all of $S^2$. Explicitly one has

$$
\begin{align*}
\phi_S(x, y, z) &= \left( \frac{x}{1 - z}, \frac{y}{1 - z} \right) \\
\phi_N(x, y, z) &= \left( \frac{x}{1 + z}, \frac{y}{1 + z} \right)
\end{align*}
$$

By construction these maps are injective as they define a unique line and this will intersect the $z = 0$ plane at a unique point. They are continuous and their inverses (to construct them consider the line through $(u, v, 0)$ and $(0, 0, \pm 1)$ and see where it intersects $S^2$):

$$
\begin{align*}
\phi_S^{-1}(u, v) &= \left( \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{u^2 + v^2 - 1}{1 + u^2 + v^2} \right) \\
\phi_N^{-1}(u, v) &= \left( \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right)
\end{align*}
$$

These are also continuous. Finally we must simply observe that

$$
\begin{align*}
\phi_S \circ \phi_N^{-1}(u, v) &= \left( \frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right) \\
\phi_N \circ \phi_S^{-1}(u, v) &= \left( \frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right)
\end{align*}
$$
are $C^\infty$ on $\phi_N(U_N \cap U_S) = \phi_S(U_N \cap U_S) = \mathbb{R} - (0,0)$.

2. The Tangent Space

**Problem 2.1.** Consider the circle $S^1$ as above. Show that $f : S^1 \to \mathbb{R}$ with $f(x,y) = x^2 + y$ is $C^\infty$.

**Solution:**
We can take the coordinates above. We need to consider $f \circ \phi_i^{-1}(\theta) : \phi_i(U_i) \to \mathbb{R}$:

$$
\begin{align*}
  f \circ \phi_1^{-1}(\theta) &= \theta^2 + \sqrt{1-\theta^2} \\
  f \circ \phi_2^{-1}(\theta) &= \theta^2 - \sqrt{1-\theta^2} \\
  f \circ \phi_3^{-1}(\theta) &= 1 - \theta^2 + \theta \\
  f \circ \phi_4^{-1}(\theta) &= 1 - \theta^2 + \theta
\end{align*}
$$

(2.1)

clearly all these functions are $C^\infty$ on $\phi_i(U_i) = (-1,1)$.

3. Maps Between Manifolds

**Problem 3.1.** Show that $f : S^1 \to S^1$ defined by $f(e^{2\pi i \theta}) = e^{2\pi i n \theta}$ is $C^\infty$ for any $n$.

**Solution:**
Again we choose the same charts and note that $\theta \in [0,1]$. First observe that

$$
\begin{align*}
  f \circ \phi_1^{-1}(\theta) &= e^{in \arctan(\sqrt{1-\theta^2}/\theta)} \\
  f \circ \phi_2^{-1}(\theta) &= e^{-in \arctan(\sqrt{1-\theta^2}/\theta)} \\
  f \circ \phi_3^{-1}(\theta) &= e^{in \arctan(\theta/\sqrt{1-\theta^2})} \\
  f \circ \phi_4^{-1}(\theta) &= e^{-in \arctan(\theta/\sqrt{1-\theta^2})}
\end{align*}
$$

(3.1)

Note that this is well defined since $\theta \neq 0$ on $\phi_1(U_1), \phi_2(U_2)$ and $\theta \neq \pm 1$ on $\phi_3(U_3), \phi_4(U_4)$. For any function $\chi$ we also have that

$$
\begin{align*}
  \phi_1(e^{in\chi}) &= \cos(n\chi) \\
  \phi_2(e^{in\chi}) &= \cos(n\chi) \\
  \phi_3(e^{in\chi}) &= \sin(n\chi) \\
  \phi_4(e^{in\chi}) &= \sin(n\chi)
\end{align*}
$$

(3.2)
thus we see that $\phi_i \circ f \circ \phi_j^{-1}(\theta)$ has the form:
\[
\begin{align*}
\cos(n \arctan(\sqrt{1 - \theta^2}/\theta)) \\
\pm \sin(n \arctan(\sqrt{1 - \theta^2}/\theta)) \\
\cos(n \arctan(\theta/\sqrt{1 - \theta^2})) \\
\pm \sin(n \arctan(\theta/\sqrt{1 - \theta^2}))
\end{align*}
\]
(3.3)

And these are all $C^\infty$ on the appropriate range of $\theta$.

**Problem 3.2.** Show that the charts of two diffeomorphic manifolds are in a one to one correspondence.

**Solution:**

Let $\{U_i, \phi_i\}$ and $\{V_a, \psi_a\}$ be differential structures for $\mathcal{M}$ and $\mathcal{N}$ respectively and $f : \mathcal{M} \to \mathcal{N}$ a diffeomorphism.

First we show that $\{f(U_i), \phi_i \circ f^{-1}\}$ is a set of charts that cover $\mathcal{N}$. We note that these cover $\mathcal{N}$:
\[
\cup_i f(U_i) = f(\cup_i U_i) = f(\mathcal{M}) = \mathcal{N}
\]
(3.4)

Furthermore $\phi_i \circ f^{-1}$ are clearly homeomorphisms (bijective, continuous with the inverse continuous).

Similarly $\{f^{-1}(V_a), \psi_a \circ f\}$ is a set of charts that cover $\mathcal{M}$.

Now on $V_a \cap f(U_i)$ we have that
\[
\psi_a \circ f^{-1} \circ \phi_i^{-1} : \phi_i(V_a \cap f(U_i)) \to \psi_a(V_a \cap f(U_i))
\]
(3.5)

and this is $C^\infty$ as $f$ is $C^\infty$ (recall the definition). Thus the charts $\{f(U_i), \phi_i \circ f^{-1}\}$ are compatible with the charts $\{V_a, \psi_a\}$. Since we take the differential structure to be maximal we find that the charts $\{f(U_i), \phi_i \circ f^{-1}\}$ must be included in the differential structure $\{V_a, \psi_a\}$ of $\mathcal{N}$.

Similarly $\{f^{-1}(V_a), \psi_a \circ f\}$ are compatible with the charts $\{U_i, \phi_i\}$ and since we assume the differential structure to be maximal it follows that the $\{f^{-1}(V_a), \psi_a \circ f\}$ are included in $\{U_i, \phi_i\}$.

Thus it follows that $\{V_a, \psi_a\}$ and $\{U_i, \phi_i\}$ are in a one-to-one correspondence with each other.

**Problem 3.3.** Show that the set of diffeomorphisms from a manifold to itself forms a group under composition.

**Solution:**

Suppose $f, g : \mathcal{M} \to \mathcal{M}$ are diffeomorphims. Then
\[
\phi_i \circ (f \circ g) \circ \phi_j^{-1} = \phi_i \circ (f \circ \phi_k^{-1} \circ \phi_j \circ g) \circ \phi_j^{-1} = (\phi_i \circ f \circ \phi_k^{-1}) \circ (\phi_k \circ g \circ \phi_j^{-1})
\]
(3.6)

is $C^\infty$ for all $i, j, k$ since $f$ and $g$ are $C^\infty$. Similarly for $g^{-1} \circ f^{-1}$. Furthermore $f \circ g$ is a bijection if both $f$ and $g$ are. Thus $f \circ g$ is a diffeomorphism.

Clearly $id : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism and by definition (the properties are symmetric between $f$ and $f^{-1}$) if $f$ is a diffeomorphism then so is $f^{-1}$. 


4. Vector Fields

Problem 4.1. What goes wrong if try to define \((X \cdot Y)(f) = X(f) \cdot Y(f)\)?

Solution:

With this definition we find that
\[
X \cdot Y(f + g) = (X(f) + X(g))(Y(f) + Y(g))
= X(f)Y(f) + X(f)Y(g) + X(g)Y(f) + X(g)Y(g)
= X \cdot Y(f) + X \cdot Y(g) + X(f)Y(g) + X(g)Y(f)
\]

(4.1)

and the last two terms are unwanted.

If we try the same trick that we used for the commutator and define \([X, Y](f) = X(f)Y(f) - Y(f)X(f)\) then this clearly vanishes identically.

Problem 4.2. Show that, if in a particular coordinate system,
\[
X = \sum_{\mu} X^\mu(x) \frac{\partial}{\partial x^\mu}_p, \quad Y = \sum_{\mu} Y^\mu(x) \frac{\partial}{\partial x^\mu}_p
\]

(4.2)

then
\[
[X, Y] = \sum_{\mu} \sum_{\nu} \left( X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu \right) \frac{\partial}{\partial x^\nu}_p
\]

(4.3)

Solution:

We simply calculate:
\[
X(Y)(f) = X \left( \sum_{\nu} Y^\nu \frac{\partial}{\partial x^\nu} (f \circ \phi_i^{-1}) \circ \phi_i \right)
= \sum_{\mu} \left( X^\mu \frac{\partial}{\partial x^\mu} \left( \sum_{\nu} Y^\nu \frac{\partial}{\partial x^\nu} (f \circ \phi_i^{-1}) \circ \phi_i \circ \phi_i^{-1} \right) \circ \phi_i \right)
= \sum_{\mu} \sum_{\nu} \left( X^\mu \partial_\mu Y^\nu \frac{\partial}{\partial x^\nu} (f \circ \phi_i^{-1}) + X^\mu Y^\nu \frac{\partial^2}{\partial x^\mu \partial x^\nu} (f \circ \phi_i^{-1}) \right) \circ \phi_i
\]

(4.4)

Since the second term is symmetric in \(X^\mu\) and \(Y^\nu\) we find that
\[
X(Y)(f) - Y(X)(f) = \sum_{\mu} \sum_{\nu} \left( X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu \right) \frac{\partial}{\partial x^\nu} (f \circ \phi_i^{-1}) \circ \phi_i
\]

(4.5)

and we prove the theorem.

Problem 4.3. Show that for three vector fields \(X, Y, Z\) on \(M\) the Jacobi identity holds:
\[
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0
\]

(4.6)
Solution:
Here we simply expand things out; suppose \( f \in C^\infty(M) \), then

\[
\left( [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \right)(f) = X([Y, Z]f) - [Y, Z](Xf) + Y([Z, X]f)
\]

\[
-[Z, X](Yf) + Z([X, Y]f) - [X, Y](Zf)
\]

\[
= X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf)) + Y(Z(\mathcal{X})) - Z(Y(\mathcal{X})) + X(\mathcal{X})
\]

\[
= 0
\]

(4.7)

where we used the linearity properties of vectors, i.e. \( X(Y + Z)f = X(Yf) + X(Zf) \).

**Problem 4.4.** Consider a manifold with a local coordinate system \( \phi_i = (x^1, \ldots, x^n) \).

i) Show that \[ \frac{\partial}{\partial x^i} \bigg|_p, \frac{\partial}{\partial x^v} \bigg|_p \] = 0

ii) Evaluate \[ \frac{\partial}{\partial x^i} \bigg|_p, \varphi(x^1, x^2) \frac{\partial}{\partial x^v} \bigg|_p \] where \( \varphi(x^1, x^2) \) is a \( C^\infty \) function of \( x^1, x^2 \).

**Solution:**
In the first case we find

\[
\left[ \frac{\partial}{\partial x^i}, \varphi \frac{\partial}{\partial x^v} \right](f) = \frac{\partial}{\partial x^i} \bigg|_p \left( \frac{\partial}{\partial x^v} (f \circ \phi_i^{-1}) \circ \phi_i \right) - \frac{\partial}{\partial x^v} \bigg|_p \left( \frac{\partial}{\partial x^i} (f \circ \phi_i^{-1}) \circ \phi_i \right)
\]

\[
= \frac{\partial^2}{\partial x^i \partial x^v} (f \circ \phi_i^{-1}) \circ \phi_i - \frac{\partial^2}{\partial x^v \partial x^i} (f \circ \phi_i^{-1}) \circ \phi_i
\]

\[
= 0
\]

(4.8)

And in the second case:

\[
\left[ \frac{\partial}{\partial x^i}, \varphi \frac{\partial}{\partial x^v} \right](f) = \varphi \frac{\partial^2}{\partial x^v \partial x^i} (f \circ \phi_i^{-1}) \circ \phi_i + \left( \frac{\partial}{\partial x^i} \bigg|_p \varphi \right) \frac{\partial}{\partial x^v} \bigg|_p (f)
\]

\[
- \varphi \frac{\partial^2}{\partial x^i \partial x^v} (f \circ \phi_i^{-1}) \circ \phi_i
\]

\[
= \left( \frac{\partial}{\partial x^i} \bigg|_p \varphi \right) \frac{\partial}{\partial x^v} \bigg|_p (f)
\]

(4.9)

so

\[
\left[ \frac{\partial}{\partial x^i}, \varphi \frac{\partial}{\partial x^v} \right] = \left( \frac{\partial}{\partial x^i} \bigg|_p \varphi \right) \frac{\partial}{\partial x^v} \bigg|_p
\]

(4.10)